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A CHARACTERISTIC PROPERTY OF THE SEMICIRCLE LAW

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Abstract. Let $\mathscr{F}_+(\sigma) = \{\mathbf{Q}_m^\sigma(dy);\ \ m \in (m_1^\sigma, m_+^\sigma)\}$ be the Cauchy-Stieltjes Kernel (CSK) family induced by a (non-degenerate) probability measure σ possessing a one sided support boundary from above. For $\tau \geq 0$, denote by $\widetilde{U}^\tau$ the τ-deformation of measures introduced in [6, Section 5]. In this article, a property is presented for the CSK families based on the stability under τ-deformation of measures. A CSK family satisfying such stability property is nothing but the CSK family generated by the Semicircle type law, up to affinity.

***Keywords**:* variance function, Cauchy-Stieltjes transformation, Semicircle law.

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1. INTRODUCTION

The Semicircle law plays in non-commutative probability the role played in classical probability by the Normal law. The Semicircle law also appears as the limiting law of the eigenvalues of many random symmetric matrices when the dimensions of the random matrix goes to infinity. Some properties related Semicircle law are presented in the literature by the use of the moment and the resolvent approaches respectively, see [7]. In this article our study linked to the Semicircle law is different and it is connected to the framework of Cauchy-Stieltjes Kernel (CSK) families. We give a characterization of the Semicircle law using the notion of τ-transformation of measures introduced in [6, Section 5]. For the clarity of the result to be presented, let's first recall some facts on CSK families and present the notion of τ-transformation of measures which will be used in this article.

The concept of CSK families in free probability is recently introduced. It is defined similarly to natural exponential families by exploring the Cauchy-Stieltjes kernel $(1-\zeta y)^{-1}$ replacing the exponential kernel $\exp(\zeta y)$. The CSK families have been studied in [3] for probability measures with compact support. Further results are presented in [4] involving probability measures with one sided support boundary, say from above. Denote by $\mathscr{P}_{ba}$ the set of (non-degenerate) probability measures possessing support bounded from above. For $\mu \in \mathscr{P}_{ba}$, the function

$$\mathscr{L}_\mu(\zeta) = \int \frac{1}{1-\zeta y}\mu(\mathrm{d}y) \tag{1}$$

converges $\forall\ \zeta \in [0, \zeta_+^\mu)$ with $\frac{1}{\zeta_+^\mu} = \max\{0, \sup\operatorname{supp}(\mu)\}$. The CSK family induced by μ is

$$\mathscr{F}_+(\mu) = \{\mathbf{P}_\zeta^\mu(\mathrm{d}y) = \frac{1}{\mathscr{L}_\mu(\zeta)(1-\zeta y)}\mu(\mathrm{d}y) :\ \ \zeta \in (0, \zeta_+^\mu)\}.$$

The map $\zeta \mapsto \mathbb{K}_\mu(\zeta) = \int y\mathbf{P}_\zeta^\mu(\mathrm{d}y)$, (said mean function) is strictly increasing on $(0, \zeta_+^\mu)$, see [4]. The interval $(m_1^\mu, m_+^\mu) = \mathbb{K}_\mu((0, \zeta_+^\mu))$ is said the mean domain of $\mathscr{F}_+(\mu)$. A mean parametrization of $\mathscr{F}_+(\mu)$ is then provided: Let $\phi_\mu(\cdot)$ denote the inverse function for $\mathbb{K}_\mu(\cdot)$. For $s \in (m_1^\mu, m_+^\mu)$, consider $\mathbf{Q}_s^\mu(\mathrm{d}y) = \mathbf{P}_{\phi_\mu(s)}^\mu(\mathrm{d}y)$, we obtain

$$\mathscr{F}_+(\mu) = \{\mathbf{Q}_s^\mu(\mathrm{d}y);\ \ s \in (m_1^\mu, m_+^\mu)\}.$$

It is demonstrated in [4] that

$$m_1^\mu = \lim_{\zeta \to 0^+} \mathbb{K}_\mu(\zeta) \qquad \text{and} \qquad m_+^\mu = \mathfrak{B} - \lim_{\vartheta \to \mathfrak{B}^+} \frac{1}{\mathscr{G}_\mu(\vartheta)}, \tag{2}$$

with

$$\mathfrak{B} = \mathfrak{B}(\mu) = \max\{\sup \mathrm{supp}(\mu), 0\} = \frac{1}{\zeta_+^\mu}, \tag{3}$$

and

$$\mathscr{G}_\mu(\vartheta) = \int \frac{1}{\vartheta - y}\mu(\mathrm{d}y), \qquad \text{for } \vartheta \in \mathbb{C} \setminus \mathrm{supp}(\mu). \tag{4}$$

