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BINOMIAL SUMMATION FORMULAE CONCERNING HARMONIC NUMBERS

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Abstract. This study establishes various summation formulas involving harmonic and generalized harmonic numbers, as well as binomial coefficients. Additionally, several finite series identities and their limits related to H_{2k}, $H_{2k}^{(2)}$ and O_k are derived as specific instances.

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1. INTRODUCTION AND MOTIVATION

It is well-known that the classical harmonic numbers H_n are defined as

$$H_0 = 0 \quad \text{and} \quad H_n = \sum_{k=1}^{n} \frac{1}{k} = \gamma + \psi(n+1) \quad \text{with} \quad n \in \mathbb{N} = \{1, 2, \cdots\},$$

where $\gamma = 0.5772\cdots$ denotes Euler's constant and $\psi(z) = \Gamma'(z)/\Gamma(z)$ represents the logarithmic derivative of Euler's gamma function explicitly given by

$$\psi(z) = -\gamma + \sum_{k=0}^{\infty} \left\{ \frac{1}{k+1} - \frac{1}{k+z} \right\}, \quad \text{with} \quad z \in \mathbb{C} \backslash \{0, -1, -2, \cdots\}.$$

Among the various generalized harmonic numbers, two notable examples are:

$$H_n^{(m)} = \sum_{k=1}^{n} \frac{1}{k^m} \quad \text{and} \quad O_n^{(m)} = \sum_{k=1}^{n} \frac{1}{(2k-1)^m}, \quad \text{with} \quad m, n \in \mathbb{N},$$

which have been extensively explored in literature, such as [2–4, 8, 10, 11].

There exist numerous elegant identities involving harmonic numbers. For instance, Sun and Zhao [9] proved

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \binom{n}{k} H_k = H_n^{\langle 2 \rangle}$$

via mathematical induction (see also [7]).

To address Problem B-818 posed by Hsu, Kuppus et al. [6] derived the following summation formula:

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}H_{\lambda k}=\lambda^{n-1}(n-1)!\sum_{i=1}^{\lambda}\prod_{j=0}^{n-1}\frac{1}{j\lambda+i}\quad \text{for}\quad \lambda\in\mathbb{N}.$$

In this paper, we further evaluate the following five binomial sums with a parameter $\lambda\in\mathbb{N}$:

$$\begin{aligned}
\mathscr{A}(\lambda,n)&:=\sum_{k=1}^{n}\frac{(-1)^k}{k}\binom{n}{k}H_{\lambda k},\\
\mathscr{B}(\lambda,n)&:=\sum_{k=1}^{n}(-1)^k\binom{n}{k}H_{\lambda k}^{(2)},\\
\mathscr{C}(\lambda,n)&:=\sum_{k=1}^{n}(-1)^k\frac{1}{k}\binom{n}{k}H_{\lambda k}^{(2)},\\
\mathscr{D}(\lambda,n)&:=\sum_{k=1}^{n}(-1)^k\binom{n}{k}kH_{\lambda k},\\
\mathscr{E}(\lambda,n)&:=\sum_{k=1}^{n}(-1)^k\binom{n}{k}kH_{\lambda k}^{(2)}.
\end{aligned}$$

Additionally, several specific cases for $\lambda=1$ and $\lambda=2$ are also presented.

2. FINITE SERIES OF THE FIRST FORM

In this section, we begin to deal with the first sum $\mathscr{A}(\lambda,n):=\sum\limits_{k=1}^{n}\frac{(-1)^{k-1}}{k}\binom{n}{k}H_{\lambda k}$.

THEOREM 1. *For $\lambda\in\mathbb{N}$, it holds that*

$$\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\binom{n}{k}H_{\lambda k}=\sum_{i=1}^{\lambda}\sum_{k=1}^{n}\frac{\lambda^{k-1}}{k^2}k!\prod_{j=0}^{k-1}\frac{1}{\lambda j+i}.\tag{1}$$

Two special cases, for $\lambda=1$ and $\lambda=2$, are highlighted below.

