THE PUBLISHING HOUSE
OF THE ROMANIAN ACADEMY

PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A
Volume 25, Number 3/2024, pp. 181–185
DOI: 10.59277/PRA-SER.A.25.3.03

LYNCH'S CONJECTURE AND ANNIHILATORS OF LOCAL COHOMOLOGY MODULES

Mahnaz AMANALAHZADEH, Jafar A'ZAMI, Kamal BAHMANPOUR

University of Mohaghegh Ardabili, Faculty of Sciences, Department of Mathematics, Ardabil, Iran.
Corresponding author: Jafar A'ZAMI, E-mail: jafar.azami@gmail.com
Mahnaz AMANALAHZADEH, E-mail: m.amanalahzade@gmail.com
Kamal BAHMANPOUR, E-mail: bahmanpour.k@gmail.com

Abstract. Let R be a commutative Noetherian ring and I be an ideal of R with $\mathrm{cd}(I,R)=t\geq 1$. In this paper, we consider the Lynch's conjecture and we obtain a partial answer for this conjecture. More precisely, we show that if M is an R-module such that $0\neq H_I^t(M)$ is I-cofinite, then $\mathrm{Ann}_R H_I^t(R)\subseteq \mathfrak{p}$ for some minimal prime ideal $\mathfrak{p}$ of R.

***Keywords**:* Lynch's conjecture, cofinite module, local cohomology module.
Mathematics Subject Classification (MSC2020): Primary: 13D45,14B15; Secondary: 13E05.

1. INTRODUCTION

Let R denote a commutative Noetherian ring (with identity) and I an ideal of R. For an R-module M, the ith local cohomology module of M with support in $V(I)$ is defined as:

$$H_I^i(M)=\varinjlim_{n\geq 1}\mathrm{Ext}_R^i(R/I^n,M).$$

Local cohomology was first defined and studied by Grothendieck. The cohomological dimension of M with respect to I, denoted by $\mathrm{cd}(I,M)$, is the integer t such that $H_I^t(M)\neq 0$ while $H_I^i(M)=0$ for $i>t$. We refer the reader to [6] or [9] for the basic properties of local cohomology modules. One of the interesting question in the study of local cohomology modules is calculating the annihilator of this modules. This problem has been studied by several authors, see, for example, [2, 3, 10–14, 18], and has led to some interesting results. One of the interesting result shows that if R is a regular local ring containing a field, then $\mathrm{Ann}_R H_I^i(R)=0$ if and only if $H_I^i(R)\neq 0$, cf. [12] and [14]. The following conjecture of Laura Lynch [13] has attracted considerable interest in the literature:

Conjecture. Let R be a Noetherian ring and let I be a proper ideal of R. Let $\mathrm{cd}(I,R)=t$. Then $\dim R/\mathrm{Ann}_R H_I^t(R)=\dim R/H_I^0(R)$.

In general this conjecture is false. The first known counterexample was given by Bahmanpour [5]. When R is a Noetherian equidimensional ring, this conjecture is true if and only if the ideal $\mathrm{Ann}_R H_I^t(R)$ of R has height zero, cf. [2]. As the first main result of the second section of this paper, we provide a proof of this conjecture for a complete Noetherian Local ring R, in the case that, there exists an R-module M such that $H_I^t(M)$ is I-cofinite, where $\mathrm{cd}(I,R)=t$. An R-module M is said to be I-cofinite, if $\mathrm{Supp}\,(M)\subseteq V(I)$ and $\mathrm{Ext}_R^i(R/I,M)$ is finitely generated for each $i\geq 0$. Finally we prove this result for arbitrary Noetherian ring. More precisely, we prove the following:

THEOREM 1.1. *Let R be a Noetherian ring, I an ideal of R with $cd(I,R)=t\geq 1$. Suppose that there exists an R-module M such that $0\neq H_I^t(M)$ is I-cofinite. Then there exists a minimal prime ideal of R such that* $\mathrm{Ann}_R H_I^t(R)\subseteq \mathfrak{p}$.

We note that if R is a domain, it is easy to see that the Lynch's conjecture is equivalent to $H_I^t(R)$ being faithful. As an immediate consequence of Theorem 1.1, we derive the following corollary which shows that the Lynch's conjecture is true in this case.