If μ possess a one sided support boundary from below, the CSK family is denoted by $\mathscr{F}_-(\mu)$. Then $\zeta_-^\mu < \zeta < 0$, where ζ_-^μ is either $-\infty$ or $1/\mathfrak{A}(\mu)$ with $\mathfrak{A} = \mathfrak{A}(\mu) = \min\{\inf supp(\mu), 0\}$. Furthermore, (m_-^μ, m_1^μ) is the mean domain with $m_-^\mu = \mathfrak{A} - 1/\mathscr{G}_\mu(\mathfrak{A})$. When the support of μ is compact, the two-sided CSK family is $\mathscr{F}(\mu) = \mathscr{F}_+(\mu) \cup \mathscr{F}_-(\mu) \cup \{\mu\}$.

The map

$$s \mapsto \mathbf{V}_\mu(s) = \int (y-s)^2 \mathbf{Q}_s^\mu(\mathrm{d}y), \tag{5}$$

is the variance function of $\mathscr{F}_+(\mu)$, see [3]. If the first moment of $\mu \in \mathscr{P}_{ba}$ does not exists, all the laws in $\mathscr{F}_+(\mu)$ have infinite variance. A new concept called pseudo-variance function $\mathbb{V}_\mu(.)$ is defined in [4], as

$$\mathbb{V}_\mu(s) = s\left(\frac{1}{\phi_\mu(s)} - s\right). \tag{6}$$

If $-\infty < m_1^\mu = \int y\mu(\mathrm{d}y) < +\infty$, then $\mathbf{V}_\mu(\cdot)$ exists and we have (see [4])

$$\mathbb{V}_\mu(s) = \frac{s}{s - m_1^\mu}\mathbf{V}_\mu(s). \tag{7}$$

Remark 1. (i) $\mathbf{Q}_s^\mu(\mathrm{d}y)$ can be presented as $\mathbf{Q}_s^\mu(\mathrm{d}y) = l_\mu(y,s)\mu(\mathrm{d}y)$ so that

$$l_\mu(y,s) := \begin{cases} \frac{\mathbb{V}_\mu(s)}{\mathbb{V}_\mu(s) + s(s-y)}, & s \neq 0 \quad ; \\ 1, & s = 0,\ \ \mathbb{V}_\mu(0) \neq 0 \quad ; \\ \frac{\mathbb{V}'_\mu(0)}{\mathbb{V}'_\mu(0) - y}, & s = 0,\ \ \mathbb{V}_\mu(0) = 0 \quad . \end{cases} \tag{8}$$

(ii) Let $\rho \neq 0$ and $\beta \in \mathbb{R}$ and consider $f(\mu)$ the image of μ by $f : y \mapsto \rho y + \beta$. $\forall$ s close enough to $m_1^{f(\mu)} = f(m_1^\mu) = \rho m_1^\mu + \beta$, we get

$$\mathbb{V}_{f(\mu)}(s) = \frac{\rho^2 s}{s - \beta}\mathbb{V}_\mu\left(\frac{s-\beta}{\rho}\right). \tag{9}$$

In case of existence of $\mathbf{V}_\mu(\cdot)$, we have

$$\mathbf{V}_{f(\mu)}(s) = \rho^2 \mathbf{V}_\mu\left(\frac{s-\beta}{\rho}\right). \tag{10}$$

We come now to the concept of τ-deformation of measures. Krystek and Yoshida [6] have considered a transformation of the Cauchy-Stieltjes transform of a measure σ (with finite mean) in the following way: Let $\mathbf{t} = (a,b)$, for $a \in \mathbb{R}$ and $b > 0$. Introduce the $(\mathbf{t} = (a,b))$-deformation by

$$\frac{1}{\mathscr{G}_{\widetilde{U}^{\mathbf{t}}(\sigma)}(\xi)} = \frac{b}{\mathscr{G}_{\sigma}(\xi)} + (1-b)\xi + (b-a)m_1^{\sigma}, \tag{11}$$

where m_1^{σ} denote the first moment of σ. If $t = a = b > 0$, the (a,b)-deformation is reduced to the t-deformation considered in [1] and [2]. The τ-deformation of measures, which we denote by $\widetilde{U}^{\tau}$, is a particular case of the $(\mathbf{t} = (a,b))$-deformation by considering $b = 1$ and $a = \tau \geq 0$. In that case, relation (11) reduces to

$$\frac{1}{\mathscr{G}_{\widetilde{U}^{\tau}(\sigma)}(\xi)} = \frac{1}{\mathscr{G}_{\sigma}(\xi)} + (1-\tau)m_1^{\sigma}. \tag{12}$$

