THE PUBLISHING HOUSE
OF THE ROMANIAN ACADEMY

PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A
Volume 26, Number 1/2025, pp. 11–20
DOI: 10.59277/PRA-SER.A.26.1.02

SEVERAL BINOMIAL SUMMATION FORMULAE

Yulei CHEN [1], Dongwei GUO [2]

[1] Zhoukou Normal University, School of Mathematics and Statistics, Zhoukou (Henan), P.R. China
[2] Nanjing University of Science and Technology, School of Economics and Management, Nanjing (Jiangsu), P.R. China
Corresponding author: Yulei CHEN, E-mail: chenyulei225@yeah.net
Dongwei GUO, E-mail: guo.dongwei@njust.edu.cn

Abstract. In this work, by means of the Omega operator we establish a binomial summation formula. Its particular cases result in several interesting summation formulae. One of them solves a monthly problem proposed recently by Ohtsuka and Tauraso (2021).

***Keywords**:* triple sums, Omega operator, binomial coefficients, telescoping.
Mathematics Subject Classification (MSC2020): Primary 11B65, Secondary 05A19.

1. INTRODUCTION AND MOTIVATION

In [3], Calkin proved the following curious binomial identity:

$$\sum_{k=0}^{n}\left(\sum_{j=0}^{k}\binom{n}{j}\right)^3 = n2^{3n-1}+2^{3n}-3n2^{n-2}\binom{2n}{n}.$$

In 1996, Hirschhorn [5] re-proved Calkin's identity with a direct method. Furthermore, he also established recurrence relations on the sum

$$S_p=\sum_{k=0}^{n}\left(\sum_{j=0}^{k}\binom{n}{j}\right)^p,$$

and got the identity below:

$$\sum_{k=0}^{n}\left(\sum_{j=0}^{k}\binom{n}{j}\right)^2 = (n+2)2^{2n-1}-\frac{1}{2}n\binom{2n}{n}.$$

Zhang [9, 10] discussed the recurrence formulae about the alternating case

$$R_p=\sum_{k=0}^{n}(-1)^k\left(\sum_{j=0}^{k}\binom{n}{j}\right)^p$$

and obtained the following identities

$$\sum_{k=0}^{n}(-1)^k\left(\sum_{j=0}^{k}\binom{n}{j}\right)^2 = \begin{cases}1, & n=0;\\ 2^{2n-1}, & n\equiv_2 0;\\ -2^{2n-1}-(-1)^{\frac{n-1}{2}}\binom{n-1}{\frac{n-1}{2}}, & n\equiv_2 1,\end{cases}$$

and

$$\sum_{k=0}^{n}(-1)^k\left(\sum_{j=0}^{k}\binom{n}{j}\right)^3=-2^{3n-1}-3(-1)^{\frac{n-1}{2}}2^{n-1}\binom{n-1}{\frac{n-1}{2}},\ \text{with } n\equiv_2 1,$$

where $n\equiv_p m$ stands for "n is congruent to m modulo p".

In 2004, Wang and Zhang [8] considered the following summations:

$$\sum_{k=0}^{n}k^t\left(\sum_{j=0}^{k}\binom{n}{j}\right),\quad \sum_{k=0}^{n}k^t\left(\sum_{j=0}^{k}\binom{n}{j}\right)^2 \quad\text{and}\quad \sum_{k=0}^{n}k^t\left(\sum_{j=0}^{k}\binom{n}{j}\right)^3,$$

which extends Calkin's formula.

Andrews and Paule [1] proved Calkin's identity by using the Omega operator $\Omega_{\geq}$ [2,6]. In 2006, by the same approach, Zhang [10] got the calculation formula of the general case

$$\sum_{k=0}^{n}f_k\left(\sum_{j=0}^{k}\binom{n}{j}g_j\right)^2,$$

where f_k and g_k are two real number sequences.

In this paper, we shall consider the following triple sum involving binomial coefficients:

$$\sum_{k=0}^{n}A_k\left(\sum_{i=0}^{k}B_i\binom{n}{i}\right)\left(\sum_{j=0}^{k}C_j\binom{n}{j}\right), \tag{1}$$

where A_k, B_k and C_k are real number sequences. When taking $A_k=(-1)^k$, $B_k=r^k$ and $C_k=(-r)^k$, we give a solution of the monthly problem proposed recently by Ohtsuka and Tauraso [7]:

$$\sum_{k=0}^{n}(-1)^k\left(\sum_{j=0}^{k}r^j\binom{n}{j}\right)\left(\sum_{j=0}^{k}(-r)^j\binom{n}{j}\right)=\left(\frac{(r+1)^n+(r-1)^n}{2}\right)^2. \tag{2}$$

In the next section, we shall, by using the Omega operator $\Omega_{\geq}$, establish a formula to calculate the sum (1). In section 3, we shall use the theorem established in section 2 to obtain some spacial cases, including (2). For convenience, we will use of the following notation: $[x^n]f(x)$ denotes the coefficient of x^n in the formal power series $f(x)$.

