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MOTION OF A CHARGE IN A CONSTANT AND UNIFORM FIELD: A DESCRIPTION BASED ON THE EIGENVALUES

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Abstract. The motion of a charged particle in a constant and uniform electromagnetic field is investigated. This problem, on its own, has been extensively studied. The novelty is in using an argument based on eigenvalues to arrive at a description of the motion, specifically the way that the velocity behaves at large times.

***Keywords**:* charge, constant and uniform field, eigenvalues.
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1. INTRODUCTION

The nonrelativistic motion of an electric charge in an electromagnetic field is governed by the following equation [1, 2]

$$m\,\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = q\,(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}). \tag{1}$$

m and q are the mass and charge of the particle, respectively; $\boldsymbol{v}$ is the particle's velocity; t is the time; and $\boldsymbol{E}$ and $\boldsymbol{B}$ are the electric and magnetic fields, respectively.

The relativistic equation is [3, 4]

$$m\,\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = q\,(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}). \tag{2}$$

where

$$\boldsymbol{u} = \gamma\,\boldsymbol{v}, \tag{3}$$

$$\gamma = \left(1 - \frac{\boldsymbol{v}\cdot\boldsymbol{v}}{c^2}\right)^{-1/2}. \tag{4}$$

γ is the Lorentz factor, and $\boldsymbol{u}$ is the space part of the four-velocity.

This equation of motion has been investigated in many texts. Some recent sources include [5–7].

A special case is when the electromagnetic field is constant (time-independent) and uniform (space-independent), [8, 9]. In [9], the cases of no magnetic field and no electric field are discussed separately. Then a case is studied that the electric field and the magnetic field are normal to each other.

If there is no magnetic field, the part of $\boldsymbol{u}$ which is normal to the electric field remains constant. This part of $\boldsymbol{u}$ corresponds to two components. The Lorentz factor γ as well as the part of $\boldsymbol{u}$ which is parallel to the electric field, are exponential functions (in fact linear combinations of two exponential functions) of the proper time. In this case $\boldsymbol{u}$ and γ grow indefinitely with the proper time.

If there is no electric field, the Lorentz factor γ as well as the part of $\boldsymbol{u}$ which is parallel to the magnetic field, remain constant. The part of $\boldsymbol{u}$ which is normal to the magnetic field rotates along the magnetic field. That is, γ remains constant and $\boldsymbol{u}$ rotates along the magnetic field. So γ and $\boldsymbol{u}$ remain finite.

When the electric field and the magnetic field are both nonzero but normal to each other, it is shown in [9] that using proper boosts, the problem is reduced to one of the previous cases, depending on the ratio of the magnetic field to the electric field. It is seen there that when the electric and magnetic fields are normal to each other (which also covers the cases of no magnetic field or no electric field as special cases); the velocity four vector goes to infinity when the field is *essentially electric*, and remains finite when the field is *essentially magnetic*. These correspond to the ratio of the magnetic field to the electric field being smaller or larger than c^{-1}, respectively.

The problem of motion of a charge in a constant uniform electromagnetic field is not difficult. The corresponding differential equation is a simple one. However, tackling the problem using tools of linear algebra may bring some new insight. Here such an approach is used to study the problem. The equation is written in terms of the velocity four vector u, the electromagnetic field tensor F, and the proper time τ. The result is a linear first order differential equation for u. It is shown that the behavior of the solution is determined by the eigenvalues of Θ (the generator of evolution), which is (q/m) times F.

The final result is this:

- If both the electric and magnetic fields are nonzero, and the fields are not normal to each other (that is, if $\boldsymbol{B}\cdot\boldsymbol{E}$ is nonzero), then one of the eigenvalues of Θ is real and positive. And the velocity doesn't remain finite.
- If $\boldsymbol{B}\cdot\boldsymbol{E}$ is zero and the field is *essentially electric* (the ratio of the magnetic field to the electric field is smaller than c^{-1}), then again one of the eigenvalues of Θ is real and positive. And the velocity doesn't remain finite.
- If $\boldsymbol{B}\cdot\boldsymbol{E}$ is zero and the field is *essentially magnetic* (the ratio of the magnetic field to the electric field is larger than c^{-1}), then all of the eigenvalues of Θ are pure imaginary. And the velocity remains finite.
- If $\boldsymbol{B}\cdot\boldsymbol{E}$ is zero and the ratio of the magnetic field to the electric field is equal to c^{-1}, then all of the eigenvalues of Θ are zero. In this case the velocity four vector becomes a polynomial function of the proper time.

The scheme of the paper is as follows. In section 2, a general setup is introduced, including some notation conventions and the differential equation for the evolution of velocity. In section 3, different possibilities for the eigenvalues of the generator of the evolution are discussed and the resulting motion in each case is described. Section 4 is devoted to the concluding remarks.

