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NECESSARY AND SUFFICIENT CONDITIONS FOR BOUNDEDNESS OF COMMUTATORS OF PARAMETRIC MARCINKIEWICZ INTEGRALS WITH WEIGHTED LIPSCHITZ FUNCTIONS

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Abstract. In this paper, we obtain the sharp maximal function estimate for the commutator $\mathscr{M}_{\Omega,b}^{\rho,m}$ generated by the parametric Marcinkiewicz integral $\mathscr{M}_{\Omega}^{\rho,m}$ and the locally integrable function b, where $\rho > 0$, $m > 1$ and Ω satisfies certain log-type regularity condition. Meanwhile, we prove the commutator $\mathscr{M}_{\Omega,b}^{\rho,m}$ is bounded from $L^p(\mu)$ to $L^q(\mu^{1-q})$ if and only if $b \in Lip_\beta(\mu)$, where $\mu \in A_1, 0 < \beta < 1, 1 < p < n/\beta$ and $1/q = 1/p - \beta/n$.

Keywords*:* parametric Marcinkiewicz integral, commutator, Lipschitz space.
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1. INTRODUCTION AND MAIN RESULTS

Let $\mathbb{R}^n (n \geq 2)$ be the n-dimensional Euclidean space and $\mathbb{S}^{n-1}$ be the unit sphere in $\mathbb{R}^n$ equipped with the normalized Lebesgue measure $\mathrm{d}\sigma = \mathrm{d}\sigma(\cdot)$. Let Ω be a homogeneous function of degree zero and have mean value zero, namely,

$$\Omega(\lambda x') = \Omega(x') \text{ for any } \lambda \in (0,\infty) \text{ and } x' \in \mathbb{S}^{n-1}, \tag{1}$$

and

$$\int_{S^{n-1}} \Omega\left(x'\right) \mathrm{d}\sigma\left(x'\right) = 0. \tag{2}$$

The parametric Marcinkiewicz integral was first introduced by Hörmander [6]. For any $x \in \mathbb{R}^n$, $\rho > 0$ and $m > 1$, the parametric Marcinkiewicz integral $\mathscr{M}_{\Omega}^{\rho,m}$ is defined by

$$\mathscr{M}_{\Omega}^{\rho,m}(f)(x) = \left(\int_0^\infty \left|F_{\Omega,t}^{\rho}(f)(x)\right|^m \frac{\mathrm{d}t}{t}\right)^{\frac{1}{m}},$$

where

$$F_{\Omega,t}^{\rho}(f)(x) = \frac{1}{t^\rho} \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}} f(y)\mathrm{d}y.$$

In 1958, Stein [13] showed that if $\Omega \in \mathrm{Lip}_\alpha(\mathbb{S}^{n-1})$ $(0 < \alpha \leq 1)$, then $\mathscr{M}_{\Omega}^{1,2}$ is bounded on the Lebesgue space $L^p(\mathbb{R}^n)$ for $1 < p \leq 2$. In 1962, Benedeck, Calderón and Panzone [2] obtained that if $\Omega \in C^1(\mathbb{S}^{n-1})$, then $\mathscr{M}_{\Omega}^{1,2}$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. In 2002, Al-Salman, Al-Qassem, Cheng and Pan [1] proved that if $\Omega \in L(\log L)^{1/2}(\mathbb{S}^{n-1})$, then $\mathscr{M}_{\Omega}^{1,2}$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. In 2016, Lu and Tao obtained that $\mathscr{M}_{\Omega}^{\rho,m}$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Moreover, many prominent results about the parametric Marcinkiewicz integral $\mathscr{M}_{\Omega}^{\rho,m}$ are widely investigated, we can see [9, 16–18].

The commutator $\mathscr{M}_{\Omega,b}^{\rho,m}$ generated by the parametric Marcinkiewicz integral $\mathscr{M}_{\Omega}^{\rho,m}$ and the locally integrable function b is defined by

$$\mathscr{M}_{\Omega,b}^{\rho,m}(f)(x) = \left(\int_0^\infty \left|[b, F_{\Omega,t}^{\rho}](f)(x)\right|^m \frac{\mathrm{d}t}{t}\right)^{\frac{1}{m}},$$

where

$$[b, F_{\Omega,t}^{\rho}](f)(x) = \frac{1}{t^\rho}\int_{|x-y|\le t}[b(x)-b(y)]\frac{\Omega(x-y)}{|x-y|^{n-\rho}}f(y)\mathrm{d}y.$$