2. MAIN THEOREM

In this section, we shall establish a theorem by means of the Omega operator $\Omega_{\geq}$, which was introduced by MacMahon in [6]. Here we repeat it as follows (also see [1,2,10]).

Definition 1. *The Omega operator $\Omega_{\geq}$ is defined by*

$$\Omega_{\geq}\sum_{s_1=-\infty}^{\infty}\cdots\sum_{s_r=-\infty}^{\infty}A_{s_1,\cdots,s_r}\lambda_1^{s_1}\cdots\lambda_r^{s_r}=\sum_{s_1=0}^{\infty}\cdots\sum_{s_r=0}^{\infty}A_{s_1,\cdots,s_r},$$

where the domain of the terms $A_{s_1,\cdots,s_r}$ (e.g., functions of power series) is such that the action is well-defined in some suitable analytic or algebraic context.

By means of the definition we have the following lemma [10] to be used later.

LEMMA 1. *For a function $f(x)$ defined on the nonnegative integers, it holds that*

$$f(\max\{k_1,k_2\})-f(\min\{k_1,k_2\})=\Omega_{\geq}\left\{\big(f(k_1)-f(k_2)\big)\lambda^{k_1-k_2}+\big(f(k_2)-f(k_1)\big)\lambda^{k_2-k_1}\right\}.$$

For sequences $\{A_k\}$ and $\{B_k\}$, adopting the representation used in [10]: $S_A(n)=\sum\limits_{k=0}^{n-1}A_k$, $T_B(n)=\sum\limits_{k=0}^{n}\binom{n}{k}B_k$, $P_B(n,k)=\sum\limits_{j=k}^{n}\binom{n}{j}B_jS_A(j)$, and $Q_B(n,k)=\sum\limits_{j=k}^{n}\binom{n}{j}B_{n-j}S_A(n-j)$, we have the following main theorem in this paper.

THEOREM 2.

$$\begin{aligned}\sum_{k=0}^{n}A_k\left(\sum_{i=0}^{k}B_i\binom{n}{i}\right)\left(\sum_{j=0}^{k}C_j\binom{n}{j}\right)&=S_A(n+1)T_B(n)T_C(n)\\&-\frac{1}{2}\left\{T_B(n)P_C(n,0)+T_C(n)P_B(n,0)+\sum_{k=0}^{n}\binom{n}{k}\big(\Delta_1(n,k)+\Delta_2(n,k)\big)\right\},\end{aligned}$$

where

$$\Delta_1(n,k)=C_kP_B(n,k)-B_{n-k}Q_C(n,k),\ \Delta_2(n,k)=B_kP_C(n,k)-C_{n-k}Q_B(n,k).$$

Proof. By exchanging summation order, we have

$$\begin{aligned}\sum_{k=0}^{n}A_k\left(\sum_{i=0}^{k}B_i\binom{n}{i}\right)\left(\sum_{j=0}^{k}C_j\binom{n}{j}\right)&=\sum_{k\geq i,j=0}B_iC_j\binom{n}{i}\binom{n}{j}\sum_{k=\max\{i,j\}}^{n}A_k\\&=\sum_{k\geq i,j=0}B_iC_j\binom{n}{i}\binom{n}{j}\big\{S_A(n+1)-S_A(\max\{i,j\})\big\}\\&=S_A(n+1)T_B(n)T_C(n)-M_{BC}(n),\end{aligned}$$

where

$$M_{BC}(n)=\sum_{k\geq i,j=0}B_iC_j\binom{n}{i}\binom{n}{j}S_A(\max\{i,j\}).$$

Denoting $N_{BC}(n)$ by

$$N_{BC}(n)=\sum_{k\geq i,j=0}B_iC_j\binom{n}{i}\binom{n}{j}S_A(\min\{i,j\}),$$

then $M_{BC}(n)$ plus $N_{BC}(n)$ gives

$$\begin{aligned}M_{BC}(n)+N_{BC}(n)&=\sum_{k\geq i,j=0}B_iC_j\binom{n}{i}\binom{n}{j}\big\{S_A(\max\{i,j\})+S_A(\min\{i,j\})\big\}\\&=\sum_{k\geq i,j=0}B_iC_j\binom{n}{i}\binom{n}{j}\big\{S_A(i)+S_A(j)\big\}\\&=T_BP_C(n,0)+T_CP_B(n,0),\end{aligned}$$

and $M_{BC}(n)$ minus $N_{BC}(n)$ yields

$$M_{BC}(n)-N_{BC}(n)=\sum_{k\geq i,j=0}B_iC_j\binom{n}{i}\binom{n}{j}\big\{S_A(\max\{i,j\})-S_A(\min\{i,j\})\big\}.$$

