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BIHARMONIC SLANT SUBMANIFOLDS IN SPACE FORMS

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Abstract. In this research work, bi-harmonic slant sub-manifolds in space forms have been studied. Bi-harmonic maps are critical points of the bi-energy functional and are of great importance due to both analytical and geometrical characteristics. After the study of Riemannian Manifolds with complex and contact structure and s-space form structure along with slant sub-manifolds theory, some significant developments related to bi-harmonic slant submanifolds in space forms have been established. Manifolds with constant sectional curvature is known as space form. As s-space form is the generalization of complex and contact structure, we get generalized results. Different aspects of results have discussed.

***Keywords**:* isometric immersion, complex structure, contact structure, generalized s-space form, invariant and anti-invariant sub-manifolds, slant sub-manifold, curvature tensor, biharmonic maps, sub-manifold theory.
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1. INTRODUCTION

Theory of bi-harmonic maps has great importance in various fields of differential geometry. The theory of bi-harmonic maps is an ancient and rich subject. They have been studied since 1862 by Maxwell and Airy to describe a mathematical model of elasticity. It has its roots in 3-D Euclidean space, analysis of curves and surfaces. Due to their analytic and geometric effects, bi-harmonic maps have become an attractive and important field of research. In the form of their Riemannian curvature, invariants, anti-invariant and slant sub-manifolds [4, 21].

Sub-manifolds play a fundamental role in differential geometry, providing a frame work for studying curved spaces and enabling the development of powerful mathematical tools. They are essential in understanding the geometry of higher-dimensional spaces and are a key ingredient in various branches of mathematics and physics. Calabi, initiated the study of real and complex sub-manifolds of complex structure manifolds [5]. After this many authors started research on sub-manifolds of almost contact metric manifolds known as odd dimensional structure manifolds analogue to complex structure manifolds that are of even dimension.

A slant sub-manifold is a sub-manifold whose tangent spaces make a constant angle with a given vector field. Slant sub-manifolds are relevant in complex analysis and complex differential geometry. They provide a way to study the interplay between complex structure and sub-manifolds, leading to a rich geometric theory.

Slant immersion into almost contact metric manifold was first presented by A. Lotta in 1988. Later on, further expansion of the results for contact slant sub-manifolds in Sasakian and in Kenmotsue space forms were established in [1, 19].

The study of bi-harmonic slant sub-manifolds in space forms involves investigating their geometric properties, such as their curvature, shape and deformation behaviour. Researchers are interested in understanding the interplay between the bi-harmonic and slant properties and how they influence the overall geometry of the sub-manifold.

Blair introduced manifolds with f-structure named as s-manifolds that are the generalization of complex and contact structures. For further detail of s-space form see [21]. A space form is a completely Riemannian manifold with constant sectional curvature. In present research work we will study slant sub-manifolds regarding space forms within the context of bi-harmonic maps and will establish significant generalized results.

2. PRELIMINARIES

In this section, we explore some well-known conditions concerning bi-harmonic slant sub-manifold for complex manifold, almost contact metric manifold and S-Manifold.

2.1. Complex manifold

Let N be a 2n-dimensional Riemannian manifold and M be an slant sub-manifold with almost complex structure f and almost metric h which satisfy the following conditions

$$f^2 = -Id, \quad h(fZ_1, fZ_2) = h(Z_1, Z_2) \quad \text{and} \quad \nabla f = 0$$

for any $Z_1, Z_2 \in TM$. Where ∇ denote the covariant derivative with respect to Levi-Civita connection. An isometric immersion $f : M \to N$ is called holomorphic if, for any point x in M, we have $f(T_xM) = T_xM$ where T_xM denotes the tangent space of M at x. If we have $f(T_xM) \subseteq T_x^{\perp}M$ for each $x \in M$, where $T_x^{\perp}M$ denote the normal space of M in N at x.

