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WEAK TYPE ESTIMATES FOR SOME MAXIMAL OPERATOR ON THE p-ADIC VECTOR SPACE

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Abstract. In this paper, the main aim is to prove the weak type estimates for (nonlinear) commutators of p-adic Hardy-Littlewood maximal function and sharp maximal function in the Lebesgue spaces, where the symbols of the commutators belong to the BMO space. Moreover, we also obtain the weak estimate for commutator of p-adic fractional maximal function, where the symbols of the commutators belong to the Lipschitz space.

***Keywords**:* Hardy-Littlewood maximal function, p-adic fractional maximal function, Lipschitz space, BMO space, commutator.

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1. INTRODUCTION AND MAIN RESULTS

For a prime number p. The p-adic field is consist of $\mathbb{Q}$ with respect to non-Archimedean p-adic norm. Let $x = p^{\gamma}\frac{a}{b}$, where $x \in \mathbb{Q}$ and $\gamma \in \mathbb{Z}$, a and b are integers which are not divisible by p, then the p-adic norm is defined by $|x|_p = p^{-\gamma}$ and satisfies $|xy|_p = |x|_p|y|_p$ and $|x+y|_p \leq \max\{|x|_p, |y|_p\}$, where $|x|_p \neq |y|_p \Rightarrow |x+y|_p = \max\{|x|_p, |y|_p\}$.

By virtue of the standard p-adic analysis, non-zero p-adic number

$$x = p^{\gamma}(a_0 + a_1 p + a_2 p^2 + \cdots) = p^{\gamma} \sum_{j=0}^{\infty} a_j p^j,\ a_j = 0, \cdots, p-1,\ a_0 \neq 0,$$

then the above series converges in the p-adic norm.

For any $x = (x_1, x_2, \cdots, x_n)$, where $x_i \in \mathbb{Q}_p$ $(i = 1, \ldots, n)$, the p-adic norm is defined by $|x|_p = \max_{1\leq j\leq n} |x_j|_p$. Moreover, the p-adic ball is denoted by $B_{\gamma}(a) = \{x \in \mathbb{Q}_p^n : |x-a|_p \leq p^{\gamma}\}$, where the center of p-adic ball $a \in \mathbb{Q}_p^n$ and radius p^{γ} with $\gamma \in \mathbb{Z}$. The p-adic sphere is written as $S_{\gamma}(a) = \{x \in \mathbb{Q}_p^n : |x-a|_p = p^{\gamma}\} = B_{\gamma}(a) \setminus B_{\gamma-1}(a)$. It is easy to see that $B_{\gamma}(a) = \bigcup_{k\leq\gamma} S_k(a)$.

Since $\mathbb{Q}_p^n$ is a locally compact commutative group under addition, there exists Haar measure on $\mathbb{Q}_p^n$, it is easy to know that unique Haar measure dx on $\mathbb{Q}_p^n$ satisfies translation invariant. Normalizing the measure dx by $\int_{B_0(0)} dx = |B_0(0)|_h = 1$, where $|B_0(0)|_h$ is denoted by the Haar measure of p-adic unit ball. Besides, the normalized Haar measure $\int_{B_{\gamma}(a)} dx = |B_{\gamma}(a)|_h = p^{n\gamma}$ and $\int_{S_{\gamma}(a)} dx = |S_{\gamma}(a)|_h = p^{\gamma n}(1-p^{-n})$. For more details about the p-adic analysis, see [12, 13]. Moreover, p-adic analysis played role in many aspects, for instant mathematical physical, biological [2, 6, 9].

The Coifman-Rochberg-Weiss type commutator $[b,T]$ generated by classical singular integral operator T and a suitable function b is defined by

$$[b,T]f = bT(f) - T(bf). \tag{1}$$

(1) is bounded on $L^s(\mathbb{R}^n)$ $(1 < s < \infty)$ if and only if $b \in \mathrm{BMO}(\mathbb{R}^n)$ in [5,8]. Besides, (1) is bounded from $L^s(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ when $1 < s < \frac{n}{\beta}$ and $\frac{1}{s} - \frac{1}{q} = \frac{\beta}{n}$ $(0 < \beta < 1)$ if and only if $b \in \Lambda_\beta(\mathbb{R}^n)$ in [8, 11].