By means of Lemma 1, we get

$$\begin{aligned}M_{BC}(n)-N_{BC}(n)=\Omega_{\geq}&\sum_{k\geq i,j=0}B_iC_j\binom{n}{i}\binom{n}{j}\\&\times\big\{\big(S_A(i)-S_A(j)\big)\lambda^{i-j}+\big(S_A(j)-S_A(i)\big)\lambda^{j-i}\big\}\end{aligned}$$

$$= \Omega_1 + \Omega_2,$$

where

$$\Omega_1 = \Omega_{\geq} \sum_{k \geq i, j=0} B_i C_j \binom{n}{i} \binom{n}{j} \big(S_A(i) - S_A(j)\big) \lambda^{i-j},$$
$$\Omega_2 = \Omega_{\geq} \sum_{k \geq i, j=0} B_i C_j \binom{n}{i} \binom{n}{j} \big(S_A(j) - S_A(i)\big) \lambda^{j-i}.$$

Now we calculate Ω_1 and Ω_2. According to the operating rules of the Omega operator $\Omega_{\geq}$, we have

$$\Omega_1 = \Omega_{\geq} \left\{ \left(\sum_{j=0}^{n} C_j \binom{n}{j} \lambda^{-j} \right) \left(\sum_{k=0}^{n} B_k S_A(k) \lambda^{k} \right) - \left(\sum_{j=0}^{n} B_j \binom{n}{j} \lambda^{j} \right) \left(\sum_{k=0}^{n} C_k S_A(k) \lambda^{-k} \right) \right\}.$$

By making replacement $j \to n-j$ for the first double sum and $k \to n-k$ for the second double sum, we have

$$\Omega_1 = \Omega_{\geq} \lambda^{-n} \left\{ \sum_{j=0}^{n} \binom{n}{j} C_{n-j} \sum_{k=0}^{n} \binom{n}{k} B_k S_A(k) \lambda^{k+j} - \sum_{j=0}^{n} \binom{n}{j} B_j \sum_{k=0}^{n} \binom{n}{k} C_{n-k} S_A(n-k) \lambda^{k+j} \right\}.$$

Replacing j by $\ell - k$, the above equation becomes

$$\Omega_1 = \Omega_{\geq} \lambda^{-n} \sum_{\ell=0}^{2n} \sum_{k=0}^{\ell} \binom{n}{\ell-k} \binom{n}{k} \big\{ C_{n-\ell+k} B_k S_A(k) - B_{\ell-k} C_{n-k} S_A(n-k) \big\} \lambda^{\ell}$$
$$= \sum_{\ell=0}^{2n} \sum_{k=0}^{\ell} \binom{n}{\ell-k} \binom{n}{k} \big\{ C_{n-\ell+k} B_k S_A(k) - B_{\ell-k} C_{n-k} S_A(n-k) \big\}.$$

Making replacement $\ell \to \ell + n$, we have

$$\Omega_1 = \sum_{\ell=0}^{n} \sum_{k=\ell}^{n+\ell} \binom{n}{k-\ell} \binom{n}{k} \big\{ C_{k-\ell} B_k S_A(k) - B_{n+\ell-k} C_{n-k} S_A(n-k) \big\}.$$

Replacing k by $k+\ell$, Ω_1 becomes

$$\Omega_1 = \sum_{\ell=0}^{n} \sum_{k=0}^{n} \binom{n}{k} \binom{n}{k+\ell} \big\{ C_k B_{k+\ell} S_A(k+\ell) - B_{n-k} C_{n-k-\ell} S_A(n-k-\ell) \big\}.$$

Finally, making replacement $\ell \to j-k$, we have

$$\Omega_1 = \sum_{k=0}^{n} \binom{n}{k} \sum_{j=k}^{n} \binom{n}{j} \big\{ C_k B_j S_A(j) - B_{n-k} C_{n-j} S_A(n-j) \big\}$$
$$= \sum_{k=0}^{n} \binom{n}{k} \big\{ C_k P_B(n,k) - B_{n-k} Q_C(n,k) \big\} = \sum_{k=0}^{n} \binom{n}{k} \Delta_1(n,k).$$