For any vector Z_1 tangent to M, we put

$$fZ_1 = \mathscr{T}Z_1 + \mathscr{N}Z_1$$

where $\mathscr{T}Z_1$ and $\mathscr{N}Z_1$ denote the tangential and normal components of fZ_1 respectively. Then $\mathscr{T}$ is an endomorphism of the tangent bundle TM and $\mathscr{N}$ is normal bundle valued 1-form on TM. For each non-zero vector Z_1 tangent to M at x the angle $\theta(Z_1)$ between fZ_1 and T_xM is called the Wirtinger angle of Z_1. In the following, we call an immersion $f : M \to N$ a general slant immersion if the Wirtinger angle $\theta(Z_1)$ is constant. Holomorphic and totally real immersion are general slant immersions with Wirtinger angle θ equal to 0 and $\frac{\pi}{2}$. A general slant immersion which is not holomorphic is simply called slant immersion. A slant immersion is said to be proper if it is not totally real.

A complex manifold N is called a complex space form and is denoted by $N(c)$ if it has constant f-sectional curvature c. The curvature tensor R of a complex space form $N(c)$ is stated as

$$R(Z_1, Z_2)Z_3 = \frac{c}{4}\{h(Z_2, Z_3)Z_1 - h(Z_1, Z_3)Z_2 + h(Z_1, fZ_3)fZ_2 - h(Z_2, fZ_3)fZ_1 + 2h(Z_1, fZ_2)fZ_3\}$$

for any vector fields Z_1, Z_2, Z_3 on M [25].

2.2. Sasakian manifold

Let N be a $(2n+1)$-dimensional Riemannian manifold and M be an slant sub-manifold with almost contact structure f, almost metric h and a global vector field ξ (structure vector field) such that, if η is the dual 1-form of ξ which satisfy the following conditions

$$f^2 Z_1 = -Z_1 + \eta(Z_1)\xi, \quad h(Z_1, \xi) = \eta(Z_1)$$
$$h(fZ_1, fZ_2) = h(Z_1, Z_2) - \eta(Z_1)\eta(Z_2)$$

For any $Z_1, Z_2 \in TM$. In this case,

$$h(fZ_1, Z_2) + h(Z_1, fZ_2) = 0,$$

for any $Z_1, Z_2 \in TM$. Let f denote the 2-form in M given by $f(Z_1,Z_2) = h(Z_1, fZ_2)$ for all $Z_1, Z_2 \in TM$. The 2-form f is called the fundamental 2-form in M.

For any $Z_1 \in TM$, we write

$$fZ_1 = \mathscr{T}Z_1 + \mathscr{N}Z_1$$

where $\mathscr{T}Z_1$ and $\mathscr{N}Z_1$ denote the tangential and normal components of fZ_1 respectively. Similarly, for any $V \in T^{\perp}M$, we have

$$fV = tV + nV$$

where tV and nV denote the tangential and normal components of fV respectively.

The submanifold M is said to be invariant if normal component is identically zero, that is fZ_1 is tangent to M, for any $Z_1 \in T_xM$. On the other hand, M is said to be an anti-invariant submanifold if T is identically zero, that is fZ_1 is normal to M, for any $Z_1 \in T_xM$. Next, for each nonzero vector $Z_1 \in T_xM$, such that Z_1 is not proportional to ξ, we denote by $\theta(Z_1)$ the angle between fZ_1 and T_xM. Then, M is said to be slant if the angle $\theta(Z_1)$ is a constant. Invariant and anti-invariant immersions are slant immersions with slant angle $\theta(Z_1) = 0$ and $\theta(Z_1) = \frac{\pi}{2}$ respectively. A slant immersion which is not invariant nor anti-invariant is called a proper slant immerssion.

Further the slant submanifold M of an almost contact metric manifold (N,h) staisfies

$$h(TZ_1, TZ_2) = \cos^2\theta(h(Z_1,Z_2) - \eta(Z_1)\eta(Z_2))$$
$$h(NZ_1, NZ_2) = \sin^2\theta(h(Z_1,Z_2) - \eta(Z_1)\eta(Z_2)),$$

for Z_1, $Z_2 \in TM$.