We focus on proving the weak type BMO estimates for (nonlinear) commutators of p-adic Hardy-Littlewood maximal function and sharp maximal function in the Lebesgue spaces. Moreover, we also introduce the weak Lipschitz estimate for commutator of p-adic fractional maximal function. In this paper, the constant C always takes the place of a constant independent of the primary parameters involved and whose value may differ from line to line. At first, for the sake of claiming the following theorems, we define the following functions

$$\log^+ t = \begin{cases} \log t & \text{if } t \geq 1, \\ 0 & \text{if } 0 < t < 1. \end{cases}, \quad b^-(x) = \begin{cases} 0, & \text{if } b(x) \geq 0, \\ |b(x)|, & \text{if } b(x) < 0, \end{cases}, \quad b^+(x) = |b(x)| - b^-(x).$$

THEOREM 1. *Let $\Phi(t) = t(1+\log^+ t)$, then $b \in \mathrm{BMO}(\mathbb{Q}_p^n)$ if and only if for all $f \in L(1+\log^+ L)(\mathbb{Q}_p^n)$ and $\lambda > 0$, the following inequality holds.*

$$\left|\{x \in \mathbb{Q}_p^n : M_p^b f(x) > \lambda\}\right|_h \leq C \|\Phi\left(\frac{|f(\cdot)|}{\lambda}\right)\|_{L^1(\mathbb{Q}_p^n)}. \tag{2}$$

Remark 1. In the Euclidean setting, we refer reader to see [1].

THEOREM 2. *Let $\Phi(t) = t(1+\log^+ t)$. If $b \in \mathrm{BMO}(\mathbb{Q}_p^n)$* and *$b^- \in L^\infty(\mathbb{Q}_p^n)$, then there exists a constant $C > 0$, such that*

$$\left|\{x \in \mathbb{Q}_p^n : \left|[b, M_p] f(x)\right| > \lambda\}\right|_h \leq C \|\Phi\left(\frac{|f(\cdot)|}{\lambda}\right)\|_{L^1(\mathbb{Q}_p^n)},$$

for all $f \in L(1+\log^+ L)(\mathbb{Q}_p^n)$ and $\lambda > 0$.

Remark 2. In the Euclidean setting, we also see [1].

For $f \in L^1_{\mathrm{loc}}(\mathbb{Q}_p^n)$, the p-adic version of sharp maximal function $M_p^\sharp$ is defined by

$$M_p^\sharp(f)(x) = \sup_{\gamma \in \mathbb{Z}} \frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |f(y) - f_{B_\gamma(x)}| \mathrm{d}y,$$

where the supremum is taken over all p-adic balls $B_\gamma(x) \subset \mathbb{Q}_p^n$ and $f_{B_\gamma(x)} = \frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} f(y) \mathrm{d}y$. And the nonlinear commutator produced by b with $M_p^\sharp$ is defined by

$$[b, M_p^\sharp](f)(x) = b(x) M_p^\sharp(f)(x) - M_p^\sharp(bf)(x). \tag{3}$$

THEOREM 3. *Let $\Phi(t) = t(1+\log^+ t)$. If $b \in \mathrm{BMO}(\mathbb{Q}_p^n)$* and *$b^- \in L^\infty(\mathbb{Q}_p^n)$, then there exists a constant $C > 0$, such that*

$$\left|\{x \in \mathbb{Q}_p^n : |[b, M_p^\sharp] f(x)| > \lambda\}\right|_h \leq C \|\Phi\left(\frac{|f(\cdot)|}{\lambda}\right)\|_{L^1(\mathbb{Q}_p^n)},$$

for all $f \in L(1+\log^+ L)(\mathbb{Q}_p^n)$ and $\lambda > 0$.