In 1990, Torchinsky and Wang [15] proved that the boundedness of the commutator $\mathscr{M}_{\Omega,b}^{1,2}$ generated by parametric Marcinkiewicz integral $\mathscr{M}_{\Omega}^{1,2}$ and the locally integrable function $b \in BMO(\mathbb{R}^n)$ on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, if $\Omega \in Lip_\alpha(\mathbb{S}^{n-1})$ $(0 < \alpha < 1)$. In 2015, Chen and Ding [3] also showed that $b \in BMO(\mathbb{R}^n)$ is necessary for the boundedness of the commutator $\mathscr{M}_{\Omega,b}^{1,2}$ on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, if Ω satisfies the following logarithm type regularity:

$$\left|\Omega(x') - \Omega(y')\right| \lesssim \left(\log\frac{2}{|x'-y'|}\right)^{-\gamma} \text{ for any } x', y' \in S^{n-1}, \text{ and some } \gamma > 1. \tag{3}$$

Recently, the theory of commutators is still extensively studied in various function spaces, we refer the readers to see [5, 11, 19, 20] and therein references.

To state our main results, we need some basic definitions and notions.

For a locally integrable function f, the Hardy-Littlewood maximal function Mf is given by

$$Mf(x) = \sup_{B \ni x}\frac{1}{|B|}\int_B |f(y)|\mathrm{d}y,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$ containing x.

The sharp maximal function $M^\sharp f$ is defined by

$$M^\sharp f(x) = \sup_{B \ni x}\frac{1}{|B|}\int_B |f(y) - f_B|\,\mathrm{d}y,$$

where $f_B = \frac{1}{|B|}\int_B f(y)\mathrm{d}y$ and the supremum is taken over all cubes $B \subset \mathbb{R}^n$ containing x.

Definition 1 [4]. A non-negative function μ defined on $\mathbb{R}^n$ is called weight if it is locally integral. A weight μ is said to belong to the Muckenhoupt class $A_p(\mathbb{R}^n)$ for $1 < p < \infty$, if there exists a constant C such that

$$\frac{1}{|B|}\int_B \mu(x)\mathrm{d}x\left(\frac{1}{|B|}\int_B (\mu(x))^{-\frac{1}{p-1}}\mathrm{d}x\right)^{p-1} \le C$$

for any ball $B \subset \mathbb{R}^n$. The class $A_1(\mathbb{R}^n)$ is defined replacing the above inequality by

$$\frac{1}{|B|}\int_B \mu(x)\mathrm{d}x \le C\mu(x), \quad \text{a.e. } x \in \mathbb{R}^n,$$

for any ball $B \subset \mathbb{R}^n$.

Definition 2 [4]. A locally integral non-negative function μ is said to belong to $A_{p,q}(\mathbb{R}^n)\,(1 < p,q < \infty)$, if there exists a constant C such that

$$\left(\frac{1}{|B|}\int_B \mu(x)^q\,\mathrm{d}x\right)^{\frac{1}{q}}\left(\frac{1}{|B|}\int_B (\mu(x))^{-p'}\,\mathrm{d}x\right)^{\frac{1}{p'}} \le C$$

for any ball $B \subset \mathbb{R}^n$ and $1/p' + 1/p = 1$.

Definition 3. Given a weight function μ. For $1 < p < \infty$, the weighted Lebesgue space $L^p(\mu)$ is the space of all functions f such that

$$||f||_{L^p(\mu)} = \left(\int_{R^n} |f(x)|^p \mu(x)\mathrm{d}x\right)^{\frac{1}{p}} < \infty.$$

Definition 4 [7]. A locally integrable function f belongs to weighted Lipschitz space $\mathrm{Lip}^p_{\beta,\mu}$ for $1 \le p < \infty, 0 < \beta < 1$ and $\mu \in A_\infty(\mathbb{R}^n)$, that is

$$\sup_B \frac{1}{\mu(B)^{\frac{\beta}{n}}}\left[\frac{1}{\mu(B)}\int_B |f(x) - f_B|^p\,\mu(x)^{1-p}\mathrm{d}x\right]^{\frac{1}{p}} \le C,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$. The smallest constant C is taken to be the norm of f and is denoted by $\|f\|_{Lip^p_{\beta,\mu}}$.

Let $\mathrm{Lip}_{\beta,\mu} = \mathrm{Lip}^1_{\beta,\mu}$. Obviously, for the case $\mu = 1$, then weighted Lipschitz space $\mathrm{Lip}_{\beta,\mu}$ is classical Lipschitz space Lip_β. For $\delta > 0$, let $M_\delta f(x) = M(|f|^\delta)^{1/\delta}(x)$ and $M^\sharp_\delta f(x) = M^\sharp(|f|^\delta)^{1/\delta}(x)$. Throughout this paper, C denotes the constant that is independent of the main parameters involved but whose value may differ from line to line. For a measurable set E, denote by χ_E the characteristic function of E. The symbol $f \lesssim g$ means that $f \le Cg$. If $f \lesssim g$ and $g \lesssim f$, then $f \approx g$.