Similarly, we can get

$$\Omega_2 = \sum_{k=0}^{n} \binom{n}{k} \big\{ B_k P_C(n,k) - C_{n-k} Q_B(n,k) \big\} = \sum_{k=0}^{n} \binom{n}{k} \Delta_2(n,k).$$

Therefore, we have

$$M_{BC}(n)-N_{BC}(n)=\sum_{k=0}^{n}\binom{n}{k}\{\Delta_1(n,k)+\Delta_2(n,k)\}.$$

Combining $M_{BC}(n)-N_{BC}(n)$ and $M_{BC}(n)+N_{BC}(n)$, we obtain

$$M_{BC}(n)=\frac{1}{2}\left\{T_B(n)P_C(n,0)+T_C(n)P_B(n,0)+\sum_{k=0}^{n}\binom{n}{k}\big(\Delta_1(n,k)+\Delta_2(n,k)\big)\right\},$$

which completes the proof. □

3. APPLICATION

3.1. Case of $A_k=x^k$, $B_k=r^k$ and $C_k=(-r)^k$

PROPOSITION 3. *For $x\neq 1$, the following identity holds true:*

$$\sum_{k=0}^{n}x^k\left(\sum_{j=0}^{k}r^j\binom{n}{j}\right)\left(\sum_{j=0}^{k}(-r)^j\binom{n}{j}\right)=\frac{x^{n+1}}{x-1}(1-r^2)^n$$
$$-\frac{1}{x-1}\left\{\sum_{k=0}^{n}\binom{n}{k}r^k\sum_{j=k}^{n}\binom{n}{j}(-xr)^j+\sum_{k=0}^{n}\binom{n}{k}(-r)^k\sum_{j=k+1}^{n}\binom{n}{j}(xr)^j\right\}.$$

Proof. When $A_k=x^k$, $B_k=r^k$ and $C_k=(-r)^k$, it's easy to get that

$$S_A(n+1)T_B(n)T_C(n)=\frac{1-x^{n+1}}{1-x}(1-r^2)^n,$$
$$T_B(n)P_C(n,0)=\frac{(1-r^2)^n-(1+r)^n(1-xr)^n}{1-x},$$
$$T_C(n)P_B(n,0)=\frac{(1-r^2)^n-(1-r)^n(1+xr)^n}{1-x}.$$

By making replacement $j\to n-j$ and $k\to n-k$, we have

$$\sum_{k=0}^{n}\binom{n}{k}r^{n-k}\sum_{j=k}^{n}\binom{n}{j}(-r)^{n-j}\sum_{i=0}^{n-j-1}x^i=\sum_{k=0}^{n}\binom{n}{k}r^k\sum_{j=0}^{k}\binom{n}{j}(-r)^j\sum_{i=0}^{j-1}x^i.$$

Therefore

$$\sum_{k=0}^{n}\binom{n}{k}\Delta_1(n,k)=\sum_{k=0}^{n}\binom{n}{k}(-r)^k\sum_{j=k}^{n}\binom{n}{j}r^j\sum_{i=0}^{j-1}x^i-\sum_{k=0}^{n}\binom{n}{k}r^{n-k}\sum_{j=k}^{n}\binom{n}{j}(-r)^{n-j}\sum_{i=0}^{n-j-1}x^i$$
$$=\sum_{k=0}^{n}\binom{n}{k}(-r)^k\sum_{j=k}^{n}\binom{n}{j}r^j\sum_{i=0}^{j-1}x^i-\sum_{k=0}^{n}\binom{n}{k}r^k\sum_{j=0}^{k}\binom{n}{j}(-r)^j\sum_{i=0}^{j-1}x^i.$$

Keeping in mind that

$$\sum_{i=0}^{j-1}x^i=\frac{1-x^j}{1-x}\quad\text{and}\quad\sum_{k=0}^{n}\binom{n}{k}(-r)^k\sum_{j=k}^{n}\binom{n}{j}r^j=\sum_{k=0}^{n}\binom{n}{k}r^k\sum_{j=0}^{k}\binom{n}{j}(-r)^j,$$

we have

$$\sum_{k=0}^{n}\binom{n}{k}\Delta_1(n,k)=\frac{1}{1-x}\sum_{k=0}^{n}\binom{n}{k}r^k\sum_{j=0}^{k}\binom{n}{j}(-rx)^j-\frac{1}{1-x}\sum_{k=0}^{n}\binom{n}{k}(-r)^k\sum_{j=k}^{n}\binom{n}{j}(xr)^j.$$