Let $0 \leq \alpha < n$, for $f \in L^1_{\rm loc}(\mathbb{Q}_p^n)$, the p-adic fractional maximal function of f is defined by

$$M_{\alpha,p}(f)(x) = \sup_{\gamma \in \mathbb{Z}} \frac{1}{|B_\gamma(x)|_h^{1-\frac{\alpha}{n}}} \int_{B_\gamma(x)} |f(y)|\mathrm{d}y,$$

where the supremum is taken over all p-adic balls $B_\gamma(x) \subset \mathbb{Q}_p^n$. For $\alpha = 0$, we write M_p=$M_{0,p}$. If $b \in L^1_{\rm loc}(\mathbb{Q}_p^n)$, the fractional maximal commutator produced by b with $M_{\alpha,p}$ is provided by

$$M^b_{\alpha,p}(f)(x) = \sup_{\gamma \in \mathbb{Z}} \frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |b(x) - b(y)||f(y)|\mathrm{d}y. \tag{4}$$

For $\alpha = 0$, we write $M^b_p = M^b_{0,p}$. And the commutator produced by b with $M_{\alpha,p}$ is defined by

$$[b, M_{\alpha,p}](f)(x) = b(x)M_{\alpha,p}(f)(x) - M_{\alpha,p}(bf)(x), \tag{5}$$

we write $[b,M_p] = [b,M_{0,p}]$(Theorem 2). (3)-(5) have been studied in [3, 7, 14–17].

THEOREM 4. *Let* $0 < \alpha < n,\ 0 < \beta < 1,\ 0 < \alpha < \alpha + \beta < n$. *If* $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ *and* $b \geq 0$, *then there exists a constant* $C > 0$, *such that for any* $\lambda > 0$,

$$\left|\{x \in \mathbb{Q}_p^n : |[b, M_{\alpha,p}](f)(x)| > \lambda\}\right|_h \leq C \left(\frac{\|f\|_{L^1(\mathbb{Q}_p^n)}}{\lambda}\right)^{n/(n-\alpha-\beta)}.$$

THEOREM 5. *Let* $0 < \alpha < n,\ 0 < \beta < 1,\ 0 < \alpha < \alpha + \beta < n$, *then* $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ *if and only if there exists a constant* $C > 0$, *such that for any* $\lambda > 0$,

$$\left|\{x \in \mathbb{Q}_p^n : M^b_{\alpha,p}(f)(x) > \lambda\}\right|_h \leq C \left(\frac{\|f\|_{L^1(\mathbb{Q}_p^n)}}{\lambda}\right)^{n/(n-\alpha-\beta)}. \tag{6}$$

2. p-ADIC FUNCTION SPACE

Assume that $1 \leq q < \infty$, we denote $L^q(\mathbb{Q}_p^n)$ as the p-adic Lebesgue space, the space of all functions f is in the L^q space with finite norm

$$\|f\|_{L^q(\mathbb{Q}_p^n)} = \left(\int_{\mathbb{Q}_p^n} |f(x)|^q \mathrm{d}x\right)^{\frac{1}{q}}.$$

For $q = \infty$ and denote $L^\infty(\mathbb{Q}_p^n)$ as the set of all measurable real-valued functions f satisfying

$$\|f\|_{L^\infty(\mathbb{Q}_p^n)} = \operatorname*{ess\,sup}_{x \in \mathbb{Q}_p^n} |f(x)| = \inf\{\lambda > 0 : |\{x \in \mathbb{Q}_p^n : |f(x)| > \lambda\}|_h = 0\} < \infty.$$

Here, if the limit exists, the integral in above equation is defined as follows:

$$\int_{\mathbb{Q}_p^n} |f(x)|^q \mathrm{d}x = \lim_{\gamma \to \infty} \int_{B_\gamma(0)} |f(x)|^q \mathrm{d}x = \lim_{\gamma \to \infty} \sum_{-\infty < k \leq \gamma} \int_{S_k(0)} |f(x)|^q \mathrm{d}x.$$

The Zygmund space can be introduced in [3, 17]

$$L(1 + \log^+ L)(\mathbb{Q}_p^n) = \{f \text{ is measurable function} : \int_{\mathbb{Q}_p^n} |f(y)|(1 + \log^+ |f(y)|)\mathrm{d}y < \infty\}.$$

