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A NOTE ON NON-SOLVABLE GROUPS WITH GIVEN NUMBER OF PARTICULAR SUBGROUPS

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Abstract. Considering the quantitative properties of some particular subgroups of a finite group, we prove that (1) a non-solvable group G has exactly 5 non-subnormal non-supersolvable proper subgroups if and only if $G \cong A_5$ or $SL_2(5)$. (2) a non-solvable group G has exactly 5 non-subnormal non-2-nilpotent proper subgroups if and only if $G \cong A_5$ or $SL_2(5)$. (3) a non-solvable group G has exactly 16 non-subnormal non-2-closed proper subgroups (or two same order classes of non-subnormal non-2-closed proper subgroups) if and only if $G \cong A_5$ or $SL_2(5)$. Our results improve some known related research.

***Keywords**:* non-solvable group, non-subnormal, non-supersolvable subgroup, non-2-nilpotent subgroup, non-2-closed subgroup.
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1. INTRODUCTION

All groups are assumed to be finite. Huppert [2] showed that a group all of whose proper subgroups are supersolvable is solvable. As a generalization, Shi and Zhang [6, Theorem 1.1] had the following result.

THEOREM 1.1 [6, Theorem 1.1]. *Let G be a group.*
(1) *If G has at most* 4 *non-supersolvable proper subgroups, then G is solvable.*
(2) *G is a non-solvable group with exactly* 5 *non-supersolvable proper subgroups if and only if all non-supersolvable proper subgroups are conjugate maximal subgroups and* $G/\Phi(G) \cong A_5$.

In Section 2 of this paper, we prove the following result which provides a complete improvement of [6, Theorem 1.1 (2)].

THEOREM 1.2. *Suppose that G is a non-solvable group having exactly* 5 *non-supersolvable proper subgroups, then* $G \cong A_5$ *or* $SL_2(5)$.

As an extension of [6, Theorem 1.1] and Theorem 1.2, considering non-subnormal non-supersolvable proper subgroups, we have the following result, the proof of which is given in Section 3.

THEOREM 1.3. (1) *Suppose that a group G has at most* 4 *non-subnormal non-supersolvable proper subgroups, then G is solvable.*
(2) *A non-solvable group G has exactly* 5 *non-subnormal non-supersolvable proper subgroups if and only if* $G \cong A_5$ *or* $SL_2(5)$.

A group G is called a 2-nilpotent group if G has a normal 2-complement. It is easy to see that a 2-nilpotent group is solvable by Feit-Thompson theorem and a minimal non-2-nilpotent group is also solvable by [4, Proposition 2]. As a generalization, Shi and Liu [7, Theorem 1.1] had the following result.

THEOREM 1.4 [7, Theorem 1.1]. (1) *If a group G has at most 4 non-2-nilpotent proper subgroups, then G is solvable.*

(2) *If G is a non-solvable group having exactly 5 non-2-nilpotent proper subgroups, then $G/\Phi(G) \cong A_5$, where $\Phi(G) = Z(G)$ is a 2-group.*

Arguing as in the proof of Theorem 1.2, we can obtain the following result which provides a complete improvement of [7, Theorem 1.1 (2)].

THEOREM 1.5. *Suppose that G is a non-solvable group having exactly 5 non-2-nilpotent proper subgroups, then $G \cong A_5$ or $SL_2(5)$.*

As an extension of [7, Theorem 1.1] and Theorem 1.5, considering non-subnormal non-2-nilpotent proper subgroups, arguing as in the proof of Theorem 1.3, we have the following result.

THEOREM 1.6. (1) *Suppose that a group G has at most 4 non-subnormal non-2-nilpotent proper subgroups, then G is solvable.*

(2) *A non-solvable group G has exactly 5 non-subnormal non-2-nilpotent proper subgroups if and only if $G \cong A_5$ or $SL_2(5)$.*

A group G is said to be 2-closed if the Sylow 2-subgroup of G is normal. Observe that a 2-closed group is solvable by Feit-Thompson theorem and a minimal non-2-closed group is also solvable by [1]. As a generalization, [8, Theorem 1.1] proved that a group having at most 15 non-2-closed proper subgroups is solvable, and if G is a non-solvable group having exactly 16 non-2-closed proper subgroups, then $G/\Phi(G) \cong A_5$, where $\Phi(G) = Z(G) = 1$ or a cyclic 2-group.

In Section 4 of this paper, we have the following result which provides a complete improvement of [8, Theorem 1.1 (2)].