Our main results can be stated as follows.

THEOREM 1. *Let* $\mathscr{M}^{\rho,m}_\Omega$ *be a parametric Marcinkiewicz integral with the rough kernel* Ω *satisfying (1) and (2). Let* $\mu \in A_1(\mathbb{R}^n)$, $b \in \mathrm{Lip}_{\beta,\mu}$, $1/q = 1/p - \beta/n$ *for* $0 < \beta < 1$ *and* $0 < \delta < 1 < r < n/\beta$. *Then there exists a constant C such that*

$$M^\sharp_\delta\left(\mathscr{M}^{\rho,m}_{\Omega,b}(f)\right)(x) \le C\mu(x)\|b\|_{\mathrm{Lip}_{\beta,\mu}}\left(M_{\beta,\mu,r}(\mathscr{M}^{\rho,m}_\Omega(f))(x) + M_{\beta,\mu,r}(f)(x)\right)$$

for all function f *and* $x \in \mathbb{R}^n$,*and where*

$$M_{\beta,\mu,r}f(x) = \sup_{B\ni x}\left(\frac{1}{\mu(B)^{1-\frac{r\beta}{n}}}\int_B |f(y)|^r \mu(y)\mathrm{d}y\right)^{\frac{1}{r}}.$$

THEOREM 2. *Let* $\mathscr{M}^{\rho,m}_\Omega$ *be a parametric Marcinkiewicz integral with the rough kernel* Ω *satisfying (1), (2) and (3). Let* $\mu \in A_1(\mathbb{R}^n)$ *and* $1/q = 1/p - \beta/n$ *for* $0 < \beta < 1$ *and* $1 < p < q < \infty$. *Then*

(a) If $b \in \mathrm{Lip}_{\beta,\mu}$, *then commutator* $\mathscr{M}^{\rho,m}_{\Omega,b}$ *is bounded from* $L^p(\mu)$ *to* $L^q(\mu^{1-q})$.

(b) If commutator $\mathscr{M}^{\rho,m}_{\Omega,b}$ *is bounded from* $L^p(\mu)$ *to* $L^q(\mu^{1-q})$, *then* $b \in \mathrm{Lip}_{\beta,\mu}$.

Remark 1. When $\mu = 1$ and $m = 2$, Lu [9] showed that $b \in \mathrm{Lip}_\beta$ if and only if the commutator $\mathscr{M}^{\rho,2}_{\Omega,b}$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, where $1/q = 1/p - \beta/n$, $0 < \beta < 1$ and $1 < p < q < \infty$. When $\mu = 1$, $1 < m < 2$ and $\mu = 1$, $m > 2$, our results are also new.

2. PRELIMINARIES

We present some necessary lemmas in this section, which is important to prove our main results.

LEMMA 1 [14]. *Let* $0 < p, \delta < \infty$ *and* $\mu \in \bigcup_{1\le r<\infty} A_r(\mathbb{R}^n)$. *There exists a constant* C *such that*

$$\int_{\mathbb{R}^n} M_\delta f(x)^p \mu(x)\mathrm{d}x \le C\int_{\mathbb{R}^n} M_\delta^\sharp f(x)^p \mu(x)\mathrm{d}x.$$

LEMMA 2 [12] . *Let* $\mu \in A_1(\mathbb{R}^n)$ *and* $0 < \beta < 1$. *If* $1 \le p < \infty$, *then*

$$||b||_{\mathrm{Lip}_{\beta,\mu}} \approx \sup_B \frac{1}{\mu(B)^{\frac{\beta}{n}}}\left(\frac{1}{\mu(B)}\int_B |b(x)-b_B|^p \mu(x)^{1-p}\mathrm{d}x\right)^{\frac{1}{p}}.$$

LEMMA 3 [12]. *Suppose that* $1 \le r < p < n/\beta, 1/q = 1/p-\beta/n$ *and* $\mu \in A_{p,q}(\mathbb{R}^n)$. *Then there exists a constant* C *such that*

$$||M_{\beta,\mu,r}(f)||_{L^q(\mu)} \le C||f||_{L^p(\mu)}.$$

LEMMA 4 [10]. *Let* $\mathscr{M}_\Omega^{\rho,m}$ *be a parametric Marcinkiewicz integral with the rough kernel* Ω *satisfying (1) and (2). For* $1 < p < \infty$, *there exists a constant* C *such that*

$$\int_{\mathbb{R}^n} |\mathscr{M}_\Omega^{\rho,m}(f)(x)|^p \mathrm{d}x \le C\int_{\mathbb{R}^n} |f(x)|^p \mathrm{d}x.$$