In this article, a property is presented for the CSK families based on the stability under τ-deformation of measures. A CSK family satisfying such stability property is nothing but the CSK family induced by the Semicircle law (up to affinity), which is provided by

$$\mathbf{sc}(\mathrm{d}t) = \frac{\sqrt{4-t^2}}{2\pi}\mathbf{1}_{(-2,2)}(t)\mathrm{d}t, \tag{13}$$

with $m_1^{\mathbf{sc}} = 0$. We have $\mathbf{V}_{\mathbf{sc}}(s) = 1 = \mathbb{V}_{\mathbf{sc}}(s)$, $\forall\, s \in (m_-^{\mathbf{sc}}, m_+^{\mathbf{sc}}) = (-1,\ 1)$.

2. MAIN RESULT

This section is devoted to state and prove the article's main result.

THEOREM 1. (i) *Let $\nu \in \mathscr{P}_{ba}$ with finite first moment $m_1^{\nu} < 0$. Assume furthermore that the domain of means of $\mathscr{F}_+(\nu)$ contains an interval $[0,\delta)$ for some $\delta > 0$. Fix $\tau \geq 0$ such that $\tau \neq 1$ and assume that $\widetilde{U}^{\tau}(\boldsymbol{Q}_m^{\nu}) \in \mathscr{F}_+(\nu)$ for all $m \in (0,\delta)$. Then ν is a Semicircle-type law.*

(ii) *Suppose ν is a Semicircle law with mean m_1^{ν} and variance $\alpha > 0$ so that the two-sided domain of means is $(m_1^{\nu} - \sqrt{\alpha}, m_1^{\nu} + \sqrt{\alpha})$. If $\boldsymbol{Q}_m^{\nu}$ is a Marchenko-Pastur type law in $\mathscr{F}_+(\nu)$ and $\tau \geq 0$ is such that $\tau m \in (m_1^{\nu} - \sqrt{\alpha}, m_1^{\nu} + \sqrt{\alpha})$, then $\widetilde{U}^{\tau}(\boldsymbol{Q}_m^{\nu}) \in \mathscr{F}_+(\nu)$.*

Proof. (i) Fix $m \in (m_1^{\nu}, m_+^{\nu})$ and measure $\mathbf{Q}_m^{\nu} \in \mathscr{F}_+(\nu)$. Then, the mean and the variance of $\mathbf{Q}_m^{\nu}$ are m and $\mathbf{V}_{\nu}(m)$. Denote $\mu = \widetilde{U}^{\tau}(\mathbf{Q}_m^{\nu})$. Lemma 1.3 in [6] relates moments of $\mathbf{Q}_m^{\nu}$ and moments of its image μ. A calculation based on the these moment formulas in [6] identifies the mean and variance of this measure $\mu = \widetilde{U}^{\tau}(\mathbf{Q}_m^{\nu})$ as τm and $\mathbf{V}_{\nu}(m)$. So, if we assume that $\mu = \widetilde{U}^{\tau}(\mathbf{Q}_m^{\nu}) \in \mathscr{F}_+(\nu)$ then $\mu = \mathbf{Q}_{\tau m}^{\nu}$. The value of τm as well as the value of m must be in the domain of means, and hence μ has variance $\mathbf{V}_{\nu}(\tau m)$. This shows that

$$\mathbf{V}_{\nu}(\tau m) = \mathbf{V}_{\nu}(m). \tag{14}$$

Note that, relation (14) hold only for one fixed value of $\tau \neq 1$ and only for m such that both m and τm are in the domain of means. Next we consider two cases: $\tau < 1$ and $\tau > 1$. Suppose $0 \leq \tau < 1$ and $m \in (0,\delta)$. Then $\tau m \in (0,\delta)$ is in the domain of means and we can use (14) recursively:

$$\mathbf{V}_{\nu}(m) = \mathbf{V}_{\nu}(\tau m) = \ldots = \mathbf{V}_{\nu}(\tau^k m).$$

Passing to the limit as $k \to +\infty$ and using continuity of $\mathbf{V}_{\nu}(.)$, we see that $\mathbf{V}_{\nu}(m) = \mathbf{V}_{\nu}(0)$. From the fact that $\mathbf{V}_{\nu}(.)$ is analytic on its mean domain, we conclude that $\mathbf{V}_{\nu}(.)$ is constant on its domain of mean. From [3, Theorem 3.2] and up to affinity, measure ν is of the Semicircle type law. This ends the proof if $0 \leq \tau < 1$.