THEOREM 1.7. *Suppose that G is a non-solvable group having exactly 16 non-2-closed proper subgroups, then $G \cong A_5$ or $SL_2(5)$.*

Shi and Liu [8, Theorem 1.2] showed that a group in which all non-2-closed proper subgroups have the same order is solvable and if G is a non-solvable group having exactly two same order classes of non-2-closed proper subgroups, then $G/\Phi(G) \cong A_5$, where $\Phi(G) = Z(G) = 1$ or a cyclic 2-group. Arguing as in the proof of Theorem 1.7, we can obtain the following result which is a complete improvement of [8, Theorem 1.2 (2)].

THEOREM 1.8. *Suppose that G is a non-solvable group having exactly two same order classes of non-2-closed proper subgroups, then $G \cong A_5$ or $SL_2(5)$.*

The following corollary is a direct consequence of Theorem 1.8.

COROLLARY 1.9. *Suppose that G is a non-solvable group having exactly two conjugacy classes of non-2-closed proper subgroups, then $G \cong A_5$ or $SL_2(5)$.*

As an extension of [8, Theorem 1.1], Theorem 1.7, [8, Theorem 1.2] and Theorem 1.8, considering non-subnormal non-2-closed proper subgroups, arguing as in the proof of Theorem 1.3, we have the following result.

THEOREM 1.10. (1) *If a group G has at most 15 non-subnormal non-2-closed proper subgroups, then G is solvable.*

(2) *A non-solvable group G has exactly 16 non-subnormal non-2-closed proper subgroups if and only if $G \cong A_5$ or $SL_2(5)$.*

(3) *If all non-subnormal non-2-closed proper subgroups have the same order, then G is solvable.*

(4) *A non-solvable group G has exactly two same order classes of non-subnormal non-2-closed proper subgroups if and only if $G \cong A_5$ or $SL_2(5)$.*

2. PROOF OF THEOREM 1.2

Proof. We first have $G/\Phi(G) \cong A_5$ by [6, Theorem 1.1 (2)].

Let L be a maximal subgroup of G such that $L/\Phi(G) \cong A_4$ and M be a maximal subgroup of G such that $M/\Phi(G) \cong Z_5 \rtimes Z_2$, where L is a minimal non-supersolvable group and M is a supersolvable group. Let $\pi_e(G)$ be the set of all prime divisors of $|G|$. Then $\pi_e(G) = \pi_e(G/\Phi(G)) = \{2, 3, 5\}$.

Claim: $5 \nmid |\Phi(G)|$.

Otherwise, assume $5 \mid |\Phi(G)|$. Then $5 \mid |L|$. It follows that $|L|$ has 3 distinct prime divisors. Since L is a minimal non-supersolvable group, one has $|L| = p^{\alpha}qr^{p}$ by [5], where p, q and r are distinct primes, $\alpha \geq 1$, $p^{\alpha}q \mid r-1$ and $p \mid q-1$. It is easily seen that $r > q > p$. One has $r = 5$. However, $2^{\alpha} \cdot 3 \nmid 4$, a contradiction. Therefore, $5 \nmid |\Phi(G)|$.

Let $Q \in \mathrm{Syl}_5(M)$. It is obvious that Q is also a Sylow 5-subgroup of G. Since M is supersolvable and 5 is the largest prime divisor of $|M|$, Q is normal in M. Note that $5 \nmid |\Phi(G)$. It follows that $Q\Phi(G) = Q \times \Phi(G)$ is nilpotent. Then $\Phi(G) \leq C_G(Q)$. Furthermore, one has $\Phi(G) \leq C_G(Q^G)$, where Q^G is the normal closure of Q in G. Since $Q \nleq \Phi(G)$ and $G/\Phi(G)$ is a non-abelian simple group, $Q^G = G$. It follows that $\Phi(G) \leq C_G(G) = Z(G)$. Moreover, since G is non-abelian and $G/\Phi(G)$ is a non-abelian simple group, one has $Z(G) \leq \Phi(G)$. Thus $\Phi(G) = Z(G)$.

Claim: $3 \nmid |\Phi(G)|$.

Otherwise, assume $3 \mid |\Phi(G)|$. Let T be a maximal subgroup of $\Phi(G)$ such that $\Phi(G)/T \cong Z_3$. Then $(G/T)/\Phi(G/T) = (G/T)/(\Phi(G)/T) \cong G/\Phi(G) \cong A_5$. It follows that $G/T \cong A_5 \times Z_3$. However, $A_5 \times Z_3$ has more than 5 non-supersolvable proper subgroups which implies that G has more than 5 non-supersolvable proper subgroups, a contradiction. Therefore, $3 \nmid |\Phi(G)|$.

Hence $\Phi(G) = Z(G) = 1$ or a 2-group.

Let $P \in \mathrm{Syl}_2(G)$. Then $P/\Phi(G) = P/Z(G) \cong Z_2 \times Z_2$.