3. PROOFS OF MAIN RESULTS

Proof of Theorem 1. Using the vector-valued singular integral notation of Benedek, Calderón and Panzone in [2], let $\mathscr{H}$ be the Hilbert space defined by

$$\mathscr{H} = \left\{h : \|h\|_{\mathscr{H}} = \left(\int_0^\infty \frac{|h(t)|^m}{t}\mathrm{d}t\right)^{\frac{1}{m}} < \infty\right\}.$$

Then, we can write

$$\mathscr{M}_\Omega^{\rho,m}(f)(x) = \|F_{\Omega,t}^\rho(f)(x)\|_{\mathscr{H}}, \quad \mathscr{M}_{\Omega,b}^{\rho,m}(f)(x) = \|[b, F_{\Omega,t}^\rho](f)(x)\|_{\mathscr{H}}.$$

For $x \in \mathbb{R}^n$, let B be a ball centered at x . Take $B^* = 2B$. We decompose $f = f\chi_{B^*} + f\chi_{(B^*)^\complement} := f_1 + f_2$ and write

$$\begin{aligned}\mathscr{M}_{\Omega,b}^{\rho,m}(f)(y) &= \mathscr{M}_{\Omega,b-b_{B^*}}^{\rho,m}(f)(y) = \|[b-b_{B^*}, F_{\Omega,t}^\rho](f)(y)\|_{\mathscr{H}} := \|F_{\Omega,t}^{\rho,b-b_{B^*}}(f)(y)\|_{\mathscr{H}} \\ &= \|(b(y)-b_{B^*})F_{\Omega,t}^\rho(f)(y) - F_{\Omega,t}^\rho((b-b_{B^*})f_1)(y) - F_{\Omega,t}^\rho((b-b_{B^*})f_2)(y)\|_{\mathscr{H}}.\end{aligned}$$

Let $C_B = \mathscr{M}_\Omega^{\rho,m}((b-b_{B^*})f_2)(x) = \|F_{\Omega,t}^\rho((b-b_{B^*})f_2)(x)\|_{\mathscr{H}}$. Then, for $y \in B$, we get

$$\begin{aligned}\left|\mathscr{M}_{\Omega,b}^{\rho,m}(f)(y) - C_B\right| &= \left|\|F_{\Omega,t}^{\rho,b-b_{B^*}}(f)(y)\|_{\mathscr{H}} - \|F_{\Omega,t}^\rho((b-b_{B^*})f_2)(x)\|_{\mathscr{H}}\right| \\ &\le |b(y)-b_{B^*}|\|F_{\Omega,t}^\rho(f)(y)\|_{\mathscr{H}} + \|F_{\Omega,t}^\rho((b-b_{B^*})f_1)(y)\|_{\mathscr{H}} \\ &\quad + \|F_{\Omega,t}^\rho(b-b_{B^*})f_2(y) - F_{\Omega,t}^\rho((b-b_{B^*})f_2)(x)\|_{\mathscr{H}} \\ &= I_1(y) + I_2(y) + I_3(y).\end{aligned}$$

Next, we estimate each term separately. Let $1/r' + 1/r = 1$ and $1 < r < n/\beta$. By Hölder's inequality and

Lemma 2, we have

$$\begin{aligned}\frac{1}{|B|}\int_B |I_1(y)|\mathrm{d}y &= \frac{1}{|B|}\int_B |(b(y)-b_{B^*})\mathscr{M}_\Omega^{\rho,m}(f)(y)|\mathrm{d}y\\
&\le C\left(\frac{1}{|B^*|}\int_{B^*}|b(y)-b_{B^*}|^{r'}\mu(y)^{1-r'}\mathrm{d}y\right)^{\frac{1}{r'}}\left(\frac{1}{|B|}\int_B\left|\mathscr{M}_\Omega^{\rho,m}(f)(y)\right|^r\mu(y)\mathrm{d}y\right)^{\frac{1}{r}}\\
&\le C\frac{\mu(B)}{|B|}\|b\|_{\mathrm{Lip}_{\beta,\mu}}M_{\beta,\mu,r}(\mathscr{M}_\Omega^{\rho,m}(f))(x)\\
&\le C\mu(x)\|b\|_{\mathrm{Lip}_{\beta,\mu}}M_{\beta,\mu,r}(\mathscr{M}_\Omega^{\rho,m}(f))(x).\end{aligned}$$

