



NUMERICAL SOLUTION TO THE NEUMANN PROBLEM IN A LIPSCHITZ DOMAIN, BASED ON RANDOM WALKS

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Abstract. We deal with probabilistic numerical solutions for linear elliptic equations with Neumann boundary conditions in a Lipschitz domain, by using a probabilistic numerical scheme introduced by Milstein and Tretyakov based on new numerical layer methods.

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1. INTRODUCTION

Getting the solution of boundary value problems for partial differential equations (PDEs) is a general topic in applied mathematics. Analytical methods (e.g., separation of variables and transform techniques) are valued for their exactness and the insight they provide; however, the range of problems they solve is limited. Numerical schemes (e.g., finite element, finite difference, and spectral methods) solve a much wider range of problems. Some methods (e.g., boundary integral methods) combine specific analytical information about the solution with numerical approximations.

In this paper we solve numerically a class of linear elliptic equations with Neumann boundary conditions in a Lipschitz domain, by using a probabilistic numerical scheme introduced by Milstein and Tretyakov [12, 13] to solve the linear elliptic PDEs, together with some results obtained in [3]. It is to be noted that as in [13] these methods are probabilistic ones. Probabilistic representations of the solutions with new numerical layer methods for semilinear parabolic equations with Neumann conditions are constructed in [12] and the convergence of the algorithm is proved. This technique was used in [10] in order to give a probabilistic numerical approach for the nonlinear Dirichlet problem associated with a branching process.

Monte Carlo methods to compute the solution of elliptic equations with Neumann boundary conditions are introduced in [11]. The algorithm is based on the Euler scheme coupled with a local time approximation method and numerical examples are given on the Laplace operator on a square domain. Moreover, the authors have introduced a walk-on spheres approximation in the inhomogeneous Neumann boundary conditions. The article [15] points out some advances in probabilistic approximation methods to Neumann problem, including some numerical examples.

A gradient Newton Galerkin numerical method to obtain a positive solution to an elliptic equation with Neumann boundary condition is introduced in [2] while discrete boundary element methods with error estimates

for Neumann problems for the heat equation are presented in [16]. The authors in [1] develop a numerical method to solve Neumann problem with discontinuous coefficients, which is illustrated through numerical examples with given exact solutions. In [9] it is developed a wavelet method with explicit C^1 wavelet bases on Powell-Sabin triangulations to approximate the solution of the Neumann boundary problem for partial differential equations. In [8] a modified implicit prediction domain decomposition algorithm is used to solve parabolic partial differential equations with Neumann boundary conditions.

The structure of the paper is the following. Section 2 presents several preliminary results from [3], on the Neumann boundary value problem in a bounded Lipschitz domain. In particular, Theorem 1 gives the probabilistic representation of the solution to the Neumann problem by Brosamler's formula. This provides the existence of a unique generalized solution to the Neumann problem. Next, we construct a Markov chain approximation for the solution using a boundary layer method inspired by [13] and we write down the corresponding algorithm. We give a convergence result in the dimensional space $d \geq 5$ for a function $f \in C^4(\bar{O})$, see Theorem 2. Finally, we presented some final remarks on our results.

2. THE NEUMANN PROBLEM IN A LIPSCHITZ DOMAIN

Setting of the problem and existence result. Consider the following Neumann boundary value problem on a bounded Lipschitz domain O from $\mathbb{R}^d$, $d \geq 1$,

$$\begin{cases} \Delta w = 0 & \text{in } O, \\ \frac{\partial w}{\partial \nu} = f & \text{on } \partial O, \end{cases} \quad (1)$$

where $\frac{\partial}{\partial \nu}$ is the outward normal derivative to the boundary ∂O of O , f is a bounded measurable functions on ∂O , and we assume that $\int_{\partial O} f d\sigma = 0$; we denoted by σ the surface measure on the boundary ∂O . Recall that this is a necessary condition for the existence of a solution to the problem (1), according to the Green's first identity.

A generalized solution to the Neumann boundary value problem (1) is a function $w \in C(\bar{O})$ such that

$$\int_O w(x) \Delta \varphi(x) dx + \int_{\partial O} f(x) \varphi(x) \sigma(dx) = \int_{\partial O} w(x) \frac{\partial \varphi}{\partial \nu}(x) \sigma(dx) \quad \text{for any } \varphi \in C^2(\bar{O})$$

(cf. e.g. Definition 5.1 from [3]).

We recall now the existence result of the generalized solution to the Neumann boundary value problem, its probabilistic representation using the reflecting Brownian motion X on O , and some estimates for the transition density $p(t, x, y)$ of the reflecting Brownian motion; cf. [3], Theorem 5.3 and Theorem 3.1.