Claim: $Z(G) = 1$ or Z_2.

Otherwise, assume $|\Phi(G)| = |Z(G)| = 2^s$, where $s \geq 2$. For any maximal subgroup N of $\Phi(G)$, $\Phi(G/N) = \Phi(G)/N = Z(G)/N \cong Z_2$. Since $(G/N)/(\Phi(G)/N) \cong G/\Phi(G) \cong A_5$ and $(G/N)/(\Phi(G)/N) = (G/N)/\Phi(G/N)$, one has $G/N \cong SL_2(5)$. It follows that $P/N \cong Q_8$, which implies that P is non-abelian.

It is obvious that $Z(G) \leq Z(P)$. If $Z(G) < Z(P)$. Then $P/Z(P) \cong 1$ or Z_2, which implies that P is abelian, a contradiction. Therefore, $Z(G) = Z(P)$. It follows that $P/Z(P) \cong Z_2 \times Z_2$ and for any maximal subgroup N of $Z(P)$ one has $P/N \cong Q_8$.

It is easy to see that $\Phi(P) \leq Z(P)$. If $\Phi(P) < Z(P)$. Take a maximal subgroup N of $Z(P)$ such that $\Phi(P) \leq N < Z(P)$, one has $P/N \cong Z_2 \times Z_2 \times Z_2$, this contradicts $P/N \cong Q_8$ for any maximal subgroup N of $Z(P)$. Therefore, $\Phi(P) = Z(P)$.

Since $P/Z(P) = P/\Phi(P) \cong Z_2 \times Z_2$, P is a minimal non-abelian 2-group of order 2^{s+2}, where $s \geq 2$. By [9, Theorem 1.7.10], $P \cong M_2(n,m) = \langle a, b \mid a^{2^n} = b^{2^m} = 1, a^b = a^{1+2^{n-1}} \rangle$, where $n \geq 2$, $m \geq 1$; or $P \cong M_2(n,m,1) = \langle a, b, c \mid a^{2^n} = b^{2^m} = c^2 = 1, [a,b] = c, [c,a] = [c,b] = 1 \rangle$, where $n \geq m \geq 1$.

If $P \cong M_2(n,m) = \langle a \rangle \rtimes \langle b \rangle$. Then $M_2(n,m)/N = (\langle a \rangle N)/N \rtimes (\langle b \rangle N)/N$ which cannot be isomorphic to Q_8 for any normal subgroup N of $M_2(n,m)$, a contradiction.

If $P \cong M_2(n,m,1)$. One has $Z(P) = \langle a^2 \rangle \times \langle b^2 \rangle$. Take $N = \langle a^4 \rangle \times \langle b^2 \rangle$, then $P/N \cong Z_4 \rtimes Z_2 \ncong Q_8$, also a contradiction.

Hence $Z(G) = 1$ or Z_2.

If $\Phi(G) = Z(G) = 1$. Then $G \cong A_5$.

If $\Phi(G) = Z(G) = Z_2$. One has $G \cong SL_2(5)$.

□

3. PROOF OF THEOREM 1.3

Proof. (1) Suppose that G has at most 4 non-subnormal non-supersolvable proper subgroups. We will show that G is solvable.

Let G be a counterexample of minimal order. Then G is a minimal non-solvable group which implies that $G/\Phi(G)$ is a minimal non-abelian simple group. By [6, Theorem 1.1], $G/\Phi(G)$ has at least 5 non-supersolvable proper subgroups. Note that for any non-supersolvable proper subgroup $H/\Phi(G)$ of $G/\Phi(G)$, H is a non-subnormal non-supersolvable proper subgroup of G. Then G has at least 5 non-subnormal non-supersolvable proper subgroups, a contradiction.

Therefore, the counterexample of minimal order does not exist and then G is solvable.

(2) We only need to prove the necessity part. Since G is non-solvable, there exists a subgroup N of G such that N is a minimal non-solvable group. It follows that $N/\Phi(N)$ is a minimal non-abelian simple group. It is easy to see that any non-supersolvable proper subgroup of N is a non-subnormal non-supersolvable proper subgroup of N, which is also a non-subnormal non-supersolvable proper subgroup of G. By the hypothesis and [6, Theorem 1.1], N has exactly 5 non-supersolvable proper subgroups. Then $N \cong A_5$ or $SL_2(5)$ by Theorem 1.2.

In the following we show that $N = G$.

Otherwise, assume $N < G$. By the hypothesis, N is subnormal in G. Let M be a subgroup of G such that N is maximal in M. Then N is normal in M. Let H be any non-supersolvable proper subgroup of N. One has $|N : N_N(H)| = 5$.