Definition 1. *Let* $0 < \beta < 1$*, the p-adic Lipschitz spaces* $\Lambda_\beta(\mathbb{Q}_p^n)$ *is defined by [4]*

$$\Lambda_\beta(\mathbb{Q}_p^n) := \left\{ f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{Q}_p^n) : \sup_{x,y\in\mathbb{Q}_p^n,\ x\neq y} \frac{|f(x)-f(y)|}{|x-y|_p^\beta} < \infty \right\}.$$

Remark 3. (1) Assume that $1 \le q < \infty$, the p-adic version of homogeneous Lipschitz spaces $Lip^q_\beta(\mathbb{Q}_p^n)$ is defined by

$$Lip^q_\beta(\mathbb{Q}_p^n) := \{ f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{Q}_p^n) :\ \|f\|_{Lip^q_\beta(\mathbb{Q}_p^n)} < \infty \},$$

where

$$\|f\|_{Lip^q_\beta(\mathbb{Q}_p^n)} = \sup_{\substack{\gamma\in\mathbb{Z}\\ x\in\mathbb{Q}_p^n}} \frac{1}{|B_\gamma(x)|_h^{\frac{\beta}{n}}} \left(\frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |f(y) - f_{B_\gamma(x)}|^q \mathrm{d}y \right)^{\frac{1}{q}}.$$

(2) (see Lemma 6 of [7]) By virtue of Definition 1, for all $0 < \beta < 1$ and $1 \le q < \infty$, $\Lambda_\beta(\mathbb{Q}_p^n) \approx Lip^q_\beta(\mathbb{Q}_p^n)$ with equivalent norms.

Definition 2. *Let* $f \in L^1_{\mathrm{loc}}(\mathbb{Q}_p^n)$ *be given. If* $\|M_p^\sharp(f)\|_{L^\infty(\mathbb{Q}_p^n)} < \infty$*, then we say that* f *is a BMO function on* $\mathbb{Q}_p^n$ *with the finite norm [10]*

$$\|f\|_{*,p} = \|M_p^\sharp(f)\|_{L^\infty(\mathbb{Q}_p^n)} = \sup_{\substack{\gamma\in\mathbb{Z}\\ x\in\mathbb{Q}_p^n}} \frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |f(y) - f_{B_\gamma(x)}|\mathrm{d}y.$$

3. PROOF OF THE PRRINCIPAL RESULTS

LEMMA 1. *There exists a constant* $C > 0$ *such that for all function* f *and for all* $\lambda > 0$,

$$\left|\{x \in \mathbb{Q}_p^n : M_p(M_p(f))(x) > \lambda\}\right|_h \le C \int_{\mathbb{Q}_p^n} \frac{|f(x)|}{\lambda} \left(1 + \log^+ \left(\frac{|f(x)|}{\lambda} \right) \right) \mathrm{d}x.$$

Proof. First we note that the following estimate holds (see Lemma 2.3 of [17])

$$M_p(M_p(f))(x) \le C M_{L\log L,p} f(x),$$

where $M_{L\log L,p}$ is a well-known p-adic maximal function of $L\log L$ type. For its definition and properties, we refer to [3, 17]. Therefore it suffices to show that for all $\lambda > 0$, we have

$$\left|\{x \in \mathbb{Q}_p^n : M_{L\log L,p} f(x) > \lambda\}\right|_h \le C \int_{\mathbb{Q}_p^n} \frac{|f(x)|}{\lambda} \left(1 + \log^+ \left(\frac{|f(x)|}{\lambda} \right) \right) \mathrm{d}x.$$

We consider $E_\lambda = \{x \in \mathbb{Q}_p^n : M_{L\log L,p} f(x) > \lambda\}$. Thus, we take any $x \in E_\lambda$, then, there exists a p-adic ball $B_{\gamma_x}(x)$ such that

$$\frac{1}{|B_{\gamma_x}(x)|_h} \int_{B_{\gamma_x}(x)} \frac{|f(y)|}{\lambda} \left(1 + \log^+ \left(\frac{|f(y)|}{\lambda} \right) \right) \mathrm{d}y > 1. \tag{7}$$