COROLLARY 1.2. *Let R be a Noetherian integral domain, I an ideal of R with $cd(I,R)=t\geq 1$. Let M be an R-module such that $0\neq H_I^t(M)$ is I-cofinite. Then* $\mathrm{Ann}_R H_I^t(R)=0$.

Throughout this paper, R will always be a commutative Noetherian ring with non-zero identity and I will be an ideal of R. We shall use $\mathrm{Max}\,(R)$ to denote the set of all maximal ideals of R. Also, for any ideal I of R, we denote $\{\mathfrak{p}\in\mathrm{Spec}\,(R):\mathfrak{p}\supseteq I\}$ by $V(I)$. Finally, for any ideal I of R, the radical of I, denoted by $Rad(I)$ or $\sqrt{I}$, is defined to be the set $\{x\in R: x^n\in I,\, for\ some\ n\in\mathbb{N}\}$. For any unexplained notation and terminology we refer the reader to [6] and [16].

2. SECOND SECTION

The following Lemma plays a key role in the proof of the main results.

LEMMA 2.1. *Let $(R,\mathfrak{m})$ be a Noetherian local ring and let I be a proper ideal of R. Let M be an I-cofinite R-module of dimension $n\geq 1$. Then there exists $x\in\mathfrak{m}$ such that M/xM is I-cofinite and non-zero of dimension $n-1$.*

Proof. Since M is I-cofinite, it follows that $\mathrm{Ass}_R M$ is a finite set. As $\mathfrak{m}\nsubseteq\bigcup_{\mathfrak{p}\in\mathrm{Ass}_R M\setminus\{\mathfrak{m}\}}\mathfrak{p}$, so there exists $x\in\mathfrak{m}$ such that $x\notin\bigcup_{\mathfrak{p}\in\mathrm{Ass}_R M\setminus\{\mathfrak{m}\}}\mathfrak{p}$. Therefore $V(Rx)\cap\mathrm{Ass}_R M\subseteq\{\mathfrak{m}\}$. Hence $\Gamma_{Rx}(M)\subseteq\Gamma_{\mathfrak{m}}(M)$ and so $\Gamma_{Rx}(M)=\Gamma_{\mathfrak{m}}(M)$. The exact sequence

$$0\to 0:_{\Gamma_{\mathfrak{m}}(M)} I\to 0:_M I$$

shows that $0:_{\Gamma_{\mathfrak{m}}(M)} I$ is finitely generated with support in $\{\mathfrak{m}\}$ and so by Melkersson's Theorem [17, Theorem1.6], $\Gamma_{\mathfrak{m}}(M)$ is I-cofinite and Artinian. Consequently $\Gamma_{\mathfrak{m}}(M)/x\Gamma_{\mathfrak{m}}(M)$ is Artinian and I-cofinite. Set $\bar{M}:=M/\Gamma_{\mathfrak{m}}(M)$. From the exact sequence

$$0\to\Gamma_{\mathfrak{m}}(M)\to M\to\bar{M}\to 0$$

we conclude that $\bar{M}$ is I-cofinite and $\dim\bar{M}=n$. Also

$$\mathrm{Ass}_R\bar{M}=\mathrm{Ass}_R M/\Gamma_{Rx}(M)=Ass_R M\setminus V(Rx)$$

so that $x\notin\bigcup_{\mathfrak{p}\in\mathrm{Ass}_R\bar{M}}\mathfrak{p}$. Hence the sequence

$$0\to\bar{M}\xrightarrow{x}\bar{M}\to\bar{M}/x\bar{M}\to 0\ (*)$$

is exact and by [15, Theorem 2.9], $H_{\mathfrak{m}}^n(\bar{M})\neq 0$. The exact sequence $(*)$ iduces the following exact sequence of local cohomology modules

$$\cdots\to H_{\mathfrak{m}}^{n-1}(\bar{M}/x\bar{M})\to H_{\mathfrak{m}}^n(\bar{M})\xrightarrow{x}H_{\mathfrak{m}}^n(\bar{M})$$

which shows that $H_{\mathfrak{m}}^{n-1}(\bar{M}/x\bar{M}) \neq 0$ and by Grothendieck's vanishing Theorem [6, Theorem 6.1.2], $\dim \bar{M}/x\bar{M} \geq n-1$. On the other hand $x \notin Z_R(\bar{M})$ yields that $\dim \bar{M}/x\bar{M} \leq \dim \bar{M} - 1 = n-1$. Therefore $\dim \bar{M}/x\bar{M} = n-1$ and $\bar{M}/x\bar{M} \neq 0$. The exact sequence