A Sasakian manifold $N(\xi,\eta,h)$ is said to be a Sasakian space form is denoted by $N(c)$ if it has a constant f-sectional curvature $c(Z_1 \wedge fZ_1)$, where $c(Z_1 \wedge fZ_1)$ denotes the sectional curvature of the section spanned by the unit vector field Z_1, orthogonal to ξ and fZ_1. The curvature tensor R of a Sasakian space form $N(c)$ is stated as

$$R(Z_1,Z_2)Z_3 = \frac{c+3}{4}\{h(Z_2,Z_3)Z_1 - h(Z_1,Z_3)Z_2\} + \frac{c-1}{4}\{h(Z_1,fZ_3)fZ_2 - h(Z_2,fZ_3)fZ_1 + 2h(Z_1,fZ_2)fZ_3\}$$
$$+ \frac{c-1}{4}\{\eta(Z_1)\eta(Z_3)Z_2 - \eta(Z_2)\eta(Z_3)Z_1 + h(Z_1,Z_3)\eta(Z_2)\xi - h(Z_2,Z_3)\eta(Z_1)\xi\}$$

For any vector fields Z_1, Z_2, Z_3 on M [25].

2.3. Generalized S-manifold

Let $f : (M,h) \to (N,g)$ be a smooth map between two Riemannian manifolds and (N,g) be a Riemannian $(2n+s)$-dimensional manifold and express the vector fields with Lie algebra in N by TN and is defined as a metric f-manifold if there is a $(1,1)$ tensor field f, $\xi_1,...,\xi_s$ as a structure vector fields and $\eta_1,...,\eta_s$ are s 1-forms on TN so that:

$$f^2Z_1 = -Z_1 + \sum_{\alpha=1}^{s}\eta_\alpha(Z_1)\xi_\alpha, \quad h(Z_1,\xi_\alpha) = \eta_\alpha(Z_1),$$
$$f\xi_\alpha = 0, \qquad \eta_\alpha \circ f = 0$$
$$h(fZ_1, fZ_2) = h(Z_1,Z_2) - \sum_{\alpha=1}^{s}\eta_\alpha(Z_1)\eta_\alpha(Z_2),$$

for $Z_1, Z_2 \in TN$ and $\alpha = 1,...,s$.

Let F be the fundamental 2-form in TN given by $F(Z_1,Z_2) = h(Z_1, fZ_2)$ for any $Z_1, Z_2 \in TN$.

Let a $(2n+s)$-dimensional slant submanifold M is isometrically immersed in N. We denote the Lie algebra of tangent vector fields by TN and $T^{\perp}N$ shows the set of all those vector fields that are normal to N , such that

$$fZ_1 = \mathscr{T}Z_1 + \mathscr{N}Z_1$$

where $\mathscr{T}Z_1$ (and resp. $\mathscr{N}Z_1$) denotes tangential (and resp. normal) component of fZ_1.

If the normal component $\mathscr{N}$ vanishes identically then the sub-manifold M is said to be invariant, i.e., if $fZ_1 \in TN$, for any $Z_1 \in TN$ and if the tangential component $\mathscr{T}$ vanishes identically then M is said to be anti-invariant, i.e., $fZ_1 \in T^{\perp}N$, for any $Z_1 \in TN$.

Consider the vector fields ξ_α, where $\alpha = 1, ..., s$ are tangent to M . Then, the orthogonal distribution to ξ_α, in TM is denoted by $\mathscr{D}$. Hence, $TM = \mathscr{D} \oplus \xi_\alpha$ is considered to be orthogonal direct decomposition .

We define the Wirtinger angle $\theta(Z_1)$ for any non-zero vector field Z_1 which is in between fZ_1 and TN, where Z_1 is not proportional to ξ_α .

If the Wirtinger angle $\theta(Z_1)$ is a constant then the sub-manifold M is said to be θ-slant sub-manifold. The Wirtinger angle θ of a slant immersion is known as slant angle. The sub-manifold M is said to be invariant (resp. anti invariant) if the slant angle $\theta = 0$ (resp. $\theta = \frac{\pi}{2}$). A slant immersion is known as proper slant immersion if $\theta \neq 0, \frac{\pi}{2}$.