Similarly, we can get

$$\sum_{k=0}^{n}\binom{n}{k}\Delta_2(n,k)=\frac{1}{1-x}\sum_{k=0}^{n}\binom{n}{k}(-r)^k\sum_{j=0}^{k}\binom{n}{j}(rx)^j-\frac{1}{1-x}\sum_{k=0}^{n}\binom{n}{k}r^k\sum_{j=k}^{n}\binom{n}{j}(-xr)^j.$$

Thus, we can evaluate

$$\sum_{k=0}^{n}\binom{n}{k}\big(\Delta_1(n,k)+\Delta_2(n,k)\big)=\frac{(1+r)^n(1-rx)^n+(1-r)^n(1+rx)^n}{1-x}$$
$$-\frac{2}{1-x}\left\{\sum_{k=0}^{n}\binom{n}{k}r^k\sum_{j=k}^{n}\binom{n}{j}(-xr)^j+\sum_{k=0}^{n}\binom{n}{k}(-r)^k\sum_{j=k+1}^{n}\binom{n}{j}(xr)^j\right\}.$$

Based on the above conclusions, we can get the desired result. □

For the case of $x=1$, it is not hard to get the following proposition.

PROPOSITION 4.

$$\sum_{k=0}^{n}\left(\sum_{j=0}^{k}r^j\binom{n}{j}\right)\left(\sum_{j=0}^{k}(-r)^j\binom{n}{j}\right)=(1-r^2)^n+n(1+r^2)(1-r^2)^{n-1}$$
$$+\sum_{k=0}^{n}\binom{n}{k}(-r)^k\sum_{j=0}^{k}j\binom{n}{j}r^j+\sum_{k=1}^{n}\binom{n}{k}r^k\sum_{j=0}^{k-1}j\binom{n}{j}(-r)^j.$$

LEMMA 5.

$$\sum_{k=0}^{n}\binom{n}{k}r^k\sum_{j=0}^{k}\binom{n}{j}r^j+\sum_{k=1}^{n}\binom{n}{k}(-r)^k\sum_{j=0}^{k-1}\binom{n}{j}(-r)^j=\frac{(r+1)^{2n}+(r-1)^{2n}}{2}.$$

Proof. The left-hand side of the equation can be rewritten as follows:

$$LHS=\sum_{k=0}^{n}\binom{n}{k}r^k\sum_{j=0}^{k}\binom{n}{j}r^j+\sum_{k=0}^{n}\binom{n}{k}(-r)^k\sum_{j=0}^{k}\binom{n}{j}(-r)^j-\sum_{k=0}^{n}\binom{n}{k}^2r^{2k}.$$

It is obvious, for any integer ℓ, that the coefficient of $r^{2\ell+1}$ is 0, and

$$[r^{2\ell}]LHS=2\sum_{k=0}^{\ell}\binom{n}{k}\binom{n}{2\ell-k}-\binom{n}{\ell}^2.$$

On the other hand, for the right hand side we have

$$[r^{2\ell}]\frac{(r+1)^{2n}+(r-1)^{2n}}{2}=\binom{2n}{2\ell}\quad\text{and}\quad[r^{2\ell+1}]\frac{(r+1)^{2n}+(r-1)^{2n}}{2}=0.$$

In addition, we know the following identity [4, 3.6]

$$\sum_{k=0}^{\ell}\binom{n}{k}\binom{n}{2\ell-k}=\frac{1}{2}\left\{\binom{2n}{2\ell}+\binom{n}{\ell}^2\right\}.$$

Therefore, by comparing the coefficients of r^ℓ on both sides, we can complete the proof. □

For the particular case $x=-1$ in Proposition 3, by using the above Lemma 5, we have immediately the following identity proposed recently by Ohtsuka and Tauraso [7] as a monthly problem.

PROPOSITION 6.

$$\sum_{k=0}^{n}(-1)^k\Big(\sum_{j=0}^{k}r^j\binom{n}{j}\Big)\Big(\sum_{j=0}^{k}(-r)^j\binom{n}{j}\Big)=\Big(\frac{(r+1)^n+(r-1)^n}{2}\Big)^2.$$

3.2. Case of $A_k=(-1)^k\binom{n}{k}^{-1}$, $B_k=r^k$ and $C_k=(-r)^k$

LEMMA 7. *For integers $0\le\lambda\le m\le n$, the following identity holds true*

$$\sum_{k=\lambda}^{m}\frac{(-1)^k}{\binom{n}{k}}=\frac{n+1}{n+2}\Big\{\frac{(-1)^\lambda}{\binom{n+1}{\lambda}}+\frac{(-1)^m}{\binom{n+1}{m+1}}\Big\}.$$