For the second term $I_2(y)$, choose $v\in(1,r)$ and let $1/v=1/u+1/r$. Then, by the boundedness of $\mathscr{M}_\Omega^{\rho,m}$ on $L^v(\mathbb{R}^n)$, Hölder's inequality and Lemma 2, we obtain

$$\begin{aligned}\frac{1}{|B|}\int_B I_2(y)\mathrm{d}y &= \frac{1}{|B|}\int_B \mathscr{M}_\Omega^{\rho,m}\left((b-b_{B^*})f_1\right)(y)\mathrm{d}y\\
&\le\left(\frac{1}{|B|}\int_B|\mathscr{M}_\Omega^{\rho,m}\left((b-b_{B^*})f_1\right)(y)|^v\mathrm{d}y\right)^{\frac{1}{v}}\\
&\le\left(\frac{1}{|B|}\int_{B^*}|\left(b(y)-b_{B^*}\right)f(y)|^v\mathrm{d}y\right)^{\frac{1}{v}}\\
&\le C\left(\frac{1}{|B^*|}\int_{B^*}|b(y)-b_{B^*}|^u\mu(y)^{1-u}\mathrm{d}y\right)^{\frac{1}{u}}\left(\frac{1}{|B|}\int_B|f(y)|^r\mu(y)\mathrm{d}y\right)^{\frac{1}{r}}\\
&\le C\mu(x)\|b\|_{\mathrm{Lip}_{\beta,\mu}}M_{\beta,\mu,r}(f)(x),\end{aligned}$$

Finally, for $I_3(y)$, it is easy to see that

$$\begin{aligned}I_3(y)&=\left(\int_0^\infty\left|\frac{1}{t^\rho}\int_{|y-z|\le t}(b(z)-b_{B^*})f_2(z)\frac{\Omega(y-z)}{|y-z|^{n-\rho}}\mathrm{d}z-\frac{1}{t^\rho}\int_{|x-z|\le t}(b(z)-b_{B^*})f_2(z)\frac{\Omega(y-z)}{|y-z|^{n-\rho}}\mathrm{d}z\right|^m\frac{\mathrm{d}t}{t}\right)^{\frac{1}{m}}\\
&\le\left(\int_0^\infty\left|\frac{1}{t^\rho}\int_{|y-z|<t\le|x-z|}(b(z)-b_{B^*})f_2(z)\frac{\Omega(y-z)}{|y-z|^{n-\rho}}\mathrm{d}z\right|^m\frac{\mathrm{d}t}{t}\right)^{\frac{1}{m}}\\
&\quad+\left(\int_0^\infty\left|\frac{1}{t^\rho}\int_{|x-z|<t\le|y-z|}(b(z)-b_{B^*})f_2(z)\frac{\Omega(x-z)}{|x-z|^{n-\rho}}\mathrm{d}z\right|^m\frac{\mathrm{d}t}{t}\right)^{\frac{1}{m}}\\
&\quad+\left(\int_0^\infty\left|\frac{1}{t^\rho}\int_{|x-z|\le t,|y-z|\le t}(b(z)-b_{B^*})f_2(z)\left[\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(x-z)}{|x-z|^{n-\rho}}\right]\mathrm{d}z\right|^m\frac{\mathrm{d}t}{t}\right)^{\frac{1}{m}}\\
&:=II_1(y)+II_2(y)+II_3(y).\end{aligned}$$

In what follows, we estimate $II_1(y)$, $II_2(y)$ and $II_3(y)$, respectively. Note that, for $x,y\in B$, $z\in(B^*)^{\complement}$, we have $|x-z|\sim|y-z|$. By Hölder's inequality and Lemma 2,

$$\begin{aligned}II_1(y)&\le\int_{(B^*)^{\complement}}|b(z)-b_{B^*}||f(z)|\frac{|\Omega(y-z)|}{|y-z|^{n-\rho}}\left|\frac{1}{|y-z|^{m\rho}}-\frac{1}{|x-z|^{m\rho}}\right|^{\frac{1}{m}}\mathrm{d}z\\
&\le\int_{(B^*)^{\complement}}|b(z)-b_{B^*}||f(z)|\frac{1}{|y-z|^{n-\rho}}\frac{|x-y|^{\frac{1}{m}}}{|x-z|^{\rho+\frac{1}{m}}}\mathrm{d}z\\
&\le\sum_{j=1}^\infty\int_{2^{j+1}B\setminus2^jB}|b(z)-b_{B^*}||f(z)|\frac{|x-y|^{\frac{1}{m}}}{|x-z|^{n+\frac{1}{m}}}\mathrm{d}z\end{aligned}$$

$$\begin{aligned}
&\leq C\sum_{j=1}^{\infty}2^{-j}\frac{1}{|B|}\int_{2^{j+1}B}|b(z)-b_{B^*}||f(z)|\mathrm{d}z\\
&\leq C\sum_{j=1}^{\infty}\frac{j}{2^j}\mu(x)\|b\|_{\mathrm{Lip}_{\beta,\mu}}M_{\beta,\mu,r}(f)(x)\\
&\leq C\mu(x)\|b\|_{\mathrm{Lip}_{\beta,\mu}}M_{\beta,\mu,r}(f)(x).
\end{aligned}$$