By the inequality (7), we have $\sup_{x\in E_\lambda} \gamma_x < \infty$. Hence, by using p-adic version of covering Lemma (see Lemma 2.8 of [3] or Lemma 3.3 of [18]), we take a sequence $\{x_k\}_{k=1}^\infty \subset E_\lambda$ such that the disjoint subcollection

$\{B_{\gamma_{x_k}}(x_k)\}_{k=1}^{\infty}$ from $\{B_{\gamma_x}(x)\}_{x\in E_\lambda}$ and

$$|E_\lambda|_h \le p^n \sum_{k=1}^{\infty} |B_{\gamma_{x_k}}(x_k)|_h. \tag{8}$$

Combining (7) and (8), we can easily obtain the result of Lemma 1. □

LEMMA 2 [17]. *For any locally integrable function f on $\mathbb{Q}_p^n$. If $b \in \mathrm{BMO}(\mathbb{Q}_p^n)$, then*

$$M_p^b(f)(x) \le C\|b\|_{*,p} M_p(M_p(f))(x).$$

Proof. To Theorem 1. "only if" part: By applying Lemmas 1 and 2, we have

$$\begin{aligned}\left|\{x\in\mathbb{Q}_p^n : M_p^b f(x) > \lambda\}\right|_h &\le \left|\left\{x\in\mathbb{Q}_p^n : M_p(M_p f)(x) > \frac{\lambda}{C\|b\|_{*,p}}\right\}\right|_h \\ &\le C\int_{\mathbb{Q}_p^n} \frac{C\|b\|_{*,p}|f(x)|}{\lambda}\left(1+\log^+\left(\frac{C\|b\|_{*,p}|f(x)|}{\lambda}\right)\right)\mathrm{d}x \\ &\le C\|b\|_{*,p}(1+\log^+\|b\|_{*,p})\int_{\mathbb{Q}_p^n}\frac{|f(x)|}{\lambda}\left(1+\log^+\left(\frac{|f(x)|}{\lambda}\right)\right)\mathrm{d}x,\end{aligned}$$

the last step is obtained since $1+\log^+(ab) \le (1+\log^+ a)(1+\log^+ b)$, where $a,b>0$. Thus we finish the proof of "only if" part.

"if" part: Fix any p-adic ball $B_{\gamma'}(y)$ and assume $f=\chi_{B_{\gamma'}(y)}$, then for any $\lambda>0$, using (2)

$$\begin{aligned}&C\frac{|B_{\gamma'}(y)|_h}{\lambda}\left(1+\log^+\frac{1}{\lambda}\right) \ge \left|\left\{x\in\mathbb{Q}_p^n : M_p^b f(x) > \lambda\right\}\right|_h \\ &= \left|\left\{x\in\mathbb{Q}_p^n : \sup_{\gamma\in Z}\frac{1}{|B_\gamma(x)|_h}\int_{B_\gamma(x)\cap(B_{\gamma'}(y))}|b(x)-b(z)|\mathrm{d}z > \lambda\right\}\right|_h \\ &\ge \left|\left\{x\in B_{\gamma'}(y) : \frac{1}{|B_{\gamma'}(y)|_h}\int_{B_{\gamma'}(y)}|b(x)-b(z)|\mathrm{d}z > \lambda\right\}\right|_h \\ &\ge \left|\left\{x\in B_{\gamma'}(y) : |b(x)-b_{B_{\gamma'}(y)}| > \lambda\right\}\right|_h.\end{aligned}$$