2. THE GENERAL FRAMEWORK

The following convention is used for the Minkowski metric η:

$$\eta_{00} = -c^2, \tag{5}$$
$$\eta_{ij} = \delta_{ij}, \tag{6}$$
$$\eta_{0i} = 0. \tag{7}$$

Time is the zeroth component of the space-time four vector r :

$$t = r^0. \tag{8}$$

Greek indices run from 0 to 3 (both temporal and spatial values), while Roman indices can take only spatial (nonzero) values. The components of the velocity four vector u are

$$u^0 = \gamma, \tag{9}$$

$$u^i = \gamma v^i. \tag{10}$$

The field strength tensor F is an antisymmetric tensor with the following components.

$$F_{i0} = E_i, \tag{11}$$

$$F_{ij} = \varepsilon_{ijk} B^k. \tag{12}$$

Einstein's summation convention is used: repeated indices are summed, when one is subscript and another is superscript [10]. ε is the Levi-Civita tensor: the completely antisymmetric tensor with

$$\varepsilon_{123} = 1. \tag{13}$$

The proper time is denoted by τ:

$$\mathrm{d}\tau = \sqrt{-c^{-2}\,\eta_{\mu\nu}\,(\mathrm{d}x^\mu)\,(\mathrm{d}x^\nu)}. \tag{14}$$

So,

$$\mathrm{d}\tau = \gamma^{-1}\,\mathrm{d}t. \tag{15}$$

The relativistic equation of motion for u is

$$m\,\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = q\,F^\mu{}_\nu\,u^\nu. \tag{16}$$

In a more compact form,

$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = \Theta\,u. \tag{17}$$

Θ is called the generator (of the evolution in terms of the proper time):

$$\Theta = \kappa F, \tag{18}$$

$$\kappa = \frac{q}{m}. \tag{19}$$

When the electromagnetic field is space-time-independent, the solution to the equation of motion is

$$u(\tau) = \Upsilon(\tau)\,u(0), \tag{20}$$

$$\Upsilon(\tau) = \exp(\tau\,\Theta). \tag{21}$$

When Θ is diagonalizale, the above can be written as

$$\Upsilon(\tau) = \sum_{\lambda} \exp(\lambda\,\tau)\,\Pi_{\lambda}. \tag{22}$$

Π_{λ} is the projection operator on the eigenspace of Θ corresponding to the eigenvalue λ:

$$\Pi_{\lambda}\,\Pi_{\mu} = \delta_{\lambda\,\mu}\,\Pi_{\lambda}, \tag{23}$$
$$\mathsf{I} = \sum_{\lambda} \Pi_{\lambda}, \tag{24}$$
$$\Theta = \sum_{\lambda} \lambda\,\Pi_{\lambda}. \tag{25}$$

I is the identity. It is seen that if Θ is diagonalizable, and none of its eigenvalues have positive real parts, then Υ remains finite. So all of the components of u (and as a result the energy) remain finite.

3. THE EIGENVALUES OF THE GENERATOR

The eigenvalues of Θ are the roots of C_{Θ}, the characteristic polynomial of Θ:

$$\mathsf{C}_{\Theta}(z) = \det(z\,\mathsf{I} - \Theta). \tag{26}$$

It is seen that

$$\mathsf{C}_{\Theta}(z) = z^4 + \kappa^2\,\sigma\,z^2 - \kappa^4\,\varpi, \tag{27}$$
$$\sigma = \boldsymbol{B}\cdot\boldsymbol{B} - c^{-2}\,\boldsymbol{E}\cdot\boldsymbol{E}, \tag{28}$$
$$\varpi = c^{-2}\,(\boldsymbol{B}\cdot\boldsymbol{E})^2. \tag{29}$$

In general, the characteristic equation (C_{Θ} equal to 0) has two solutions for z^2. These are denoted by w_1 and w_2:

$$w_1 + w_2 = -\kappa^2\,\sigma, \tag{30}$$
$$w_1\,w_2 = -\kappa^4\,\varpi. \tag{31}$$

As κ^2 is fixed and positive, these solutions are determined by σ and τ. It is seen that ϖ is non-negative, but the sign of σ depends on the ratio of the magnetic field to the electric field. The following cases are possible.

Case 1.

$$\varpi > 0. \tag{32}$$

That is,

$$\boldsymbol{B}\cdot\boldsymbol{E} \neq 0. \tag{33}$$

The result is that

$$w_1\,w_2 < 0. \tag{34}$$

Here w_1 and w_2 are real, one positive and the other negative. For the four eigenvalues of the generator Θ, one is real and positive, another is real and negative, and the two remaining are pure imaginary. As one of the eigenvalues of the generator is real and positive, (a part of) the velocity four vector grows indefinitely (and exponentially) with the proper time. Hence the energy of the particle grow indefinitely.