By the similar arguments as in estimating $II_1(y)$, we obtain

$$II_2(y)\leq C\mu(x)\|b\|_{\mathrm{Lip}_{\beta,\mu}}M_{\beta,\mu,r}(f)(x).$$

For $II_3(y)$, by Minkowski's inequality, we have

$$\begin{aligned}
II_3(y)\leq &C\int_{(B^*)^{\complement}}|b(z)-b_{B^*}|\,|f(z)|\left|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(x-z)}{|x-z|^{n-\rho}}\right|\frac{1}{|x-z|^{\rho}}\mathrm{d}z\\
\leq &C\int_{(B^*)^{\complement}}|b(z)-b_{B^*}|\,|f(z)|\frac{|\Omega(x-z)|}{|x-z|^{\rho}}\left|\frac{1}{|y-z|^{n-\rho}}-\frac{1}{|x-z|^{n-\rho}}\right|\mathrm{d}z\\
&+C\int_{(B^*)^{\complement}}|b(z)-b_{B^*}|\,|f(z)|\frac{|\Omega(y-z)-\Omega(x-z)|}{|x-z|^{n}}\\
:=&III_1(y)+III_2(y).
\end{aligned}$$

As in estimating $II_1(y)$, we deduce that

$$\begin{aligned}
III_1(y)&\leq C\int_{(B^*)^{\complement}}|b(z)-b_{B^*}|\,|f(z)|\frac{|x-y|}{|x-z|^{n+1}}\mathrm{d}z\\
&\leq C\sum_{j=1}^{\infty}2^{-j}\frac{1}{|B|}\int_{2^{j+1}B}|b(z)-b_{B^*}|\,|f(z)|\mathrm{d}z\\
&\leq C\sum_{j=1}^{\infty}\frac{j}{2^j}\mu(x)\|b\|_{\mathrm{Lip}_{\beta,\mu}}M_{\beta,\mu,r}(f)(x)\\
&\leq C\mu(x)\|b\|_{\mathrm{Lip}_{\beta,\mu}}M_{\beta,\mu,r}(f)(x).
\end{aligned}$$

For $III_2(y)$, invoking the condition (3), we obtain

$$\begin{aligned}
III_2(y)&\leq C\int_{(B^*)^{C}}|b(z)-b_{B^*}|\,\frac{|f(z)|}{|x-z|^n}\left(\log\frac{2|x-z|}{|x-y|}\right)^{-\gamma}\mathrm{d}z\\
&\leq C\sum_{j=1}^{\infty}\int_{2^{j+1}B\setminus 2^jB}|b(z)-b_{B^*}|\,\frac{|f(z)|}{|x-z|^n}\left(\log\frac{2|x-z|}{|x-y|}\right)^{-\gamma}\mathrm{d}z\\
&\leq C\sum_{j=1}^{\infty}\frac{j}{(j+1)^{\gamma}}\mu(x)\|b\|_{\mathrm{Lip}_{\beta,\mu}}M_{\beta,\mu,r}(f)(x)\\
&\leq C\mu(x)\|b\|_{\mathrm{Lip}_{\beta,\mu}}M_{\beta,\mu,r}(f)(x).
\end{aligned}$$

Summing up the estimates of $II_1(y)$, $II_2(y)$, $III_1(y)$ and $III_2(y)$, we can see that

$$\frac{1}{|B|}\int_B I_3(y)\mathrm{d}y\leq C\mu(x)\|b\|_{\mathrm{Lip}_{\beta,\mu}}M_{\beta,\mu,r}(f)(x).$$

This, together with the estimates for $I_1(y)$, $I_2(y)$, immediately yields that

$$M_{\delta}^{\sharp}\left(\mathscr{M}_{\Omega,b}^{\rho,m}(f)\right)(x)\leq C\mu(x)\|b\|_{\mathrm{Lip}_{\beta,\mu}}\left(M_{\beta,\mu,r}(\mathscr{M}_{\Omega}^{\rho,m}(f))(x)+M_{\beta,\mu,r}(f)(x)\right),$$

which completes the proof of Theorem 1. □

Proof of Theorem 2. We first prove (a). From Lemma 1 and Lemma 3, since $\mu \in A_1(\mathbb{R}^n)$, then $\mu^{1-q} \in A_q(\mathbb{R}^n)$. Then by Theorem 1 with $0<\delta<1<r<p$, we have