$$0 \to \Gamma_{\mathfrak{m}}(M) \to M \to \bar{M} \to 0$$

induces the exact sequence (Snake Lemma) [4, Proposition 2.10], as follows

$$\cdots \to 0 :_{M/\Gamma_{\mathfrak{m}}(M)} x \to \Gamma_{\mathfrak{m}}(M)/x\Gamma_{\mathfrak{m}}(M) \to M/xM \to \bar{M}/x\bar{M} \to 0$$

Since $0 :_{M/\Gamma_{\mathfrak{m}}(M)} x = 0$, it follows that M/xM is I-cofinite of dimension $n-1$ and non-zero, because $\dim \Gamma_{\mathfrak{m}}(M)/x\Gamma_{\mathfrak{m}}(M) \leq 0$. □

COROLLARY 2.2. *Let $(R,\mathfrak{m})$ be a Noetherian local ring and I be an ideal of R. Let M be an I-cofinite R-module of dimension $n \geq 1$. Then there exist elements $x_1, \cdots, x_n \in \mathfrak{m}$ such that $M/(x_1, \cdots, x_n)M$ is I-cofinite and non-zero of dimension zero and consequently is Artinian.*

Proof. The assertion follows from Lemma 2.1. □

LEMMA 2.3. *Let $(R,\mathfrak{m})$ be a Noetherian local ring of dimension d. Set,*

$$T(R) := \bigcup \{J \trianglelefteq R \mid \dim J < \dim R\} = \{0\}.$$

then $\operatorname{Ann}_R H_{\mathfrak{m}}^{\dim R}(R) = 0$.

Proof. [3, Theorem 2.3]. □

LEMMA 2.4. *Let R be a Noetherian ring, I be an ideal of R and M, N be two R-modules. If $cd(I,R) = t \geq 1$ then $H_I^t(M) \otimes_R N \cong H_I^t(M \otimes_R N)$.*

Proof. By [6, Exercise 6.1.8], we have

$$\begin{aligned} H_I^t(M) \otimes_R N &\cong (H_I^t(R) \otimes_R M) \otimes_R N \\ &\cong H_I^t(R) \otimes_R (M \otimes_R N) \cong H_I^t(M \otimes_R N), \end{aligned}$$

as required. □

LEMMA 2.5. *Let $(R,\mathfrak{m})$ be a Noetherian complete local ring, I an ideal of R and let $M \neq 0$ be an Artinian and I-cofinite R-module. Then $\sqrt{I + \operatorname{Ann}_R M} = \mathfrak{m}$.*

Proof. See [1, Lemma 2.1]. □

LEMMA 2.6. *Let $(R,\mathfrak{m})$ be a Noetherian complete local ring, I an ideal of R with $cd(I,R) = t \geq 1$, and let M be an R-module such that $H_I^t(M) \neq 0$ and I-cofinite. Then $\operatorname{Ann}_R H_I^t(R) \subseteq q$ for some minimal prime ideal q of R.*

Proof. Let $n = \dim H_I^t(M)$. Since $H_I^t(M)$ is I-cofinite, it follows from Lemma 2.1, that there exist elements $x_1, \cdots, x_n \in \mathfrak{m}$ such that $H_I^t(M) \otimes_R R/(x_1, \cdots, x_n) \cong H_I^t(M)/(x_1, \cdots, x_n)H_I^t(M)$ is I-cofinite and non-zero Artinian. Therefore by Corollary 2.4, $H_I^t(M/(x_1, \cdots, x_n)M)$ is Artinian and I-cofinite. Let $\mathfrak{p} \in \operatorname{Att}_R H_I^t(M/(x_1, \cdots, x_n)M)$ and set $N := M/(x_1, \cdots, x_n)M$. Then

$$H_I^t(N)/\mathfrak{p}H_I^t(N) \cong H_I^t(N) \otimes_R R/\mathfrak{p} \cong H_I^t(N \otimes_R R/\mathfrak{p}) \cong H_I^t(N/\mathfrak{p}N).$$