Proof. Noting that the summation term can be rewritten as

$$\frac{(-1)^k}{\binom{n}{k}}=\frac{n+1}{n+2}(\tau_k-\tau_{k+1}),\ \text{ where }\ \tau_k=\frac{(-1)^k}{\binom{n+1}{k}}.$$

By means of the telescoping, we can evaluate

$$\sum_{k=\lambda}^{m}\frac{(-1)^k}{\binom{n}{k}}=\frac{n+1}{n+2}(\tau_\lambda-\tau_{m+1})=\frac{n+1}{n+2}\Big\{\frac{(-1)^\lambda}{\binom{n+1}{\lambda}}+\frac{(-1)^m}{\binom{n+1}{m+1}}\Big\}.$$ □

PROPOSITION 8. *For $r\neq\pm1$, the following identity holds true*

$$\sum_{k=0}^{n}\frac{(-1)^k}{\binom{n}{k}}\Big(\sum_{j=0}^{k}r^j\binom{n}{j}\Big)\Big(\sum_{j=0}^{k}(-r)^j\binom{n}{j}\Big)=\frac{n+1}{n+2}(r^2-1)^n+\frac{(r^2+r)^{n+2}+(r^2-r)^{n+2}}{(n+2)(r^2-1)^2}$$
$$-\frac{(r^2+1)^n\{r^4+(n+4)r^2-n-1\}}{(n+2)(r^2-1)^2}.$$

Proof. Letting $A_k=(-1)^k\binom{n}{k}^{-1}$, $B_j=r^j$ and $C_j=(-r)^j$ in Theorem 2, we have

$$S_A(n+1)T_B(n)T_C(n)=\frac{n+1}{n+2}\big\{(1-r^2)^n+(r^2-1)^n\big\}.$$

By using the Lemma 7, we can evaluate

$$T_B(n)P_C(n,0)=\frac{n+1}{n+2}(1+r)^nN_{n,r},$$
$$T_C(n)P_B(n,0)=\frac{n+1}{n+2}(1-r)^nM_{n,r},$$

where

$$N_{n,r}=(1-r)^n-\frac{1-r^{n+1}}{1-r}+\frac{r(1-r^n)}{(n+1)(1-r)^2}-\frac{nr^{n+1}}{(n+1)(1-r)},$$
$$M_{n,r}=(1+r)^n-\frac{1-(-r)^{n+1}}{1+r}-\frac{r(1-(-r)^n)}{(n+1)(1+r)^2}-\frac{n(-r)^{n+1}}{(n+1)(1+r)}.$$

Now we compute the sum $\sum_{k=0}^{n}\binom{n}{k}\big(\Delta_1(n,k)+\Delta_2(n,k)\big)$. By utilizing the Lemma 7, we get

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}\Delta_1(n,k) &= \sum_{k=0}^{n}\binom{n}{k}(-r)^k\sum_{j=k}^{n}\binom{n}{j}r^j\sum_{i=0}^{j-1}\frac{(-1)^i}{\binom{n}{i}} \\ &\quad -\sum_{k=0}^{n}\binom{n}{k}r^{n-k}\sum_{j=k}^{n}\binom{n}{j}(-r)^{n-j}\sum_{i=0}^{n-j-1}\frac{(-1)^i}{\binom{n}{i}} \\ &= \frac{n+1}{n+2}\sum_{k=0}^{n}\binom{n}{k}(-r)^k\sum_{j=k}^{n}\binom{n}{j}r^j\left\{1-\frac{(-1)^j}{\binom{n+1}{j}}\right\} \\ &\quad -\frac{n+1}{n+2}\sum_{k=0}^{n}\binom{n}{k}r^k\sum_{j=0}^{k}\binom{n}{j}(-r)^j\left\{1-\frac{(-1)^j}{\binom{n+1}{j}}\right\}.\end{aligned}$$

In the process of dealing with the second triple sum, we have replaced j by $n-j$ and k by $n-k$. For convenience, we rewrite the sum as

$$\sum_{k=0}^{n}\binom{n}{k}\Delta_1(n,k)=\frac{n+1}{n+2}(\delta_1-\delta_2-\delta_3+\delta_4),$$

where

$$\delta_1=\sum_{k=0}^{n}\binom{n}{k}(-r)^k\sum_{j=k}^{n}\binom{n}{j}r^j,\quad \delta_2=\sum_{k=0}^{n}\binom{n}{k}(-r)^k\sum_{j=k}^{n}(-r)^j\frac{\binom{n}{j}}{\binom{n+1}{j}},$$
$$\delta_3=\sum_{k=0}^{n}\binom{n}{k}r^k\sum_{j=0}^{k}\binom{n}{j}(-r)^j,\quad \delta_4=\sum_{k=0}^{n}\binom{n}{k}r^k\sum_{j=0}^{k}r^j\frac{\binom{n}{j}}{\binom{n+1}{j}}.$$