COROLLARY 1.

$$\boxed{\lambda=1}\quad \sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\binom{n}{k}H_k=H_n^{(2)};\tag{2}$$

$$\boxed{\lambda=2}\quad \sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\binom{n}{k}H_{2k}=\frac{1}{2}H_n^{(2)}+\frac{1}{2}\sum_{k=1}^{n}\frac{4^k}{k^2\binom{2k}{k}}.\tag{3}$$

Their limiting forms as $n\to\infty$ bring the infinite series identities

COROLLARY 2.

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\binom{n}{k}H_k=\frac{\pi^2}{6};\tag{4}$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\binom{n}{k}H_{2k}=\frac{\pi^2}{3}.\tag{5}$$

Proof of Theorem 1. According to the recursive relationship of binomial coefficients $\binom{n+1}{k}=\binom{n}{k}+\binom{n}{k-1}$, we have no difficult to obtain that

$$\begin{aligned}\mathscr{A}(\lambda,n+1)&=\sum_{k=1}^{n+1}\frac{(-1)^{k-1}}{k}\binom{n+1}{k}H_{\lambda k}=\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\left[\binom{n}{k}+\binom{n}{k-1}\right]H_{\lambda k}+\frac{(-1)^n}{n+1}H_{\lambda(n+1)}\\&=\mathscr{A}(\lambda,n)+\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\binom{n}{k-1}H_{\lambda k}+\frac{(-1)^n}{n+1}H_{\lambda(n+1)}.\end{aligned}$$

By replacing k with $k+1$ in the last sum, we get $\mathscr{A}(\lambda,n+1)=\mathscr{A}(\lambda,n)+\sum\limits_{k=0}^{n}\frac{(-1)^k}{k+1}\binom{n}{k}H_{\lambda(k+1)}$. Now, let us manipulate the sum on the right. By using the formula $\binom{n+1}{k+1}=\frac{n+1}{k+1}\binom{n}{k}$, and then replacing k with $k-1$, we obtain

$$\sum_{k=0}^{n}\frac{(-1)^k}{k+1}\binom{n}{k}H_{\lambda(k+1)}=\frac{1}{n+1}\sum_{k=1}^{n+1}(-1)^{k-1}\binom{n+1}{k}H_{\lambda k}=\frac{\lambda^n n!}{n+1}\sum_{i=1}^{\lambda}\prod_{j=0}^{n}\frac{1}{\lambda j+i},$$

where the second equation is based on the result in [6]:

$$\mathscr{T}(\lambda,n):=\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}H_{\lambda k}=\lambda^{n-1}(n-1)!\sum_{i=1}^{\lambda}\prod_{j=0}^{n-1}\frac{1}{\lambda j+i}.$$

Hence $\mathscr{A}(\lambda,n+1)=\mathscr{A}(\lambda,n)+\frac{1}{n+1}\lambda^n n!\sum\limits_{i=1}^{\lambda}\prod_{j=0}^{n}\frac{1}{\lambda j+i}$. Then the identity in Theorem 1 can be obtained by means of the telescoping method. □

It should be noted that when calculating the limit (5), we have used $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\frac{4^k}{k^2\binom{2k}{k}}=\frac{\pi^2}{2}$, which can be found in [4] and [1, pp. 262].

3. FINITE SERIES OF THE SECOND FORM

In this section, we will devote to the second sum involving binomial coefficients and generalized harmonic numbers defined by $\mathscr{B}(\lambda,n):=\sum\limits_{k=1}^{n}(-1)^{k-1}\binom{n}{k}H_{\lambda k}^{(2)}$.

THEOREM 2. *For $\lambda\in\mathbb{N}$, the following identity holds true:*

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}H_{\lambda k}^{(2)}=\frac{(n-1)!}{\lambda^2}\sum_{i=1}^{\lambda}\frac{\Gamma(\frac{i}{\lambda})}{\Gamma(\frac{i}{\lambda}+n)}\{\psi(\frac{i}{\lambda}+n)-\psi(\frac{i}{\lambda})\}. \tag{6}$$

For the above identity, we record two nice examples of it below.