Then taking $0<s<1$, using Lemma 2.4 in [10], we obtain

$$\begin{aligned}&\int_{B_{\gamma'}(y)}|b(z)-b_{B_{\gamma'}(y)}|^s\mathrm{d}z = s\int_0^\infty \lambda^{s-1}\left|\left\{x\in B_{\gamma'}(y) : |b(x)-b_{B_{\gamma'}(y)}| > \lambda\right\}\right|_h \mathrm{d}\lambda \\ &= s\left(\int_0^1+\int_1^\infty\right)\lambda^{s-1}\left|\left\{x\in B_{\gamma'}(y) : |b(x)-b_{B_{\gamma'}(y)}| > \lambda\right\}\right|_h \mathrm{d}\lambda \\ &\le |B_{\gamma'}(y)|_h + Cs|B_{\gamma'}(y)|_h\int_1^\infty \lambda^{s-1}\frac{1}{\lambda}\left(1+\log^+\frac{1}{\lambda}\right)\mathrm{d}\lambda = \left(1+C\frac{s}{1-s}\right)|B_{\gamma'}(y)|_h.\end{aligned}$$

It follows from Corollary 5.17 in [10] that $b\in\mathrm{BMO}(\mathbb{Q}_p^n)$. Thus we finish Theorem 1. □

Before proving Theorem 2, we need the following fact and Lemma

$$|[b,M_p]f(x)| \le |[|b|,M_p]f(x)| + 2b^-(x)M_p f(x) \le M_p^b f(x) + 2b^-(x)M_p f(x). \tag{9}$$

LEMMA 3 [10]. *If f be a integrable function on $\mathbb{Q}_p^n$, then for all $\lambda > 0$ such that*

$$\left|\left\{x \in \mathbb{Q}_p^n : \ |M_p f(x)| > \lambda\right\}\right|_h \leq \frac{p^n}{\lambda}\|f\|_{L^1(\mathbb{Q}_p^n)}.$$

Proof. To Theorem 2. Since $b \in \mathrm{BMO}(\mathbb{Q}_p^n)$ and $b^- \in L^\infty(\mathbb{Q}_p^n)$, using Theorem 1, Lemma 3 and (9), we have

$$\begin{aligned}
&\left|\left\{x \in \mathbb{Q}_p^n : |[b,M_p]f(x)| > \lambda\right\}\right|_h \\
&\leq \left|\left\{x \in \mathbb{Q}_p^n : \left|M_p^b f(x)\right| > \lambda/2\right\}\right|_h + \left|\left\{x \in \mathbb{Q}_p^n : \left|2b^-\right| M_p f(x) > \lambda/2\right\}\right|_h \\
&\leq \left|\left\{x \in \mathbb{Q}_p^n : \left|M_p^b f(x)\right| > \lambda/2\right\}\right|_h + \left|\left\{x \in \mathbb{Q}_p^n : 2\left\|b^-\right\|_{L^\infty(\mathbb{Q}_p^n)} M_p f(x) > \lambda/2\right\}\right|_h \\
&\leq C(1+\log^+ 2)\int_{\mathbb{Q}_p^n} \frac{|f(x)|}{\lambda}\left(1+\log^+\left(\frac{|f(x)|}{\lambda}\right)\right)\mathrm{d}x \\
&+ C\left\|b^-\right\|_{L^\infty(\mathbb{Q}_p^n)} \frac{1}{\lambda}\|f\|_{L^1(\mathbb{Q}_p^n)} \leq C\|\Phi(\frac{|f(\cdot)|}{\lambda})\|_{L^1(\mathbb{Q}_p^n)},
\end{aligned}$$

thus we finish Theorem 2. □

Proof. To Theorem 3. Using (3.2) of [15] and $M_p^\sharp(f)(x) \leq 2M_p(f)(x)$, we obtain

$$|[b,M_p^\sharp]f(x)| \leq 4b^-(x)M_p f(x) + 4M_p(b^- f)(x) + 2M_p^{|b|}f(x). \tag{10}$$