Analogously, we can get that

$$\sum_{k=0}^{n}\binom{n}{k}\Delta_2(n,k)=\frac{n+1}{n+2}(\delta_1'-\delta_2'-\delta_3'+\delta_4'),$$

where

$$\delta_1'=\sum_{k=0}^{n}\binom{n}{k}r^k\sum_{j=k}^{n}\binom{n}{j}(-r)^j,\quad \delta_2'=\sum_{k=0}^{n}\binom{n}{k}r^k\sum_{j=k}^{n}r^j\frac{\binom{n}{j}}{\binom{n+1}{j}},$$
$$\delta_3'=\sum_{k=0}^{n}\binom{n}{k}(-r)^k\sum_{j=0}^{k}\binom{n}{j}r^j,\quad \delta_4'=\sum_{k=0}^{n}\binom{n}{k}(-r)^k\sum_{j=0}^{k}(-r)^j\frac{\binom{n}{j}}{\binom{n+1}{j}}.$$

By exchanging the summation order, it's not difficult to get that

$$\delta_1+\delta_1'=\delta_3+\delta_3'=(1-r^2)^n-\sum_{k=0}^{n}\binom{n}{k}^2(-r^2)^k.$$

Noticing that

$$\sum_{j=0}^{k}r^j\frac{\binom{n}{j}}{\binom{n+1}{j}}=\sum_{j=0}^{k}r^j\left\{1-\frac{j}{n+1}\right\}=\frac{1-r^{k+1}}{1-r}-\frac{1}{n+1}\left\{\frac{r(1-r^k)}{(1-r)^2}-\frac{kr^{k+1}}{1-r}\right\},$$

$$\sum_{j=k}^{n} r^j \frac{\binom{n}{j}}{\binom{n+1}{j}} = \sum_{j=k}^{n} r^j \left\{1 - \frac{j}{n+1}\right\} = \frac{r^k - r^{n+1}}{1-r} - \frac{1}{n+1}\left\{\frac{kr^k - nr^{n+1}}{1-r} + \frac{r^{k+1} - r^{n+1}}{(1-r)^2}\right\},$$

we can evaluate $\delta_4 + \delta_4' - \delta_2 - \delta_2' = \frac{\delta}{(n+1)(r^2-1)^2}$, where

$$\begin{aligned}\delta = {} & 2(1+r^2)^n\{r^4 + (n+4)r^2 - n - 1\} - (r^2+r)^{n+2} - (r^2-r)^{n+2} \\ & - (r+1)^n\{(n+2)r^3 + (n+3)r^2 - nr - n - 1\} \\ & + (1-r)^n\{(n+2)r^3 - (n+3)r^2 - nr + n + 1\}.\end{aligned}$$

Combining the above consequences and making some simple simplification, we can get the desired result. □

3.3. Case of $A_k = (-1)^k \binom{2n+1}{2k+1}\binom{n}{k}^{-1}$, $B_k = 1$ and $C_k = (-1)^k$

LEMMA 9. *For integers $0 \le \lambda \le m \le n$, the following identity holds true*

$$\sum_{k=\lambda}^{m} (-1)^k \frac{\binom{2n+1}{2k+1}}{\binom{n}{k}} = (-1)^m \frac{2n-2m-1}{2n} \frac{\binom{2n+2}{2m+2}}{\binom{n+1}{m+1}} + (-1)^\lambda \frac{2n-2\lambda+1}{2n} \frac{\binom{2n+2}{2\lambda}}{\binom{n+1}{\lambda}}.$$

Proof. The summation term can be rewritten as

$$(-1)^k \frac{\binom{2n+1}{2k+1}}{\binom{n}{k}} = \tau_{k+1} - \tau_k, \quad \text{where} \quad \tau_k = (-1)^{k+1} \frac{2n-2k+1}{2n} \frac{\binom{2n+2}{2k}}{\binom{n+1}{k}}.$$

By using the telescoping, we can get the content stated in the lemma. □

PROPOSITION 10. *For $n \in \{1,2,\dots\}$, it holds*

$$\sum_{k=0}^{n}\sum_{j=0}^{k} \binom{2n}{2k+1}\binom{n}{j} = 2^{3n-2}. \tag{3}$$