COROLLARY 3.

$$\boxed{\lambda=1}\quad \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}H_{k}^{(2)}=\frac{1}{n}H_n; \tag{7}$$

$$\boxed{\lambda=2}\quad \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}H_{2k}^{(2)}=\frac{1}{4n}H_n+\frac{4^n}{2n\binom{2n}{n}}O_n. \tag{8}$$

Proof of Theorem 2. Bases on the binomial recursion relation $\binom{n+1}{k}=\binom{n}{k}+\binom{n}{k-1}$, we can evaluate that

$$\mathscr{B}(\lambda,n+1)=\sum_{k=1}^{n+1}(-1)^{k-1}\binom{n+1}{k}H_{\lambda k}^{(2)}=\sum_{k=1}^{n}(-1)^{k-1}\left[\binom{n}{k}+\binom{n}{k-1}\right]H_{\lambda k}^{(2)}+(-1)^nH_{\lambda(n+1)}^{(2)}$$

$$= \mathscr{B}(\lambda,n) + \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k-1} H_{\lambda k}^{(2)} + (-1)^n H_{\lambda(n+1)}^{(2)}.$$

By making the replacement of $k \to k+1$ for the last sum yields

$$\mathscr{B}(\lambda,n+1) = \mathscr{B}(\lambda,n) + \sum_{k=0}^{n} (-1)^k \binom{n}{k} H_{\lambda(k+1)}^{(2)}.$$

Now let us calculate the sum on the right. In view of the fact $H_{\lambda(k+1)}^{(2)} = H_{\lambda k}^{(2)} + \sum_{i=1}^{\lambda} \frac{1}{(\lambda k+i)^2}$, we can split the sum

$$\begin{aligned}\sum_{k=0}^{n} (-1)^k \binom{n}{k} H_{\lambda(k+1)}^{(2)} &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} H_{\lambda k}^{(2)} + \sum_{k=0}^{n} (-1)^k \binom{n}{k} \sum_{i=1}^{\lambda} \frac{1}{(\lambda k+i)^2} \\ &= -\mathscr{B}(\lambda,n) + \sum_{i=1}^{\lambda} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{(\lambda k+i)^2}.\end{aligned}$$

Integrating both sides of the following identity with respect to x from 0 to u

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} x^{\lambda k+i-1} = x^{i-1} \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^{\lambda k} = x^{i-1}(1-x^{\lambda})^n, \tag{9}$$

we obtain that

$$\sum_{k=0}^{n} (-1)^k \frac{1}{\lambda k+i} \binom{n}{k} u^{\lambda k+i} = \int_0^u x^{i-1}(1-x^{\lambda})^n \mathrm{d}x. \tag{10}$$

Dividing by u both sides of the above equation, and then integrating again from $u=0$ to $u=1$, we find that

$$\begin{aligned}&\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{(\lambda k+i)^2} = \int_0^1 \frac{1}{u} \int_0^u x^{i-1}(1-x^{\lambda})^n \mathrm{d}x\mathrm{d}u \\ &= \ln u \int_0^u x^{i-1}(1-x^{\lambda})^n \Big|_0^1 - \int_0^1 \ln u \,\mathrm{d}\Big(\int_0^u x^{i-1}(1-x^{\lambda})^n \mathrm{d}x\Big) \\ &= -\int_0^1 \ln u (1-u^{\lambda})^n u^{i-1} \mathrm{d}u = -\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} u^{t-1}(1-u^{\lambda})^n \mathrm{d}u \Big|_{t=i} = -\frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 u^{t-1}(1-u^{\lambda})^n \mathrm{d}u \Big|_{t=i}.\end{aligned}$$

Letting $u^{\lambda} = y$, the above formula becomes

$$\begin{aligned}&\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{(\lambda k+i)^2} = -\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{\lambda} \int_0^1 y^{\frac{t}{\lambda}-1}(1-y)^n \mathrm{d}y \Big|_{t=i} \\ &= -\frac{1}{\lambda} \frac{\mathrm{d}}{\mathrm{d}t} B\Big(\frac{t}{\lambda}, n+1\Big) \Big|_{t=i} = -\frac{1}{\lambda} \frac{\mathrm{d}}{\mathrm{d}t} \frac{n!\Gamma(\frac{t}{\lambda})}{\Gamma(\frac{t}{\lambda}+n+1)} \Big|_{t=i} = \frac{n!\Gamma(\frac{i}{\lambda})\{\psi(\frac{i}{\lambda}+n+1) - \psi(\frac{i}{\lambda})\}}{\lambda^2 \Gamma(\frac{i}{\lambda}+n+1)}.\end{aligned}$$