The curvature tensor R of a s-space form $N(c)$ is stated as

$$\begin{aligned} R(Z_1,Z_2)Z_3 = &\sum_{\alpha,\beta}\{-f^2 Z_1 \eta_\alpha(Z_2)\eta_\beta(Z_3) - h(fZ_1,fZ_3)\eta_\alpha(Z_2)\xi_\beta \\ &+ h(fZ_2,fZ_3)\eta_\alpha(Z_1)\xi_\beta + f^2 Z_2 \eta_\alpha(Z_1)\eta_\beta(Z_3)\} \\ &+ \frac{1}{4}(c+3s)\sum_{i=1}^{2n+s}\{-f^2 Z_1\ h(fZ_2,fZ_3) + f^2 Z_2\ h(fZ_1,fZ_3)\} \\ &+ \frac{1}{4}(c-s)\sum_{i=1}^{2n+s}\{-fZ_1\ h(Z_2,fZ_3) + fZ_2\ h(Z_1,fZ_3) + 2fZ_3\ h(Z_1,fZ_2\} \end{aligned}$$

for any vector fields Z_1, Z_2, Z_3 on N [25].

3. MAIN RESULTS

THEOREM 1. *Let* $(N^{2n+s}(c), f, h)$ *be a s-space form. Then* M^{2m+s} *is biharmonic slant sub-manifold of* $N^{2n+s}(c)$ *if*

$$(2m+s)\left[\sum_{k=1}^{2m+s}\left\{-\nabla_{e_k}(A_H e_k) + A_{\nabla^{\perp}_{e_k}H}(e_k) + A_H(\nabla_{e_k}e_k)\right\} + \frac{c-s}{4}\sum_{i=1}^{2m+s}\{3(te_i + ne_i)h(H, ne_i)\}\right] = 0$$

$$(2m+s)\left[\sum_{k=1}^{2m+s}\left\{-B(e_k, A_H e_k) + \nabla^{\perp}_{e_k}\nabla^{\perp}_{e_k}H - \nabla^{\perp}_{\nabla_{e_k}e_k}H\right\} + Hs + \frac{c+3s}{4}\{2mH\} - \frac{c-s}{4}\cos\theta fH\right] = 0$$

Proof. Consider orthonormal basis $\{e_i\}$ on submanifold M^{2m+s}, structure vector fields $\xi_1, \xi_2, \ldots, \xi_s$ are tangent to M^{2m+s}. For M^{2m+s} to be biharmonic submanifold we have

$$(2m+s)\overline{\Delta}H - (2m+s)\,\mathrm{trace}(R^N(e_i,H)e_i) = 0 \tag{1}$$

trace$(R^N(e_i,H)e_i)$ can be computed for slant submanifold as:

$$\sum_{i=1}^{2m+s} R(e_i,H)e_i = -Hs - \frac{c+3s}{4}\{2mH\} + \frac{c-s}{4}\cos\theta fH - \frac{c-s}{4}\sum_{i=1}^{2m+s}\{3(te_i+ne_i)h(H,ne_i)\}.$$

For isometric immersion

$$\overline{\Delta}H = \sum_{k=1}^{2m+s}\left\{\nabla_{e_k}\nabla_{e_k}H - \nabla_{\nabla_{e_k}e_k}H\right\}$$

Since

$$\nabla_{e_k}H = -A_He_k + \nabla^{\perp}_{e_k}H,$$

we can write

$$\nabla_{e_k}\nabla_{e_k}H = -\nabla_{e_k}(A_He_k) - B(e_k,A_He_k) + A_{\nabla^{\perp}_{e_k}H}(e_k) + \nabla^{\perp}_{e_k}\nabla^{\perp}_{e_k}H$$

and

$$\nabla_{\nabla_{e_k}e_k}H = -A_H(\nabla_{e_k}e_k) + \nabla^{\perp}_{\nabla_{e_k}e_k}H$$

From (1), we can write

$$(2m+s)\sum_{k=1}^{2m+s}\{-\nabla_{e_k}(A_He_k) - B(e_k,A_He_k) + A_{\nabla^{\perp}_{e_k}H}(e_k) + \nabla^{\perp}_{e_k}\nabla^{\perp}_{e_k}H + A_H(\nabla_{e_k}e_k) - \nabla^{\perp}_{\nabla_{e_k}e_k}H\}$$
$$+(2m+s)\{Hs + \frac{c+3s}{4}\{2mH\} - \frac{c-s}{4}\cos\theta fH + \frac{c-s}{4}\sum_{i=1}^{2m+s}\{3(te_i+ne_i)h(H,ne_i)\}\} = 0.$$