Case 2.

$$\varpi = 0. \tag{35}$$

That is,

$$\boldsymbol{B} \cdot \boldsymbol{E} = 0. \tag{36}$$

The result is that

$$w_1 \, w_2 = 0. \tag{37}$$

Here one of w_i's is zero, resulting in two zero eigenvalues for the generator. According to the Cayley-Hamilton theorem, any (square) matrix satisfies its characteristic equation [11]. So in this case,

$$\Theta^4 + \kappa^2 \, \sigma \, \Theta^2 = 0. \tag{38}$$

But it can be seen (through direct calculation, for example) that here Θ also satisfies the following equation.

$$\Theta^3 + \kappa^2 \, \sigma \, \Theta = 0. \tag{39}$$

Case 2.1.

$$\sigma \neq 0. \tag{40}$$

That is,

$$\boldsymbol{B} \cdot \boldsymbol{B} \neq c^{-2} \, \boldsymbol{E} \cdot \boldsymbol{E}. \tag{41}$$

Here Θ has 3 distinct eigenvalues, but (39) shows that its minimal polynomial is also of the order 3. Hence Θ is diagonalizable [12].

Case 2.1.1.

$$\sigma < 0. \tag{42}$$

That is,

$$\boldsymbol{B} \cdot \boldsymbol{B} < c^{-2} \, \boldsymbol{E} \cdot \boldsymbol{E}. \tag{43}$$

This is the *essentially electric* case. All of the eigenvalues of the generator are real: one is positive, one is negative, and two are zero. Here some components of the velocity four vector grow indefinitely (exponentially with the proper time). Hence so does the energy.

Case 2.1.2.

$$\sigma > 0. \tag{44}$$

That is,

$$\boldsymbol{B} \cdot \boldsymbol{B} > c^{-2} \boldsymbol{E} \cdot \boldsymbol{E}. \tag{45}$$

This is the *essentially magnetic* case. All of the eigenvalues of the generator are pure imaginary (two of them are zero). Here the velocity four vector remains finite. Hence so does the energy.

Case 2.2.

$$\sigma = 0. \tag{46}$$

That is,

$$\boldsymbol{B} \cdot \boldsymbol{B} = c^{-2} \boldsymbol{E} \cdot \boldsymbol{E}. \tag{47}$$

Here equation (39) becomes

$$\Theta^3 = 0. \tag{48}$$

Whether Θ^2 is also zero or not, depends on whether the electromagnetic field is zero or not:

Case 2.2.1.

$$F = 0. \tag{49}$$

This is the trivial case, resulting in

$$\Upsilon(\tau) = \mathsf{I}. \tag{50}$$

In this case, of course the velocity four vector remains constant.

Case 2.2.2.

$$F \neq 0. \tag{51}$$

In this case, it can be seen (through direct calculation, for example) that

$$\Theta^2 \neq 0. \tag{52}$$

So the generator Θ is not diagonalizable. But it is nillpotent and one arrives at

$$\exp(\tau\Theta) = \mathsf{I} + \tau\Theta + \frac{(\tau\Theta)^2}{2}. \tag{53}$$

Hence $\Upsilon(\tau)$ behaves like τ^2 for large τ. The velocity four vector grows indefinitely with τ (but like τ^2, not exponentially), and so does the energy.

4. CONCLUDING REMARKS

The problem of the motion of an electric charge in a constant and uniform electromagnetic field, in terms of the velocity four vector u is a first order linear differential equation with constant coefficients. This differential equation is characterized by a single matrix Θ: the generator. This matrix is the generator of the evolution in the proper time τ.

As in any linear (parameter-independent) system, the behavior of the system is determined by the spectrum of this generator. This spectrum was analyzed in terms of the electric and magnetic fields. The result was the following.

- In the generic case that both the electric and magnetic fields are nonzero and these fields are not normal to each other, there are two real and two pure imaginary eigenvalues, the sum of each pair being zero. So one of the eigenvalues is real and positive, hence the velocity four vector grows exponentially with the proper time.
- In the *essentially electric* case, the electric and magnetic fields are normal to each other but the electric field (in SI divided by c) is bigger than the magnetic field. Here there are two real eigenvalues, the sum of which is zero, and two zero eigenvalues. The generator is diagonalizable and again the velocity four vector grows exponentially with the proper time.
- In the *essentially magnetic* case, the electric and magnetic fields are normal to each other but the magnetic field is bigger that the electric field (in SI divided by c). Here there are two pure imaginary eigenvalues, the sum of which is zero, and two zero eigenvalues. Again the generator is diagonalizable, but the velocity four vector remains finite.
- In the marginal case, the electric and magnetic fields are normal to each other and the lengths of the electric field (in SI divided by c) and the magnetic field are the same. Here all of the eigenvalues of the generator are zero. In the trivial case of a zero electromagnetic field, of course the velocity four vector remains constant. Otherwise, the generator is not digonalizable but nilpotent. And again the velocity four vector grows indefinitely with the proper time, this time not exponentially but like the square of the proper time.

This analysis of the spectrum of the generator allows a qualitative description of the behavior of the system, without completely solving the problem.

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