Hence $\mathrm{Ann}_R H_I^t(N/\mathfrak{p}N) = \mathrm{Ann}_R H_I^t(N) \otimes R/\mathfrak{p} = \mathfrak{p}$ and $H_I^t(N) \otimes R/\mathfrak{p}$ is Artinian and I-cofinite. Therefore by Lemma 2.5, $\sqrt{I+\mathfrak{p}} = \mathfrak{m}$. Since $H_I^t(R/\mathfrak{p}) \otimes N \cong H_I^t(N/\mathfrak{p}N) \neq 0$, it follows that $H_I^t(R/\mathfrak{p}) \neq 0$ and so we have the following relations for each $i \geq 0$:

$$H_I^i(R/\mathfrak{p}) \cong H^i_{I+\mathfrak{p}/\mathfrak{p}}(R/\mathfrak{p}) \cong H^i_{\sqrt{I+\mathfrak{p}}/\mathfrak{p}}(R/\mathfrak{p}) \cong H^i_{\mathfrak{m}/\mathfrak{p}}(R/\mathfrak{p}) \cong H^i_{\mathfrak{m}}(R/\mathfrak{p})$$

Hence $H_I^t(R/\mathfrak{p}) \cong H_{\mathfrak{m}}^t(R/\mathfrak{p})$. It is clear that $\mathrm{Supp}(R/\mathfrak{p}) \subseteq \mathrm{Supp}(R)$ and consequently by [8, Theorem 2.2], $\mathrm{cd}(I,R/\mathfrak{p}) \leq \mathrm{cd}(I,R) = t$. Since $H_I^t(R/\mathfrak{p}) \neq 0$, it follows that $\mathrm{cd}(I,R/\mathfrak{p}) = t$ and therefore $\mathrm{cd}(I,R/\mathfrak{p}) = \mathrm{cd}(\mathfrak{m},R/\mathfrak{p}) = \dim R/\mathfrak{p}$ by Grothendieck's Vanishing and non-vanishing Theorems [6, Theorems 6.1.2 and 6.1.4]. Therefore by Lemma 2.3, $\mathrm{Ann}_R H_I^t(R/\mathfrak{p}^{(n)}) = \mathrm{Ann}_R H_{\mathfrak{m}}^t(R/\mathfrak{p}^{(n)}) = \mathfrak{p}^{(n)}$ where $\mathfrak{p}^{(n)}$ is the nth symbolic power of $\mathfrak{p}$. The exact sequence

$$0 \to \mathfrak{p}^{(n)} \to R \to R/\mathfrak{p}^{(n)} \to 0$$

induces the following exact sequence

$$\to H_I^t(R) \to H_I^t(R/\mathfrak{p}^{(n)}) \to 0$$

This shows that for each $n \in \mathbb{N}$, $\mathrm{Ann}_R H_I^t(R) \subseteq \mathrm{Ann}_R H_I^t(R/\mathfrak{p}^{(n)}) = \mathfrak{p}^{(n)}$ and so $\mathrm{Ann}_R H_I^t(R) \subseteq \cap_{n=1}^{\infty} \mathfrak{p}^{(n)} := J = \ker(R \to R_{\mathfrak{p}})$. Since J is finitely generated, it follows that there exists $s \in R \setminus \mathfrak{p}$ such that $sJ = 0$. Let q be a minimal prime ideal of R contained in $\mathfrak{p}$. Then $sJ = 0 \subseteq q$. Hence $s \notin q$ and so $J \subseteq q$. □

LEMMA 2.7. *Let $(R,\mathfrak{m})$ be a Noetherian local ring, I an ideal of R with $cd(I,R) = t \geq 1$. Suppose that there exists an R-module M such that $H_I^t(M) \neq 0$ and I-cofinite. Then $\mathrm{Ann}_R H_I^t(R) \subseteq q$ for some minimal prime ideal q of R.*

Proof. Since $cd(I,R) = t \geq 1$ it follows that $cd(I\hat{R},\hat{R}) = t \geq 1$. By assumption $H_I^t(M) \neq 0$ and is I-cofinite. Hence $H_I^t(M) \otimes_R \hat{R} \cong H^t_{I\hat{R}}(M \otimes_R \hat{R})$ also is $I\hat{R}$-cofinite. By Lemma 2.6, $\mathrm{Ann}_{\hat{R}} H^t_{I\hat{R}}(\hat{R}) \subseteq Q$ for some minimal prime ideal Q of $\hat{R}$. According Going-down Theorem [4, Theorem 5.11], $Q \cap R = q$ is a minimal prime ideal of R and so

$$\mathrm{Ann}_R H_I^t(R) \subseteq \mathrm{Ann}_R H_I^t(R) \otimes_R \hat{R} \subseteq \mathrm{Ann}_R H^t_{I\hat{R}}(\hat{R}) = R \cap \mathrm{Ann}_{\hat{R}} H^t_{I\hat{R}}(\hat{R}) \subseteq R \cap Q = q$$