Therefore, we have

$$\mathscr{B}(\lambda,n+1) = \sum_{i=1}^{\lambda} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{(\lambda k+i)^2} = \sum_{i=1}^{\lambda} \frac{n!\Gamma(\frac{i}{\lambda})\{\psi(\frac{i}{\lambda}+n+1) - \psi(\frac{i}{\lambda})\}}{\lambda^2 \Gamma(\frac{i}{\lambda}+n+1)},$$

which completes the proof by replacing n with $n-1$. □

Taking into account the formulas concerning the Gamma function $\Gamma(x)$ and the Digamma function $\psi(x)$

$$\psi(n+1) = H_n + \psi(1),\ \Gamma(r+\frac{1}{2}) = \frac{\Gamma(2r)\Gamma(\frac{1}{2})}{2^{2r-1}\Gamma(r)},\ \psi(r+\frac{1}{2}) = 2\psi(2r) - \psi(r) - 2\ln 2,$$

we can get the two particular cases of Corollary 3.

4. FINITE SERIES OF THE THIRD FORM

Furthermore, we compute the third binomial sum defined by $\mathscr{C}(\lambda,n) := \sum_{k=1}^{n}(-1)^{k-1}\frac{1}{k}\binom{n}{k}H_{\lambda k}^{(2)}$.

THEOREM 3. *For $\lambda \in \mathbb{N}$, we have*

$$\sum_{k=1}^{n}(-1)^{k-1}\frac{1}{k}\binom{n}{k}H_{\lambda k}^{(2)} = \sum_{i=1}^{\lambda}\sum_{k=1}^{n}\frac{(k-1)!\Gamma(\frac{i}{\lambda})\{\psi(\frac{i}{\lambda}+k)-\psi(\frac{i}{\lambda})\}}{k\lambda^2\Gamma(\frac{i}{\lambda}+k)}. \tag{11}$$

Two particular cases are given below for this theorem.

COROLLARY 4. *Let $n \in \mathbb{N}$. Then the following identities hold true:*

$$\boxed{\lambda = 1} \quad \sum_{k=1}^{n}(-1)^{k-1}\frac{1}{k}\binom{n}{k}H_{k}^{(2)} = \sum_{k=1}^{n}\frac{H_k}{k^2}; \tag{12}$$

$$\boxed{\lambda = 2} \quad \sum_{k=1}^{n}(-1)^{k-1}\frac{1}{k}\binom{n}{k}H_{2k}^{(2)} = \frac{1}{4}\sum_{k=1}^{n}\frac{H_k}{k^2} + \frac{1}{2}\sum_{k=1}^{n}\frac{4^k}{k^2\binom{2k}{k}}O_k. \tag{13}$$

Letting $n \to \infty$, we obtain the infinite series identity of (12).

COROLLARY 5.

$$\lim_{n\to\infty}\sum_{k=1}^{n}(-1)^{k-1}\frac{1}{k}\binom{n}{k}H_{k}^{(2)} = 2\zeta(3).$$

Proof of Theorem 3. Obviously, we can manipulate the sum

$$\begin{aligned}\mathscr{C}(\lambda,n+1) &= \sum_{k=1}^{n+1}(-1)^{k-1}\frac{1}{k}\binom{n+1}{k}H_{\lambda k}^{(2)} = \sum_{k=1}^{n}(-1)^{k-1}\frac{1}{k}\left[\binom{n}{k}+\binom{n}{k-1}\right]H_{\lambda k}^{(2)} + \frac{(-1)^n}{n+1}H_{\lambda(n+1)}^{(2)} \\ &= \mathscr{C}(\lambda,n) + \frac{(-1)^n}{n+1}H_{\lambda(n+1)}^{(2)} + \sum_{k=1}^{n}(-1)^{k-1}\frac{1}{k}\binom{n}{k-1}H_{\lambda k}^{(2)}.\end{aligned}$$