Proof. Letting $A_k = (-1)^k \binom{2n+1}{2k+1}\binom{n}{k}^{-1}$, $B_k = 1$ and $C_k = (-1)^k$ in Theorem 2, and by means of Lemma 9, we have

$$\begin{aligned} & S_A(n+1)T_B(n)T_C(n) = T_C(n)P_B(n,0) = 0, \\ & T_B(n)P_C(n,0) = -\frac{2n+1}{n} 2^{3n-2}, \end{aligned}$$

and

$$\begin{aligned} \sum_{k=0}^{n}\binom{n}{k}\Delta_1(n,k) &= \frac{2n+1}{2n}\sum_{k=0}^{n}\binom{n}{k}\sum_{j=0}^{k}\binom{2n}{2j} - \frac{2n+1}{2n}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\sum_{j=k}^{n}\binom{2n}{2j}(-1)^j, \\ \sum_{k=0}^{n}\binom{n}{k}\Delta_2(n,k) &= \frac{2n+1}{2n}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\sum_{j=0}^{k}\binom{2n}{2j}(-1)^j - \frac{2n+1}{2n}\sum_{k=0}^{n}\binom{n}{k}\sum_{j=k}^{n}\binom{2n}{2j}. \end{aligned}$$

Noticing that $\sum\limits_{j=0}^{k}\binom{2n}{2j} = \sum\limits_{j=k}^{n}\binom{2n}{2j}$, we get

$$\sum_{k=0}^{n}\binom{n}{k}\big(\Delta_1(n,k) + \Delta_2(n,k)\big) = 0.$$

Therefore, we have

$$\sum_{k=0}^{n}(-1)^k\frac{\binom{2n+1}{2k+1}}{\binom{n}{k}}\left(\sum_{j=0}^{k}\binom{n}{j}\right)\left(\sum_{j=0}^{k}\binom{n}{j}(-1)^j\right)=-\frac{1}{2}T_B(n)P_C(n,0)=2^{3n-3}\left(2+\frac{1}{n}\right).$$

Keeping in mind that

$$\sum_{j=0}^{k}\binom{n}{j}(-1)^j=(-1)^k\binom{n-1}{k},$$

we have

$$\sum_{k=0}^{n}(-1)^k\frac{\binom{2n+1}{2k+1}}{\binom{n}{k}}\left(\sum_{j=0}^{k}\binom{n}{j}\right)\left(\sum_{j=0}^{k}\binom{n}{j}(-1)^j\right)=\frac{2n+1}{2n}\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{2n}{2k+1}\binom{n}{j},$$

which completes the proof. □

By setting appropriate sequences A_k, B_k and C_k in Theorem 2, it is possible to obtain other interesting triple sums. The interested reader is encouraged to make further attempt.

ACKNOWLEDGEMENTS

The authors express their sincere gratitude to the anonymous referee for careful reading and valuable suggestions that have significantly contributed to improving the manuscript. This work is supported by the Henan Provincial Key Scientific and Technological Research Project (China, 252102210159), and the Key Scientific Research Project of Higher Education Institutions in Henan Province (China, 25B413006).

REFERENCES

[1] Andrews GE, Paul P. MacMahon's partition analysis IV: hypergeometric multisums. Sém. Lothar. Combin. 1999; 42: B42i.

[2] Andrews GE, Paul P, Riese A. MacMahon's partition analysis: the omega package. European J. Combin. 2001; 22(7): 887–904.

[3] Calkin NJ. A curious binomial identity. Discrete Math. 1994; 131(1–3): 335–337.

[4] Gould HW. Combinatorial identities: a standardized set of tables listing 500 binomial coefficient summations. West Virginia: Morgantown Printing and Binding Co.; 1972.

[5] Hirschhorn M. Calkin's binomial identity. Discrete Math. 1996; 159(1–3): 273–278.

[6] Macmahon PA. Combinatory analysis, 2 vols. Cambridge University Press; 1915–1916. (Reprinted) New York: Chelsea; 1960.

[7] Ohtsuka H, Tauraso R. Problem 12293. Amer. Math. Monthly 2022; 129.

[8] Wang J, Zhang Z. On extensions of Calkin's binomial identities. Discrete Math. 2004; 274(1–3): 331–342.

[9] Zhang Z. A binomial identity related to Calkin's. Discrete Math. 1999; 196(1–3): 287–289.

[10] Zhang Z. On a kind of curious binomial identity. Discrete Math. 2006; 306(21): 2740–2754.

Received July 11, 2024