Taking tangential and normal components, we get (i) and (ii)

$$(2m+s)\left[\sum_{k=1}^{2m+s}\left\{-\nabla_{e_k}(A_He_k) + A_{\nabla^{\perp}_{e_k}H}(e_k) + A_H(\nabla_{e_k}e_k)\right\} + \frac{c-s}{4}\sum_{i=1}^{2m+s}\{3(te_i+ne_i)h(H,ne_i)\}\right] = 0$$
$$(2m+s)\left[\sum_{k=1}^{2m+s}\left\{-B(e_k,A_He_k) + \nabla^{\perp}_{e_k}\nabla^{\perp}_{e_k}H - \nabla^{\perp}_{\nabla_{e_k}e_k}H\right\} + Hs + \frac{c+3s}{4}\{2mH\} - \frac{c-s}{4}\cos\theta fH\right] = 0 \quad \square$$

COROLLARY 1. *Let M^{2m+s} be bi-harmonic slant sub-manifold of $N(c)$ of dimension $(2n+s)$ with s-structure f. Then the invariant immersions are slant immersion if*

$$(2m+s)\left[\sum_{k=1}^{2m+s}\left\{-\nabla_{e_k}(A_He_k) + A_{\nabla^{\perp}_{e_k}H}(e_k) + A_H(\nabla_{e_k}e_k)\right\} + \frac{c-s}{4}\sum_{i=1}^{2m+s}\{3(te_i+ne_i)h(H,ne_i)\}\right] = 0$$
$$(2m+s)\left[\sum_{k=1}^{2m+s}\left\{-B(e_k,A_He_k) + \nabla^{\perp}_{e_k}\nabla^{\perp}_{e_k}H - \nabla^{\perp}_{\nabla_{e_k}e_k}H\right\} + Hs + \frac{c+3s}{4}\{2mH\} - \frac{c-s}{4}fH\right] = 0$$

Proof. Put $\theta = 0$ in Theorem 3.1, we obtain the required outcome. □

COROLLARY 2. *Let M^{2m+s} be bi-harmonic slant sub-manifold of $N(c)$ of dimension $(2n+s)$ with s-structure f. Then the anti-invariant immersions are slant immersion if*

$$(2m+s)\left[\sum_{k=1}^{2m+s}\left\{-\nabla_{e_k}(A_He_k)+A_{\nabla^{\perp}_{e_k}H}(e_k)+A_H(\nabla_{e_k}e_k)\right\}+\frac{c-s}{4}\sum_{i=1}^{2m+s}\{3(te_i+ne_i)h(H,ne_i)\}\right]=0$$

$$(2m+s)\left[\sum_{k=1}^{2m+s}\left\{-B(e_k,A_He_k)+\nabla^{\perp}_{e_k}\nabla^{\perp}_{e_k}H-\nabla^{\perp}_{\nabla_{e_k}e_k}H\right\}+Hs+\frac{c+3s}{4}\{2mH\}\right]=0$$

Proof. Put $\theta=\frac{\pi}{2}$ in Theorem 3.1, we obtain the required outcome. □

COROLLARY 3. *Let M^{2m} be bi-harmonic slant sub-manifold of even dimensional structure manifold N(c) of dimension $(2n)$ with complex structure f. Then the slant immersion is bi-harmonic if*

$$(2m)\left[\sum_{k=1}^{2m}\left\{-\nabla_{e_k}(A_He_k)+A_{\nabla^{\perp}_{e_k}H}(e_k)+A_H(\nabla_{e_k}e_k)\right\}+\frac{c}{4}\sum_{i=1}^{2m}\{3(te_i+ne_i)h(H,ne_i)\}\right]=0$$

$$(2m)\left[\sum_{k=1}^{2m}\left\{-B(e_k,A_He_k)+\nabla^{\perp}_{e_k}\nabla^{\perp}_{e_k}H-\nabla^{\perp}_{\nabla_{e_k}e_k}H\right\}+\frac{c}{4}\{2mH\}-\frac{c}{4}\{\cos\theta fH\}\right]=0$$