□

THEOREM 2.8. *Let R be a Noetherian ring, I an ideal of R with $cd(I,R) = t \geq 1$. Suppose that there exists an R-module M such that $H_I^t(M) \neq 0$ is I-cofinite. Then there exists a minimal prime ideal of R such that* $\mathrm{Ann}_R H_I^t(R) \subseteq \mathfrak{p}$.

Proof. Since $H_I^t(M) \neq 0$, it follows that there exists a maximal ideal $\mathfrak{m} \in R$ such that $(H_I^t(M))_{\mathfrak{m}} \neq 0$ Hence $(H_I^t(M))_{\mathfrak{m}} \neq 0$. Therefore $(H^t_{IR_{\mathfrak{m}}}(M_{\mathfrak{m}})) \neq 0$ is $IR_{\mathfrak{m}}$-cofinite. Now the assertion follows from Lemma 2.7. □

COROLLARY 2.9. *Let R be a Noetherian integral domain, I an ideal of R with $cd(I,R) = t \geq 1$. Let M be an R-module such that $H_I^t(M) \neq 0$ is I-cofinite. Then* $\mathrm{Ann}_R H_I^t(R) = 0$.

Proof. The assertion follows from Theorem 2.8. □

REFERENCES

[1] Abazari N, Bahmanpour K. A note on the Artinian cofinite modules, Comm. Algebra 2014;42:1270–1275.

[2] Atazadeh A, Naghipour R. Lynch's conjecture and attached primes of the top local cohomology modules. Arch. Math. 2022;118: 225–236.

[3] Atazadeh A, Sedghi M, Naghipour R. On the annihilators and attached primes of top local cohomology modules. Arch. Math. (Basel) 2014;102:225–236.

[4] Atiyah MF, Macdonald IG. Introduction to commutative algebra. University of Oxford: Addison-Wesley Publishing Company; 1969.

[5] Bahmanpour K. A note on Lynch's conjecture. Comm. Algebra 2017;45:2738–2745.

[6] Brodmann MP, Sharp RY. Local cohomology; an algebraic introduction with geometric applications. Cambridge, UK: Cambridge University Press; 1998.

[7] Datta R, Switala N, Zhang W. Annihilators of D-modules in mixed characteristic. arXiv 2019:1907.09948.

[8] Divaani-Aazar K, Naghipour R, Tousi M. Cohomological dimension of certain algebraic varieties. Proc. Amer. Math. Soc. 2002; 130: 3537–3544.

[9] Grothendieck A. Local cohomology. Notes by R. Hartshorne. In: Lecture Notes in Mathematics, vol. 862. New York: Springer; 1966.

[10] Hochster M. Finiteness properties and numerical behavior of local cohomology. Comm. Algebra 2019;47:1–11.

[11] Hochster M, Jeffries J. Faithfulness of top local cohomology modules in domains. Math. Res. Lett. 2020;27:1755–1765.

[12] Huneke C. Koh J. Cofiniteness and vanishing of local cohomology modules. Math. Proc. Cambridge Philos. Soc. 1991;110:421–429.

[13] Lynch LR. Annihilators of top local cohomology. Comm. Algebra 2012;40:542–551.

[14] Lyubeznik G. Finiteness properties of local cohomology modules (an application of D-modules to commutative algebra). Invent. Math. 1993;113:41–55.

[15] Mafi A. Some results on local cohomology modules, Arch. Math. 2006;87:211–216.

[16] Matsumura H. Commutative ring theory, Cambridge, UK: Cambridge University Press; 1986.

[17] Melkersson L. Properties of cofinite modules and application to local cohomology, Math. Proc. Cambridge Philos. Soc. 1999; 125:417–423.

[18] Singh AK. Walther U. On a conjecture of Lynch. Comm. Algebra 2020;48:2681–2682.

Received April 26, 2024