Since $b \in \mathrm{BMO}(\mathbb{Q}_p^n)$, it deduce that $|b| \in \mathrm{BMO}(\mathbb{Q}_p^n)$, then using Theorem 1, Lemma 3 and (10), we have

$$\begin{aligned}
&\left|\left\{x \in \mathbb{Q}_p^n : \left|[b,M_p^\sharp]f(x)\right| > \lambda\right\}\right|_h \\
&\leq \left|\left\{x \in \mathbb{Q}_p^n : \left|4b^-(x)M_p f(x) + 4M_p(b^- f)(x)\right| > \lambda/2\right\}\right|_h + \left|\left\{x \in \mathbb{Q}_p^n : 2M_p^{|b|}f(x) > \lambda/2\right\}\right|_h \\
&\leq \left|\left\{x \in \mathbb{Q}_p^n : 2M_p^{|b|}f(x) > \lambda/2\right\}\right|_h + \left|\left\{x \in \mathbb{Q}_p^n : 8\left\|b^-\right\|_{L^\infty(\mathbb{Q}_p^n)} M_p f(x) > \lambda/2\right\}\right|_h \\
&\leq C(1+\log^+ 4)\int_{\mathbb{Q}_p^n} \frac{|f(x)|}{\lambda}\left(1+\log^+\left(\frac{|f(x)|}{\lambda}\right)\right)\mathrm{d}x \\
&+ C\left\|b^-\right\|_{L^\infty(\mathbb{Q}_p^n)} \frac{1}{\lambda}\|f\|_{L^1(\mathbb{Q}_p^n)} \leq C\|\Phi(\frac{|f(\cdot)|}{\lambda})\|_{L^1(\mathbb{Q}_p^n)},
\end{aligned}$$

which implies that Theorem 3. □

LEMMA 4 [7]. *Let $0 < \alpha < n$, for any $\lambda > 0$ and all $f \in L^1(\mathbb{Q}_p^n)$, we have*

$$|\{x \in \mathbb{Q}_p^n : M_{\alpha,p} f(x) > \lambda\}|_h \leq C\left(\frac{\|f\|_{L^1(\mathbb{Q}_p^n)}}{\lambda}\right)^{\frac{n}{n-\alpha}}. \tag{11}$$

LEMMA 5 [16]. *Let $0 < \alpha < n$, $0 < \beta < 1$, $0 < \alpha+\beta < n$ and f be a locally integral function on $\mathbb{Q}_p^n$. If $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ and $b \geq 0$, then for any $x \in \mathbb{Q}_p^n$ such that $M_{\alpha,p}(f)(x) < \infty$, we have*

$$|[b,M_{\alpha,p}](f)(x)| \leq \|b\|_{\Lambda_\beta(\mathbb{Q}_p^n)} M_{\alpha+\beta,p}(f)(x). \tag{12}$$

Proof. To Theorem 4. Since $b\in\Lambda_\beta(\mathbb{Q}_p^n)$ and $b\geq 0$, it follows from (11) and (12) that

$$\left|\left\{x\in\mathbb{Q}_p^n:[b,M_{\alpha,p}](f)(x)>\lambda\right\}\right|_h\leq\left|\left\{x\in\mathbb{Q}_p^n:M_{\alpha+\beta,p}f(x)>\frac{\lambda}{\|b\|_{\Lambda_\beta(\mathbb{Q}_p^n)}}\right\}\right|_h\leq C\left(\frac{\|f\|_{L^1(\mathbb{Q}_p^n)}}{\lambda}\right)^{\frac{n}{n-\alpha-\beta}},$$

which implies that the proof of Theorem 4. □

LEMMA 6 [16]. *Let $0<\alpha<n$, $0<\beta<1$, $0<\alpha+\beta<n$ and f be a locally integral function on $\mathbb{Q}_p^n$. If $b\in\Lambda_\beta(\mathbb{Q}_p^n)$, then for any $x\in\mathbb{Q}_p^n$ such that $M_{\alpha,p}(f)(x)<\infty$, we have*

$$M_{\alpha,p}^b\leq C\|b\|_{\Lambda_\beta(\mathbb{Q}_p^n)}M_{\alpha+\beta,p}(f)(x). \tag{13}$$