THEOREM 1. *Let O be a bounded Lipschitz domain. Then the following assertions hold.*

(i) *If $f \in \mathcal{B}(\partial O)$ with $\int_{\partial O} f d\sigma = 0$ then there exists a unique generalized solution to the Neumann boundary problem (1) satisfying the condition $\int_O w(x) dx = 0$, where $\mathcal{B}(\partial O)$ is a Borel σ -algebra on ∂O . In addition we have*

$$w(x) = \lim_{t \rightarrow \infty} \frac{1}{2} \mathbb{E}^x \int_0^t f(X_s) dL_s \quad \text{for each } x \in O, \quad (2)$$

where L_t is the boundary local time for the reflecting Brownian motion X .

(ii) *There exist two positive constants $C_1, C_2 > 0$ such that for all $t > 0$ and $x, y \in \bar{O}$ we have*

$$p(t, x, y) \leq C_1 \cdot t^{-\frac{d}{2}} \cdot e^{-\frac{|x-y|^2}{C_2 t}}.$$

Relation (2) is the *Brosamler formula*, the probabilistic representation of the solution of the Neumann problem, in terms of the reflecting Brownian motion; see [7] for an approach to the Brosamler formula on balls, using the connection between the Dirichlet and the Neumann boundary problems from [5], and the explicit description of the reflecting Brownian motion and its boundary local time in terms of the free Brownian motion. See also [6] for the case of discontinuous boundary data.

Numerical approximation of the solution in a Lipschitz domain. Now let

$$Y_{t,y,h} \approx Y = y + h^{\frac{1}{2}} \xi, \quad (3)$$

where $h > 0$ is a step of integration and $\xi = (\xi^1, \dots, \xi^n)^T$, ξ^i , $i = 1, \dots, n$ are mutually independent random variables taking values $\pm \frac{1}{2}$ with probability $\frac{1}{2}$.

We define a boundary zone $S_h \subset \bar{O}$: $y \in S_h$ if at least one of the 2^n values of the vector Y is outside $\bar{O}$.

Let $\lambda > 0$ be a constant such that if the distance from $y \in O$ to the boundary ∂O is equal to or greater than $\lambda \sqrt{h}$ then y is outside the boundary zone and therefore, for such y all the realizations of the random variable Y belong to $\bar{O}$.

Let $y \in S_h$ and we construct the random vector $Y_{y,h}^\pi$ taking two values y^π and $y^\pi + \lambda \eta(y^\pi) h^{\frac{1}{2}}$ with probabilities $p = p_{Y,h}$ and $q = q_{Y,h} = 1 - p_{Y,h}$ respectively, where

$$p_{Y,h} := \frac{h^{\frac{1}{2}} \lambda}{|y + h^{\frac{1}{2}} \lambda \eta(y^\pi) - y^\pi|},$$

$y^\pi \in \partial O$ is the projection of the point y on the boundary ∂O and $\eta(y^\pi)$ is the unit vector of the internal normal to ∂O at the point y^π .

Let $h := \frac{1}{N}$, $N > 0$, integer.

Further, we approximate the solution to the Neumann boundary value problem (1), by constructing a Markov chain Y_k which stops when it reaches the boundary ∂O at a random step $\varkappa$.

We set $Y'_0 = x$. If $Y'_0 \notin S_h$ we take

$$Y_0 = Y'_0.$$

If $Y'_0 \in S_h$ then the random variable Y_0 takes two values: either $Y_0^\pi \in \partial O$ with probability $p_{Y_0^\pi}$ or $Y_0^\pi + h^{\frac{1}{2}} \lambda \eta(Y_0^\pi) \notin S_h$ with probability $q_{Y_0^\pi}$. If $Y_0 = Y_0^\pi$ we put $\varkappa = 0$, $Y_\varkappa = Y_0^\pi$ and the random walk is finished.

Let Y_k , $k < N$ from above and either $Y_k \in \partial O$ or $Y_k \notin S_h$. We assume the chain does not stopped until step k i.e. $\varkappa > k$.

We introduce Y'_{k+1} due to (3) with $t = t_k$, $y = Y_k$, $\xi = \xi_k$, as:

$$Y'_{k+1} = Y_k + h^{\frac{1}{2}} \xi_k. \quad (4)$$

Now we obtain Y_{k+1} using Y'^{k+1} as we got Y_0 using Y'_0 . More precisely, we use the following rule.