Suppose now $\tau > 1$. For $k = 0,\ 1,\ \ldots$ define $a_k = m/\tau^k$. Note that if $m \in (0,\delta)$ then $a_k \in (0,\delta)$ is in the domain of means. So again we can apply formula (14) with $m = a^k$ and $\tau m = a_{k-1}$. We get

$$\mathbf{V}_\nu(a_k) = \mathbf{V}_\nu(a_{k-1}) = \ldots = \mathbf{V}_\nu(a_0) = \mathbf{V}_\nu(m).$$

Since $a_k \xrightarrow{k\to+\infty} 0$, passing to the limit as $k \to +\infty$, we get again $\mathbf{V}_\nu(m) = \mathbf{V}_\nu(0)$.

(ii) This implication is just a calculation using the explicit formula for the Cauchy-Stieltjes transform of the Marchenko-Pastur law (up to affinity). For the simplification of the calculations, we may fix the variance of the Semicircle law to be $\alpha = 1$ and denote the mean $m_0 := m_1^\nu$. The general case of variance $\alpha > 0$ can be deduced by a scale transformation. We have

$$\mathbf{Q}_m^\nu = \frac{\sqrt{4-(x-m_0)^2}}{2\pi(1+(m-m_0)(m-x))}\mathbf{1}_{|x-m_0|<2}\,\mathrm{d}x.$$

The Cauchy-Stieltjes transform of ν is provided by

$$\mathscr{G}_\nu(z) = \frac{z-m_0-\sqrt{(z-m_0)^2-4}}{2}. \tag{15}$$

Using [5, equation 2.19] and relation (15), the Cauchy-Stieltjes transform of $\mathbf{Q}_m^\nu$ is

$$\mathscr{G}_{\mathbf{Q}_m^\nu}(z) = \frac{1}{m+\mathbb{V}_\nu(m)/m}\left(\frac{\mathbb{V}_\nu(m)}{m}\mathscr{G}_\nu(z)-1\right) = \frac{z+m_0-2m-\sqrt{(z-m_0)^2-4}}{2[1+(m-m_0)(m-z)]} = \frac{2}{\sqrt{(z-m_0)^2-4}-2m+m_0+z}. \tag{16}$$

From definition (12), we have

$$\mathscr{G}_{\widetilde{U}^\tau(\mathbf{Q}_m^\nu)}(z) = \frac{1}{\frac{1}{\mathscr{G}_{\mathbf{Q}_m^\nu}(z)}+(1-\tau)m} = \frac{2}{\sqrt{(z-m_0)^2-4}-2\tau m+m_0+z}.$$

By (16), we get $\mathscr{G}_{\widetilde{U}^\tau(\mathbf{Q}_m^\nu)}(z) = \mathscr{G}_{\mathbf{Q}_{\tau m}^\nu}(z)$ provided that $\tau m \in [m_0-1, m_0+1]$. Since Cauchy transforms are equal and identify probability measures uniquely, we see that $\widetilde{U}^\tau(\mathbf{Q}_m^\nu) = \mathbf{Q}_{\tau m}^\nu \in \mathscr{F}_+(\nu)$, ending the proof.

$\square$

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REFERENCES

[1] Bożejko M, Wysoczański J. New examples of convolutions and non-commutative central limit theorems. Banach Center Publications. 1998;43(1):95–103.

[2] Bożejko M, Wysoczański J. Remarks on t-transformations of measures and convolutions, Ann. Inst. H. Poincarobab. Statist. 2001; 37:737–761.

[3] Bryc W. Free exponential families as kernel families. Demonstr. Math. 2009;XLII(3):657–672. arxiv.org:math.PR:0601273.

[4] Bryc W, Hassairi A. One-sided Cauchy-Stieltjes kernel families. Journ. Theoret. Probab. 2011;24(2):577–594.

[5] Fakhfakh R. Characterization of quadratic Cauchy-Stieltjes Kernel families based on the orthogonality of polynomials. J. Math. Anal. Appl. 2018;459:577–589.

[6] Krystek AD, Yoshida H. Generalized t-transformation of probability measures and deformed convolutions. Probability and Mathematical Statistics. 2004;24:97–119.

[7] Wigner EP. Characteristic vectors of bordered matrices with infinite dimensions. The Annals of Mathematics. 1955;62(3):548–564.

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