$$\begin{aligned}\left\|\mathscr{M}^{\rho,m}_{\Omega,b}(f)\right\|_{L^q(\mu^{1-q})} &\leq \left\|M_\delta\left(\mathscr{M}^{\rho,m}_{\Omega,b}(f)\right)\right\|_{L^q(\mu^{1-q})}\\ &\leq \left\|M^\sharp_\delta\left(\mathscr{M}^{\rho,m}_{\Omega,b}(f)\right)\right\|_{L^q(\mu^{1-q})}\\ &\leq C\|b\|_{\mathrm{Lip}_{\beta,\mu}}\left(\|M_{\beta,\mu,r}(\mathscr{M}^{\rho,m}_{\Omega}(f))\|_{L^q(\mu)}+\|M_{\beta,\mu,r}(f)\|_{L^q(\mu)}\right)\\ &\leq C\|b\|_{\mathrm{Lip}_{\beta,\mu}}\|f\|_{L^p(\mu)}.\end{aligned}$$

Now we prove (b). We use the method given by Janson in [8]. Let $K(x):=1/|x|^n$. Choose $0\neq z_0\in\mathbb{R}^n$ and $\delta>0$, such that $1/K(z)$ can be expressed in the neighborhood $\{z:|z-z_0|<\sqrt{n}\delta\}$ as a Fourier series which is absolutely convergent, that is

$$\frac{1}{K(z)}=\sum_{n=0}^{\infty}a_n e^{iv_n\cdot z},$$

with $\sum\limits_{n=0}^{\infty}|a_n|<\infty$. Let $z_1=\frac{z_0}{\delta}$. If $|z-z_1|<2\sqrt{n}$, we obtain

$$\frac{1}{K(x)}=\frac{\delta^{-n}}{K(x\delta)}=\delta^{-n}\sum_{n=0}^{\infty}a_n e^{iv_n\cdot\delta z}.$$

For any ball $B=B(x_0,r)$. Set $y_0=x_0-rz_1$ and $B'=B(y_0,r)$. Then for $x\in B$ and $y\in B'$,

$$\left|\frac{x-y}{r}-z_1\right|=\left|\frac{x-y}{r}-\frac{x_0-y_0}{r}\right|\leq\left|\frac{x-x_0}{r}\right|+\left|\frac{y-y_0}{r}\right|\leq 2\sqrt{n}.$$

We set $s(x)=\operatorname{sgn}(b(x)-b_{B'})$, then

$$\begin{aligned}\int_B|b(x)-b_{B'}|\,\mathrm{d}x&=\int_B(b(x)-b_{B'})\,s(x)\mathrm{d}x\\ &=\frac{1}{|B'|}\int_B\int_{B'}(b(x)-b(y))s(x)\mathrm{d}y\mathrm{d}x\\ &=C\int_B\int_{\mathbb{R}^n}(b(x)-b(y))K(x-y)\sum_{n=0}^{\infty}a_n e^{iv_n\cdot\delta\frac{x-y}{r}}s(x)\chi_B(x)\chi_{B'}(y)\mathrm{d}y\mathrm{d}x\\ &=C\sum_{n=0}^{\infty}a_n\int_B\int_{\mathbb{R}^n}(b(x)-b(y))K(x-y)f_n(y)g_n(x)\mathrm{d}y\mathrm{d}x\\ &\leq C\sum_{n=0}^{\infty}|a_n|\int_B\left|\int_{\mathbb{R}^n}(b(x)-b(y))K(x-y)f_n(y)\mathrm{d}y\right|\mathrm{d}x\\ &=C\sum_{n=0}^{\infty}|a_n|\int_B\left|\int_{\mathbb{R}^n}(b(x)-b(y))\frac{1}{|x-y|^n}f_n(y)\mathrm{d}y\right|\mathrm{d}x,\end{aligned}$$

where

$$f_n(y)=e^{-i\frac{\delta}{r}v_n\cdot y}\chi_{B'}(y)\quad\text{and}\quad g_n(x)=e^{i\frac{\delta}{r}v_n\cdot x}s(x)\chi_B(x).$$

In addition, since Ω satisfies (1), (2), (3), then there exists a constant A with $0<A<1$, for $x,y\in\mathbb{R}^n$ with $x\neq y$, we deduce that

$$\Omega(x-y)=\Omega\left(\frac{x-y}{|x-y|}\right)=\Omega((x-y)')\geq\frac{C}{(\log(\frac{2}{A}))^\gamma}. \tag{4}$$