Proof. Put $s=0$ in Theorem 3.1, we obtain the required outcome. □

Remark 1. Let M^{2m} be bi-harmonic slant sub-manifold of even dimensional structure manifold $N(c)$ of dimension $(2n)$ with complex structure f. Then invariant immersions are slant immersion if

$$(2m)\left[\sum_{k=1}^{2m}\left\{-\nabla_{e_k}(A_He_k)+A_{\nabla^{\perp}_{e_k}H}(e_k)+A_H(\nabla_{e_k}e_k)\right\}+\frac{c}{4}\sum_{i=1}^{2m}\{3(te_i+ne_i)h(H,ne_i)\}\right]=0$$

$$(2m)\left[\sum_{k=1}^{2m}\left\{-B(e_k,A_He_k)+\nabla^{\perp}_{e_k}\nabla^{\perp}_{e_k}H-\nabla^{\perp}_{\nabla_{e_k}e_k}H\right\}+\frac{c}{4}\{2mH\}-\frac{c}{4}fH\right]=0$$

Proof. Put $\theta=0$ in Corollary 3.3, we obtain the required outcome. □

Remark 2. Let M^{2m} be bi-harmonic slant sub-manifold of even dimensional structure manifold $N(c)$ of dimension $(2n)$ with complex structure f. Then anti-invariant immersions are slant immersion if

$$(2m)\left[\sum_{k=1}^{2m}\left\{-\nabla_{e_k}(A_He_k)+A_{\nabla^{\perp}_{e_k}H}(e_k)+A_H(\nabla_{e_k}e_k)\right\}+\frac{c}{4}\sum_{i=1}^{2m}\{3(te_i+ne_i)h(H,ne_i)\}\right]=0$$

$$(2m)\left[\sum_{k=1}^{2m}\left\{-B(e_k,A_He_k)+\nabla^{\perp}_{e_k}\nabla^{\perp}_{e_k}H-\nabla^{\perp}_{\nabla_{e_k}e_k}H\right\}+\frac{c}{4}\{2mH\}\right]=0$$

Proof. Put $\theta=\frac{\Pi}{2}$ in Corollary 3.3, we obtain the required outcome. □

COROLLARY 4. *Let M^{2m+1} be bi-harmonic slant sub-manifold of odd dimensional structure manifold $N(c)$ of dimension $(2n+1)$ with contact structure f. Then slant immersion is bi-harmonic if*

$$(2m+1)\left[\sum_{k=1}^{2m+1}\left\{-\nabla_{e_k}(A_He_k)+A_{\nabla^{\perp}_{e_k}H}(e_k)+A_H(\nabla_{e_k}e_k)\right\}+\frac{c-1}{4}\sum_{i=1}^{2m+1}\{3(te_i+ne_i)h(H,ne_i)\}\right]=0$$

$$(2m+1)\left[\sum_{k=1}^{2m+1}\left\{-B(e_k,A_He_k)+\nabla^{\perp}_{e_k}\nabla^{\perp}_{e_k}H-\nabla^{\perp}_{\nabla_{e_k}e_k}H\right\}+H+\frac{c+3}{4}\{2mH\}-\frac{c-1}{4}\{\cos\theta fH\}\right]=0$$

Proof. Put $s=1$ in Theorem 3.1, we obtain the required outcome. □

Remark 3. Let M^{2m+1} be bi-harmonic slant sub-manifold of odd dimensional structure manifold $N(c)$ of dimension $(2n+1)$ with contact structure f. Then invariant immersions are slant immersion if

$$(2m+1)\left[\sum_{k=1}^{2m+1}\left\{-\nabla_{e_k}(A_He_k)+A_{\nabla^{\perp}_{e_k}H}(e_k)+A_H(\nabla_{e_k}e_k)\right\}+\frac{c-1}{4}\sum_{i=1}^{2m+1}\{3(te_i+ne_i)h(H,ne_i)\}\right]=0$$

$$(2m+1)\left[\sum_{k=1}^{2m+1}\left\{-B(e_k,A_He_k)+\nabla^{\perp}_{e_k}\nabla^{\perp}_{e_k}H-\nabla^{\perp}_{\nabla_{e_k}e_k}H\right\}+H+\frac{c+3}{4}\{2mH\}-\frac{c-1}{4}fH\right]=0$$