Replacing the index k of the last sum on the right with $k+1$ yields

$$\mathscr{C}(\lambda,n) = \mathscr{C}(\lambda,n) + \sum_{k=0}^{n}(-1)^k\frac{1}{k+1}\binom{n}{k}H_{\lambda(k+1)}^{(2)}.$$

By employing the binomial relation $\binom{n+1}{k+1} = \frac{n+1}{k+1}\binom{n}{k}$ and simple calculation, we can get

$$\sum_{k=0}^{n}(-1)^k\frac{1}{k+1}\binom{n}{k}H_{\lambda(k+1)}^{(2)} = \frac{1}{n+1}\mathscr{B}(\lambda,n+1).$$

Hence, we have $\mathscr{C}(\lambda,n+1) = \mathscr{C}(\lambda,n) + \frac{1}{n+1}\mathscr{B}(\lambda,n+1)$. In view of the fact that $\mathscr{C}(\lambda,1) = H_{\lambda}^{(2)}$, and by means of telescoping method, we find that $\mathscr{C}(\lambda,n) = \sum_{k=1}^{n}\frac{1}{k}\mathscr{B}(\lambda,k)$, which completes the proof. $\square$

For $\lambda = 1$, we have the special case

$$\mathscr{C}(1,n) = \sum_{k=1}^{n} \frac{1}{k}\mathscr{B}(1,k) = \sum_{k=1}^{n} \frac{1}{k^2} H_k.$$

From [5], we know that $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \frac{1}{k^2} H_k = 2\zeta(3)$, so we get the limits

$$\lim_{n\to\infty} \sum_{k=1}^{n} (-1)^{k-1} \frac{1}{k}\binom{n}{k} H_k^{(2)} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k^2} H_k = 2\zeta(3). \tag{14}$$

For $\lambda = 2$, we obtain another particular case

$$\mathscr{C}(2,n) = \sum_{k=1}^{n} \frac{1}{k}\mathscr{B}(2,k) = \frac{1}{4}\sum_{k=1}^{n} \frac{H_k}{k^2} + \frac{1}{2}\sum_{k=1}^{n} \frac{4^k}{k^2\binom{2k}{k}} O_k.$$

5. FINITE SERIES OF THE FOURTH FORM

In this section, we examine the fourth binomial sum defined by $\mathscr{D}(\lambda,n) := \sum\limits_{k=1}^{n} (-1)^k \binom{n}{k} k H_{\lambda k}$.

THEOREM 4. *For $\lambda \in \mathbb{N}$ and integer $n > 1$, we have*

$$\sum_{k=1}^{n} (-1)^k \binom{n}{k} k H_{\lambda k} = \lambda^{n-2}(n-2)! \sum_{i=1}^{\lambda} \left\{ \prod_{j=0}^{n-2} \frac{1}{\lambda j + i} + \prod_{j=1}^{n-1} \frac{n-1}{\lambda j + i} - \prod_{j=0}^{n-1} \frac{\lambda(n-1)}{\lambda j + i} \right\}. \tag{15}$$

Two special cases corresponding to $\lambda = 1$ and $\lambda = 2$ are highlighted as below.

COROLLARY 6. *For integer $n > 1$, we have*

$$\boxed{\lambda = 1} \quad \sum_{k=1}^{n} (-1)^k \binom{n}{k} k H_k = \frac{1}{n-1}; \tag{16}$$

$$\boxed{\lambda = 2} \quad \sum_{k=1}^{n} (-1)^k \binom{n}{k} k H_{2k} = \frac{1}{2(n-1)} + \frac{4^{n-1}}{(n-1)\binom{2n}{n}}. \tag{17}$$

Proof of Theorem 4. Analogously, we can evaluate that

$$\begin{aligned}\mathscr{D}(\lambda,n+1) &= \sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} k H_{\lambda k} = \sum_{k=1}^{n} (-1)^k \left\{ \binom{n}{k} + \binom{n}{k-1} \right\} k H_{\lambda k} + (-1)^{n+1}(n+1) H_{\lambda n+1} \\ &= \mathscr{D}(\lambda,n) + (-1)^{n+1}(n+1)H_{\lambda n+1} + \sum_{k=1}^{n} (-1)^k \binom{n}{k-1} k H_{\lambda k}.\end{aligned}$$