Using Minkowski's inequality, Hölder's inequality and (4), we get

$$\begin{aligned}
&\int_B \left| \int_{\mathbb{R}^n} (b(x)-b(y)) \frac{1}{|x-y|^n} f_n(y)\mathrm{d}y \right| \mathrm{d}x \\
&= \int_B \left| \int_{\mathbb{R}^n} (b(x)-b(y)) \frac{1}{|x-y|^{n-\rho}} \frac{1}{|x-y|^{\rho}} f_n(y)\mathrm{d}y \right| \mathrm{d}x \\
&\leq \int_B \frac{1}{|x-y_0|^{\rho}} \left| \int_{\mathbb{R}^n} (b(x)-b(y)) \frac{1}{|x-y|^{n-\rho}} f_n(y)\mathrm{d}y \right| \mathrm{d}x \\
&= \frac{(\log(\frac{2}{A}))^{\gamma}}{(\log(\frac{2}{A}))^{\gamma}} \int_B \frac{1}{|x-y_0|^{\rho}} \left| \int_{\mathbb{R}^n} (b(x)-b(y)) \frac{1}{|x-y|^{n-\rho}} f_n(y)\mathrm{d}y \right| \mathrm{d}x \\
&\leq C \int_B \frac{1}{|x-y_0|^{\rho}} \left| \int_{\mathbb{R}^n} \frac{1}{(\log(\frac{2}{A}))^{\gamma}} (b(x)-b(y)) \frac{1}{|x-y|^{n-\rho}} f_n(y)\mathrm{d}y \right| \mathrm{d}x \\
&\leq C \int_B \frac{1}{|x-y_0|^{\rho}} \left| \int_{\mathbb{R}^n} (b(x)-b(y)) \frac{\Omega(x-y)}{|x-y|^{n-\rho}} f_n(y)\mathrm{d}y \right| \mathrm{d}x \\
&\leq C \int_B \left| \int_{\mathbb{R}^n} (b(x)-b(y)) \frac{\Omega(x-y)}{|x-y|^{n-\rho}} f_n(y)\mathrm{d}y \left(\int_{|x-y_0|}^{\infty} \frac{\mathrm{d}t}{t^{1+m\rho}} \right) \left(\int_{|x-y_0|}^{\infty} \frac{\mathrm{d}t}{t^{1+m\rho}} \right)^{-\frac{m-1}{m}} \right| \mathrm{d}x \\
&\leq C \int_B \left| \left(\int_{|x-y_0|}^{\infty} \int_{\mathbb{R}^n} (b(x)-b(y)) \frac{\Omega(x-y)}{|x-y|^{n-\rho}} f_n(y) \chi_{\{y\in\mathbb{R}^n:|x-y|\leq t\}}(y)\mathrm{d}y \frac{\mathrm{d}t}{t^{1+m\rho}} \right) \left(\int_{|x-y_0|}^{\infty} \frac{\mathrm{d}t}{t^{1+m\rho}} \right)^{-\frac{m-1}{m}} \right| \mathrm{d}x \\
&\leq C \int_B \left(\int_0^{\infty} \left| \frac{1}{t^{\rho}} \int_{|x-y|\leq t} (b(x)-b(y)) \frac{\Omega(x-y)}{|x-y|^{n-\rho}} f_n(y)\mathrm{d}y \right|^m \frac{\mathrm{d}t}{t} \right)^{\frac{1}{m}} \mathrm{d}x \\
&= C \int_B \mathscr{M}_{\Omega,b}^{\rho,m}(f_n)(x)\mathrm{d}x.
\end{aligned}$$

Combining preceding estimates, we have

$$\begin{aligned}
\int_B |b(x)-b_{B'}|\mathrm{d}x &\leq C \sum_{n=0}^{\infty} |a_n| \int_B \mathscr{M}_{\Omega,b}^{\rho,m}(f_n)(x)\mathrm{d}x \\
&\leq C \sum_{n=0}^{\infty} |a_n| ||\mathscr{M}_{\Omega,b}^{\rho,m}(f_n)||_{L^q(\mu^{1-q})} \mu(B)^{\frac{1}{q'}} \\
&\leq C \sum_{n=0}^{\infty} |a_n| ||\chi_{B'}||_{L^p(\mu)} \mu(B)^{\frac{1}{q'}} \\
&\leq C\mu(B)^{\frac{1}{p}+\frac{1}{q'}} \\
&= C\mu(B)^{1+\frac{\beta}{n}}.
\end{aligned}$$

Therefore, we obtain

$$\frac{1}{\mu(B)^{1+\frac{\beta}{n}}} \int_B |b(y)-b_B|\mathrm{d}y \leq \frac{2}{\mu(B)^{1+\frac{\beta}{n}}} \int_B |b(y)-b_{B'}|\mathrm{d}y \leq C,$$

that is $b \in \mathrm{Lip}_{\beta,\mu}$. This completes the proof of Theorem 2. □

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