If $Y'_{k+1} \notin S_h$ then we take

$$Y_{k+1} = Y'_{k+1}.$$

If $Y'_{k+1} \in S_h$ then the random variable Y_{k+1} takes two values: either $Y_{k+1}^\pi \in \partial O$ with probability $p_{Y_{k+1}^\pi, h}$ or $Y_{k+1}^\pi + h^{\frac{1}{2}} \lambda \eta(Y_{k+1}^\pi) \notin S_h$ with probability $q_{Y_{k+1}^\pi, h}$.

If $Y_{k+1} = Y_{k+1}^\pi$ we put $\varkappa = k + 1$, $Y_\varkappa = Y_{k+1}^\pi$ and the random walk is finished.

So, the random walk Y_k has been constructed and clearly, Y_k remains in the domain $\bar{O}$ with probability 1.

Finally, we introduce an extended Markov chain defined as $Y_k = Y_\varkappa$ for $k > \varkappa$.

The constructed algorithm from above can be written as follows.

Algorithm 1**STEP 0:** $Y'_0 = x_0$ **STEP 1:** If $Y'_k \notin S_h$ then $Y_k = Y'_k$ and go to **STEP 3**.If $Y'_k \in S_h$ then either $Y_k = Y'_k{}^\pi$ with probability $p_{Y'_k, h}$
or $Y_k = Y'_k + h^{1/2} \lambda \eta(Y'_k{}^\pi)$ with probability $q_{Y'_k, h}$.**STEP 2:** If $Y_k = Y'_k{}^\pi$ then **STOP** and $\varkappa = k$, $X_\varkappa = Y'_k{}^\pi$;**STEP 3:** Sampling ξ_k and compute Y'_{k+1} by using (4), for $y = Y_k$, $\xi = \xi_k$.**STEP 4:** Set $k := k+1$ and go to **STEP 1**.**Convergence results.** The following results ensures the convergence of the above algorithm.We assume that $f \in C^4(\overline{O})$.LEMMA 1. *We have*

$$\left| \frac{1}{2} \lim_{s \rightarrow \infty} \left[\mathbb{E}^x \int_0^s f(Y_\varkappa) dL_t - \mathbb{E}^x \int_0^s f(X_t) dL_t \right] \right| \leq Ch, \quad (5)$$

where the constant C does not depend on x and h .*Proof.* The result follows reasoning as in the proof of Theorem 7.3.4 from [13]. □

The convergence result is given by the following theorem.

THEOREM 2. *Let*

$$Z_\varkappa(x) := \frac{1}{2} \lim_{s \rightarrow \infty} \mathbb{E}^x \int_0^s f(Y_\varkappa) dL_t. \quad (6)$$

Assume that $d \geq 5$. Then there exists a constant $C > 0$ which does not depend on x and h , such that

$$|Z_\varkappa(x) - w(x)| \leq Ch.$$

Proof. By assertion (i) of Theorem 1 we have

$$|Z_\varkappa(x) - w(x)| \leq \frac{1}{2} \left| \lim_{s \rightarrow \infty} \mathbb{E}^x \int_0^s [f(Y_\varkappa) - f(x)] dL_t \right| + \frac{1}{2} \left| \lim_{s \rightarrow \infty} \mathbb{E}^x \int_0^s [f(X_t) - f(x)] dL_t \right|.$$

Consequently, from Lemma 1 we obtain now

$$|Z_\varkappa(x) - w(x)| \leq Ch + \frac{1}{2} \left| \lim_{s \rightarrow \infty} \int_0^s dt \int_{\partial O} p(t, x, y) \cdot [f(y) - f(x)] \sigma(dy) \right|. \quad (7)$$

Further, using Lagrange's inequality we get

$$|Z_\varkappa(x) - w(x)| \leq Ch + \frac{1}{2} \sup_{y \in \overline{O}} |\nabla f(y)| \cdot \lim_{s \rightarrow \infty} \int_0^s \alpha t \int_{\partial O} p(t, x, y) \cdot |y - x| \sigma(dy).$$

By assertion (ii) of Theorem 1, we obtain

$$\int_{\partial O} p(t, x, y) \cdot |y - x| \sigma(dy) \leq C_1 \cdot t^{-\frac{d}{2}} \int_{\partial O} e^{-\frac{|y-x|^2}{C_2 t}} |y - x| d\sigma(y) \leq \alpha \cdot t^{-\frac{d}{2}} \cdot e^{-\frac{\beta}{t}},$$

where α and β are positive constants.