Proof. Put $\theta=0$ in Corollary 3.4, we obtain the required outcome. □

Remark 4. Let M^{2m+1} be bi-harmonic slant sub-manifold of odd dimensional structure manifold $N(c)$ of dimension $(2n+1)$ with contact structure f. Then anti-invariant immersions are slant immersion if

$$(2m+1)\left[\sum_{k=1}^{2m+1}\left\{-\nabla_{e_k}(A_He_k)+A_{\nabla^{\perp}_{e_k}H}(e_k)+A_H(\nabla_{e_k}e_k)\right\}+\frac{c-1}{4}\sum_{i=1}^{2m+1}\{3(te_i+ne_i)h(H,ne_i)\}\right]=0$$

$$(2m+1)\left[\sum_{k=1}^{2m+1}\left\{-B(e_k,A_He_k)+\nabla^{\perp}_{e_k}\nabla^{\perp}_{e_k}H-\nabla^{\perp}_{\nabla_{e_k}e_k}H\right\}+H+\frac{c+3}{4}\{2mH\}\right]=0$$

Proof. Put $\theta=\frac{\Pi}{2}$ in Corollary 3.4, we obtain the required outcome. □

In next theorem, we derive the second fundamental form expression for slant immersion but it changes only for anti-invariant case and as we get the result for $\theta=\frac{\pi}{2}$.

THEOREM 2. *Let $(N^{2n+s}(c),f,h)$ be a s-Space form. Then M^{2m+s} is biharmonic slant sub-manifold of $N^{2n+s}(c)$ with non-zero constant mean curvature H. Then M is proper biharmonic if and only if*

$$\|B\|^2=\left\{s+\frac{c+3s}{4}(2m)\right\}\quad \textit{and}\quad \cos\theta=0\Rightarrow\theta=\frac{\pi}{2}$$

or equivalently, if the scalar curvature of M satisfies,

$$Scal_M=4ms+\frac{c+3s}{4}\{4m^2-2m\}+\frac{c-s}{4}\{\cos^2\theta\}-s+9H^2$$

Proof. As M is slant sub-manifold, by Theorem 1, M is biharmonic if and only if

$$(2m+s)\left[\sum_{k=1}^{2m+s}\left\{-\nabla_{e_k}(A_He_k)+A_{\nabla^{\perp}_{e_k}H}(e_k)+A_H(\nabla_{e_k}e_k)\right\}+\frac{c-s}{4}\sum_{i=1}^{2m+s}\{3(te_i+ne_i)h(H,ne_i)\}\right]=0$$

$$(2m+s)\left[\sum_{k=1}^{2m+s}\left\{-B(e_k,A_He_k)+\nabla^{\perp}_{e_k}\nabla^{\perp}_{e_k}H-\nabla^{\perp}_{\nabla_{e_k}e_k}H\right\}+Hs+\frac{c+3s}{4}\{2mH\}-\frac{c-s}{4}\cos\theta fH\right]=0$$

The first equation is satisfied when $h(H,ne_i)=0$ and also M has constant mean curvature. Then the second equation becomes,

$$\mathrm{tr}(B(\cdot,A_H\cdot))=\left\{s+\frac{c+3s}{4}\{2m\}\right\}H-\frac{c-s}{4}\{\cos\theta fH\}$$

Moreover, for sub-manifold, $A_H=HA$, which suggests,

$$\mathrm{tr}(B(\cdot,A_H\cdot))=H\,\mathrm{tr}(B(\cdot,A\cdot))=H\|B\|^2$$

Since H is non-zero constant, we get the desired identity

$$H\|B\|^2 = \left\{s + \frac{c+3s}{4}\{2m\}\right\}H - \frac{c-s}{4}\{\cos\theta f H\}$$