For the last sum, performing the replacement $k \to k+1$, we obtain

$$\mathscr{D}(\lambda,n+1) = \mathscr{D}(\lambda,n) + \sum_{k=0}^{n} (-1)^{k-1} \binom{n}{k} (k+1) H_{\lambda(k+1)}.$$

Now manipulate the sum on the right. In view of the fact that $H_{\lambda(k+1)} = H_{\lambda k} + \sum_{i=1}^{\lambda} \frac{1}{\lambda k+i}$, we can split the sum

$$\begin{aligned}\sum_{k=0}^{n}(-1)^{k-1}\binom{n}{k}(k+1)H_{\lambda(k+1)} &= \sum_{k=0}^{n}(-1)^{k-1}\binom{n}{k}(k+1)\left\{H_{\lambda k}+\sum_{i=1}^{\lambda}\frac{1}{\lambda k+i}\right\}\\ &= -\mathscr{D}(\lambda,n)+\mathscr{T}(\lambda,n)+\sum_{k=0}^{n}(-1)^{k-1}\binom{n}{k}(k+1)\sum_{i=1}^{\lambda}\frac{1}{\lambda k+i}.\end{aligned}$$

Therefore, we obtain that

$$\begin{aligned}\mathscr{D}(\lambda,n+1) &= \mathscr{T}(\lambda,n)+\sum_{k=0}^{n}(-1)^{k-1}\binom{n}{k}k\sum_{i=1}^{\lambda}\frac{1}{\lambda k+i}+\sum_{k=0}^{n}(-1)^{k-1}\binom{n}{k}\sum_{i=1}^{\lambda}\frac{1}{\lambda k+i}\\ &= \mathscr{T}(\lambda,n)+n\sum_{k=0}^{n}(-1)^{k-1}\binom{n-1}{k-1}\sum_{i=1}^{\lambda}\frac{1}{\lambda k+i}+\sum_{k=0}^{n}(-1)^{k-1}\binom{n}{k}\sum_{i=1}^{\lambda}\frac{1}{\lambda k+i}\\ &= \mathscr{T}(\lambda,n)+n\sum_{i=1}^{\lambda}\sum_{k=0}^{n-1}(-1)^{k}\binom{n-1}{k}\frac{1}{\lambda(k+1)+i}-\sum_{i=1}^{\lambda}\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}\frac{1}{\lambda k+i}. \qquad (18)\end{aligned}$$

By the help of the relation $\frac{1}{\lambda(k+1)+i} = \int_0^1 x^{\lambda(k+1)+i-1}\mathrm{d}x$, we can evaluate that

$$\begin{aligned}\Lambda_\lambda(i,n) := \sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\frac{1}{\lambda(k+1)+i} &= \sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\int_0^1 x^{\lambda(k+1)+i-1}\mathrm{d}x\\ &= \int_0^1 x^{\lambda+i-1}\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}x^{\lambda k}\mathrm{d}x = \int_0^1 x^{\lambda+i-1}(1-x^\lambda)^{n-1}\mathrm{d}x.\end{aligned}$$

Obviously $\Lambda_\lambda(i,1) = \frac{1}{\lambda+i}$, and by means of integration by parts, we find that

$$\Lambda_\lambda(i,n) = \frac{\lambda(n-1)}{\lambda+i}\int_0^1 x^{2\lambda+i-1}(1-x^\lambda)^{n-2}\mathrm{d}x = \frac{\lambda(n-1)}{\lambda+i}\Lambda_\lambda(\lambda+i,n-1).$$

Using this recursion relation, we can get the explicit formula

$$\Lambda_\lambda(i,n) = \lambda^{n-1}(n-1)!\prod_{j=1}^{n}\frac{1}{j\lambda+i}. \qquad (19)$$

Similarly, we have

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{\lambda k+i} = \lambda^n n!\prod_{j=0}^{n}\frac{1}{j\lambda+i}. \qquad (20)$$

Combining the identities (18), (19) and (20), we can complete the proof. □

6. FINITE SERIES OF THE FIFTH FORM

Finally, we calculate the fifth binomial sum defined by $\mathscr{E}(\lambda,n) := \sum_{k=1}^{n}(-1)^k\binom{n}{k}kH_{\lambda k}^{(2)}$.