Case (i): Suppose $N_M(H) \leq N$, then $|M : N_M(H)| = |M : N_N(H)| > |N : N_N(H)| = 5$. It implies that M has more than 5 non-subnormal non-supersolvable proper subgroups, a contradiction.

Case (ii): Suppose $N_M(H) \nleq N$. If $N_M(H) = M$. Then H is normal in M, which implies that H is normal in N, a contradiction. Thus $N_M(H) < M$. It follows that $N_M(H)$ is also a non-supersolvable proper subgroup of G. In particular, $N_M(H)$ is not subnormal in G since H is not subnormal in G. It implies that G has more than 5 non-subnormal non-supersolvable proper subgroups, also a contradiction.

Hence $G = N \cong A_5$ or $SL_2(5)$.

□

4. PROOF OF THEOREM 1.7

Proof. One has $G/\Phi(G) \cong A_5$ by [8, Theorem 1.1(2)], where $\Phi(G) = Z(G) = 1$ or a cyclic 2-group.

Let H be a maximal subgroup of G such that $H/\Phi(G) \cong A_4$ and K be a maximal subgroup of G such that $K/\Phi(G) \cong S_3 = Z_3 \rtimes Z_2$, where K is a minimal non-2-closed group. Let P be a Sylow 2-subgroup of H, which is also a Sylow 2-subgroup of G. Assume $|P| = 2^m$. Obviously, $m \geq 2$. Let P_0 be a Sylow 2-subgroup of K. Then P_0 is a cyclic 2-group of order 2^{m-1} by [1], which implies that P has a cyclic maximal subgroup.

Note that $|\Phi(G)| = 2^{m-2}$. We claim $m \leq 3$.

Otherwise, assume $m \geq 4$. Take $L < \Phi(G)$ such that $\Phi(G)/L \cong Z_2$. Then $(G/L)/\Phi(G/L)$ $= (G/L)/(\Phi(G)/L) \cong G/\Phi(G) \cong A_5$. One has $G/L \cong SL_2(5)$. It follows that $P/\Phi(G) \cong Z_2 \times Z_2$ and $P/L \cong Q_8$.

Thus P is non-abelian. Since $\Phi(G) = Z(G)$, one has $\Phi(G) \leq Z(P)$. If $\Phi(G) < Z(P)$. Then $|Z(P)| \geq 2^{m-1}$, which implies that P is abelian, a contradiction. Therefore, $\Phi(G) = Z(P)$. It follows that P is a non-abelian 2-group of order 2^m $(m \geq 4)$ which has a cyclic maximal subgroup, $P/Z(P) \cong Z_2 \times Z_2$ and $P/L \cong Q_8$, where $L < Z(P)$.

By [3, Chapter I, Theorem 14.9], P might be isomorphic to one of the following groups:

(1) $P = \langle a, b \mid a^{2^{m-1}} = b^2 = 1, b^{-1}ab = a^{-1} \rangle$;

(2) $P = \langle a, b \mid a^{2^{m-1}} = 1, b^2 = a^{2^{m-2}}, b^{-1}ab = a^{-1} \rangle$;

(3) $P = \langle a, b \mid a^{2^{m-1}} = b^2 = 1, b^{-1}ab = a^{1+2^{m-2}} \rangle$;

(4) $P = \langle a, b \mid a^{2^{m-1}} = b^2 = 1, b^{-1}ab = a^{-1+2^{m-2}} \rangle$.

Note that $|Z(P)| = 2^{m-2} \geq 4$ by above argument.

It is easy to see that $|Z(P)| = |\langle a^{2^{m-2}} \rangle| = 2 < 4$ if P belongs to cases (1), (2) or (4). Therefore, P cannot belong to cases (1), (2) and (4).

If P belongs to case (3), one has $Z(P) = \langle a^2 \rangle$. Then $P/Z(P) = (\langle a \rangle \rtimes \langle b \rangle)/\langle a^2 \rangle = (\langle a \rangle/\langle a^2 \rangle) \times ((\langle b \rangle \langle a^2 \rangle)/\langle a^2 \rangle) \cong Z_2 \times Z_2$. However, $P/L = (\langle a \rangle \rtimes \langle b \rangle)/\langle a^4 \rangle = (\langle a \rangle/\langle a^4 \rangle) \times ((\langle b \rangle \langle a^4 \rangle)/\langle a^4 \rangle) \cong Z_4 \times Z_2 \ncong Q_8$. Therefore, P cannot belong to case (3), either.

Hence $2 \leq m \leq 3$. Then $\Phi(G) = 1$ or Z_2.

When $\Phi(G) = 1$, one has $G \cong A_5$.

When $\Phi(G) = Z_2$, one has $G \cong SL_2(5)$.

□

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