By comparing

$$\|B\|^2 = \left\{s + \frac{c+3s}{4}\{2m\}\right\} \quad \text{and} \quad \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$$

The Gauss equation gives us the second equivalency:

$$\mathrm{Scal}_M = \sum_{i,j=1}^{2m+s} h(R(Z_i,Z_j)Z_j,Z_i) - \|B\|^2 + 9H^2$$

Now based on the curvature tensor expression

$$\begin{aligned}
\sum_{i,j=1}^{2m+s} h(R^N(Z_i,Z_j)Z_j,Z_i) = & \sum_{i,j=1}^{2m+s}\{h(fZ_i,fZ_i)\eta_\alpha(Z_j)\eta_\beta(Z_j) - h(fZ_i,fZ_j)\eta_\alpha(Z_j)\eta_\beta(Z_i) \\
& + h(fZ_j,fZ_j)\eta_\alpha(Z_i)\eta_\beta(Z_i) - h(fZ_j,fZ_i)\eta_\alpha(Z_i)\eta_\beta(Z_j)\} \\
& + \frac{c+3s}{4}\sum_{i,j=1}^{2m+s}\{(h(fZ_i,fZ_i)h(fZ_j,fZ_j) - h(fZ_i,fZ_j)h(fZ_j,fZ_i)\} \\
& + \frac{c-s}{4}\sum_{i,j=1}^{2m+s}\{h(Z_i,fZ_i)h(Z_j,fZ_j) - h(Z_i,fZ_j)h(Z_j,fZ_i) - 2h(Z_i,fZ_j)h(Z_j,fZ_i)\}
\end{aligned}$$

$$\sum_{i,j=1}^{2m+s} h(R^N(Z_i,Z_j)Z_j,Z_i) = 4ms + \frac{c+3s}{4}\{4m^2\} + \frac{c-s}{4}\{\cos^2\theta\}$$

$$\mathrm{Scal}_M = 4ms + \frac{c+3s}{4}\{4m^2\} + \frac{c-s}{4}\{\cos^2\theta\} - s - \frac{c+3s}{4}\{2m\} + 9H^2$$

Taking component along fH, put $\theta = \frac{\pi}{2}$

$$\mathrm{Scal}_M = 4ms + \frac{c+3s}{4}\{4m^2 - 2m\} - s + 9H^2$$

Consequently, we conclude that M is appropriate bi-harmonic iff,

$$\|B\|^2 = \left\{s + \frac{c+3s}{4}(2m)\right\} \quad \text{and} \quad \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$$

that is iff,

$$\mathrm{Scal}_M = 4ms + \frac{c+3s}{4}\{4m^2 - 2m\} - s + 9H^2$$

□

COROLLARY 5. *Let M^{2m} be bi-harmonic slant sub-manifold of even dimensional structure manifold $N(c)$ of dimension $(2n)$ with complex structure f and having mean curvature H which is non-zero constant. Then M is proper bi-harmonic iff*

$$\|B\|^2 = \left\{\frac{c}{4}(2m)\right\} \quad \text{and} \quad \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$$

or equivalently, if the scalar curvature of M fulfills,

$$\mathrm{Scal}_M = \frac{c}{4}\{4m^2 - 2m\} + 9H^2$$

Proof. Put $s = 0$ in Theorem 3.2. we obtain the required outcome. □

COROLLARY 6. *Let M^{2m+1} be bi-harmonic slant sub-manifold of odd dimensional structure manifold $N(c)$ of dimension $(2n+1)$ with contact structure f and having mean curvature H which is non-zero constant. Consequently, M is proper bi-harmonic iff*

$$\|B\|^2 = \left\{1 + \frac{c+3}{4}\{2m\}\right\}, \qquad \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$$

or equivalently, if the scalar curvature of M fulfills

$$\mathrm{Scal}_M = 4m + \frac{c+3}{4}\{4m^2 - 2m\} - 1 + 9H^2$$

Proof. Put $s = 1$ in Theorem 3.2, we obtain the required outcome. □

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