THEOREM 5. *For $\lambda \in \mathbb{N}$ and integer $n > 1$, we have*

$$\sum_{k=1}^{n}(-1)^k\binom{n}{k}kH_{\lambda k}^{(2)} = \mathscr{B}(\lambda,n-1)-\mathscr{B}(\lambda,n)+\sum_{i=1}^{\lambda}\frac{(n-1)!\Gamma(\frac{i}{\lambda}+1)\{\psi(\frac{i}{\lambda}+n)-\psi(\frac{i}{\lambda}+1)\}}{\lambda^2\Gamma(\frac{i}{\lambda}+n)}.$$

For $\lambda = 1$ and $\lambda = 2$, we have the following two corresponding particular cases.

COROLLARY 7.

$$\boxed{\lambda = 1} \quad \sum_{k=1}^{n}(-1)^k\binom{n}{k}kH_k^{(2)} = \frac{1}{n-1}(H_n-1); \tag{21}$$

$$\boxed{\lambda = 2} \quad \sum_{k=1}^{n}(-1)^k\binom{n}{k}kH_{2k}^{\langle 2\rangle} = \frac{1}{4(n-1)}(H_n-1)+\frac{4^{n-1}}{(n-1)\binom{2n}{n}}(O_n-1). \tag{22}$$

Proof of Theorem 5. Similar to the proof of the theorems above, we have

$$\mathscr{E}(\lambda,n+1) = \sum_{k=1}^{n+1}(-1)^k\binom{n+1}{k}kH_{\lambda k}^{(2)} = \mathscr{E}(\lambda,n)+\sum_{k=0}^{n}(-1)^{k-1}\binom{n}{k}(k+1)H_{\lambda(k+1)}^{(2)}.$$

For the last sum, we can rewrite it as

$$\begin{aligned}\sum_{k=0}^{n}(-1)^{k-1}\binom{n}{k}(k+1)H_{\lambda(k+1)}^{(2)} &= \sum_{k=0}^{n}(-1)^{k-1}\binom{n}{k}(k+1)\left\{H_{\lambda k}^{(2)}+\sum_{i=1}^{\lambda}\frac{1}{(\lambda k+i)^2}\right\}\\ &= -\mathscr{E}(\lambda,n)+\mathscr{B}(\lambda,n)-\mathscr{B}(\lambda,n+1)+\sum_{i=1}^{\lambda}\sum_{k=0}^{n}(-1)^{k-1}\binom{n}{k}k\frac{1}{(\lambda k+i)^2}.\end{aligned}$$

Noting that the sum can be rewritten as

$$\sum_{k=0}^{n}(-1)^{k-1}\binom{n}{k}k\frac{1}{(\lambda k+i)^2} = n\sum_{k=0}^{n}(-1)^{k-1}\binom{n-1}{k-1}\frac{1}{(\lambda k+i)^2} = n\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\frac{1}{(\lambda k+\lambda+i)^2},$$

and make further efforts

$$\begin{aligned}\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\frac{1}{(\lambda k+\lambda+i)^2} &= \int_0^1\frac{1}{u}\int_0^u x^{\lambda+i-1}(1-x^\lambda)^{n-1}dxdu\\ &= \frac{(n-1)!\Gamma(\frac{i}{\lambda}+1)\{\psi(\frac{i}{\lambda}+n+1)-\psi(\frac{i}{\lambda}+1)\}}{\lambda^2\Gamma(\frac{i}{\lambda}+n+1)},\end{aligned}$$

we get the desired identity

$$\mathscr{E}(\lambda,n) = \mathscr{B}(\lambda,n-1)-\mathscr{B}(\lambda,n)+\sum_{i=1}^{\lambda}\frac{(n-1)!\Gamma(\frac{i}{\lambda}+1)\{\psi(\frac{i}{\lambda}+n)-\psi(\frac{i}{\lambda}+1)\}}{\lambda^2\Gamma(\frac{i}{\lambda}+n)}. \qquad \square$$

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