From (7) there exist $\alpha, \beta > 0$ such that

$$|Z_{\mathcal{X}}(x) - w(x)| \leq Ch + K_1 \lim_{s \rightarrow \infty} \int_0^s t^{-\frac{d}{2}} \alpha e^{-\frac{\beta}{t}} dt. \quad (8)$$

Because we supposed that $d \geq 5$, it follows that

$$0 \leq \lim_{s \rightarrow \infty} \int_0^s t^{-\frac{d}{2}} \cdot e^{-\frac{\beta}{t}} dt \leq K_2 h, \quad (9)$$

where K_2 is a positive constant which does not depend on x and h .

From (8) and (9) we deduce the claimed inequality,

$$|Z_{\mathcal{X}}(x) - w(x)| \leq (C + K_1 + K_2)h.$$

□

3. FINAL REMARKS

We complete the paper with several concluding remarks. Recall that our aim was to present a numerical treatment of some classes of Neumann problems for linear elliptic PDEs in a Lipschitz domain. The problem was considered in the paper as follows.

Revision of existence in the generalized sense for the solution to the Neumann problem for linear elliptic PDEs together with a result from Bass and Hsu (see [3]). Also, the construction of new approximations of solutions to the Neumann problem for linear elliptic PDEs was obtained by combining the result from Bass and Hsu from [3] with the random walk technique from [13].

In a forthcoming work, we intend to investigate numerical methods for the linear elliptic Neumann problem for PDEs in a unit ball, by exploiting the mentioned equivalence result together with the ideas of the simplest random walks from [13]. We shall use an equivalence result between the solution to the linear elliptic Neumann problem for PDEs and the solution to the linear elliptic Dirichlet problem; cf. [5]. It is a challenge to give approximations for the solutions for some classes of parabolic SPDEs with nonlinear Neumann boundary conditions, considered for example in [4].

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REFERENCES

- [1] Aghaeiboorkheili M, Aezeden M. Applications of mechanical engineering mathematics: Solving Neumann problem with discontinuous coefficients. *Journal of Interdisciplinary Mathematics* 2021;24:1429–1439.
- [2] Afrouzi GA, Mahdavi S, Naghizadch Z. A numerical method for finding positive solution of elliptic equation with Neumann boundary condition. *Applied Mathematics and Computation* 2021;189:23–26.
- [3] Bass RF, Hsu P. Some potential theory for reflecting Brownian motion in Hölder and Lipschitz domains. *The Annals of Probability* 1992;19:486–508.
- [4] Barbu V, Beznea L. Measure-valued branching processes associated with Neumann nonlinear semiflows. *Journal of Mathematical Analysis and Applications* 2016;167–182.

- [5] Beznea L, Pascu MN, Pascu NR. An equivalence between the Dirichlet and the Neumann problem for the Laplace operator. *Potential Analysis* 2015;44:655–672.
- [6] Beznea L, Pascu MN, Pascu NR. Connections between the Dirichlet and the Neumann problem for continuous and integrable boundary data. In: *Stochastic analysis and related topics. Progress in probability*, vol. 72. Cham: Birkhäuser; 2017, pp. 85–97.
- [7] Beznea L, Pascu MN, Pascu NR. Brosamler’s formula revisited and extensions. *Analysis and Mathematical Physics* 2019;9:747–760.
- [8] Jun Y, Mai TZ. Domain decomposition method for parabolic problems with Neumann conditions. *Applied mathematics and computation* 2006;182:1683–1695.
- [9] Li F, Sh JY, Gong WF. The C^1 wavelet method for numerical solutions of the Neumann problem. *Applicable analysis* 2007;86:1443–1454.
- [10] Lupașcu O, Stănculescu V. Numerical solution for the non-linear Dirichlet problem of a branching process. *Complex Anal. Oper. Theory* 2017;11:1895–1904.
- [11] Maire S, Tanre E. Monte Carlo approximations of the Neumann problem. *Monte Carlo Methods Appl.* 2013;19:201–236.
- [12] Milstein GN, Tretyakov MV. A probabilistic approach to the solution of the Neumann problem for nonlinear parabolic equations. *IMA Journal of numerical analysis* 2002;22:599–622.
- [13] Milstein GN, Tretyakov MV. *Stochastic numerics for mathematical physics*. Springer; 2021.
- [14] Miranda C. *Partial differential equations of elliptic type*. Springer; 1970.
- [15] Raghavan R, Brady ML. Efficient numerical solutions of Neumann problems in inhomogeneous media from their probabilistic representations and applications. *Engineering Reports* 2020;2:e12108. DOI: 10.1002/eng2.12108.
- [16] Yang HT. On the convergence of boundary element methods for initial-Neumann problems for the heat equation. *Mathematics of computation* 1999;68:547–557.

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