Proof. To Theorem 5. On the one hand, for "only if" part, similarly to Theorem 4, since $b\in\Lambda_\beta(\mathbb{Q}_p^n)$, it follows from (11) and (13) that

$$\begin{aligned}\left|\left\{x\in\mathbb{Q}_p^n:M_{\alpha,p}^b(f)(x)>\lambda\right\}\right|_h&\leq\left|\left\{x\in\mathbb{Q}_p^n:M_{\alpha+\beta,p}f(x)>\frac{\lambda}{C\|b\|_{\Lambda_\beta(\mathbb{Q}_p^n)}}\right\}\right|_h\\&\leq C\|b\|_{\Lambda_\beta(\mathbb{Q}_p^n)}^{\frac{n}{n-\alpha-\beta}}\left(\frac{\|f\|_{L^1(\mathbb{Q}_p^n)}}{\lambda}\right)^{\frac{n}{n-\alpha-\beta}}.\end{aligned}$$

On the other hand, we only need to prove "if" part, for any p-adic ball $B_\gamma(x)\subset\mathbb{Q}_p^n$, we obtain

$$|b(y)-b_{B_\gamma(x)}|\leq\frac{1}{|B_\gamma(x)|_h}\int_{B_\gamma(x)}|b(y)-b(z)|\mathrm{d}z=\frac{1}{|B_\gamma(x)|_h}\int_{B_\gamma(x)}|b(y)-b(z)|\chi_{B_\gamma(x)}\mathrm{d}z.$$

Thus for all $y\in B_\gamma(x)$,

$$M_{\alpha,p}^b(\chi_{B_\gamma(x)})(y)\geq\frac{1}{|B_\gamma(x)|_h^{1-\frac{\alpha}{n}}}\int_{B_\gamma(x)}|b(y)-b(z)|\chi_{B_\gamma(x)}\mathrm{d}z\geq|B_\gamma(x)|^{\frac{\alpha}{n}}|b(y)-b_{B_\gamma(x)}|,$$

which implies that

$$|b(y)-b_{B_\gamma(x)}|\leq|B_\gamma(x)|^{-\frac{\alpha}{n}}M_{\alpha,p}^b(\chi_{B_\gamma(x)})(y).$$

From (6) we obtain

$$\begin{aligned}|\{y\in B_\gamma(x):|b(y)-b_{B_\gamma(x)}|>\lambda\}|_h&\leq\left|\left\{y\in\mathbb{Q}_p^n:M_{\alpha,p}^b(\chi_{B_\gamma(x)})(y)>\lambda|B_\gamma(x)|_h^{\frac{\alpha}{n}}\right\}\right|_h\\&\leq C\left(\lambda^{-1}|B_\gamma(x)|_h^{1-\frac{\alpha}{n}}\right)^{\frac{n}{n-\alpha-\beta}}.\end{aligned}$$

Take $\xi>0$, similar to the "if" part of Theorem 1, since $\frac{n}{n-\alpha-\beta}>1$, we have

$$\begin{aligned}&\frac{1}{|B_\gamma(x)|_h^{1+\frac{\beta}{n}}}\int_{B_\gamma(x)}|b(y)-b_{B_\gamma(x)}|\mathrm{d}y=\frac{1}{|B_\gamma(x)|_h^{1+\frac{\beta}{n}}}\int_0^\infty|\{y\in B_\gamma(x):|b(y)-b_{B_\gamma(x)}|>\lambda\}|_h\mathrm{d}\lambda\\&=\frac{1}{|B_\gamma(x)|_h^{1+\frac{\beta}{n}}}\left(\int_0^\xi+\int_\xi^\infty\right)|\{y\in B_\gamma(x):|b(y)-b_{B_\gamma(x)}|>\lambda\}|_h\mathrm{d}\lambda\\&\leq\xi|B_\gamma(x)|_h^{-\frac{\beta}{n}}+C|B_\gamma(x)|_h^{\frac{n}{n-\alpha-\beta}(1-\frac{\alpha}{n})-(1+\frac{\beta}{n})}\xi^{1-\frac{n}{n-\alpha-\beta}}.\end{aligned}$$

Let $\xi=|B_\gamma(x)|_h^{\frac{\beta}{n}}$, we get

$$\frac{1}{|B_\gamma(x)|_h^{1+\frac{\beta}{n}}}\int_{B_\gamma(x)}|b(y)-b_{B_\gamma(x)}|\mathrm{d}y\le C.$$

Thus we finish the proof of Theorem 5. □

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