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AN APPROACH TO q-BINOMIAL THEOREM VIA m-MODULAR DIAGRAM

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Abstract. In this note, we establish a bijection between two kinds of subsets of the set of all partitions of n by extending Chapman's conjugation of 2-modular diagrams. Based on this bijection, we provide a combinatorial proof of the q-binomial theorem.

***Keywords**:* bijection, partition, q-binomial theorem, 2-modular diagram.
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1. INTRODUCTION

One of the most important summation formulae in the q-series is the following q-binomial theorem [4, (1.3.2)]:

$$\sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} z^k = \frac{(az;q)_\infty}{(z;q)_\infty}, \quad |z| < 1, |q| < 1, \tag{1}$$

which was proved by Cauchy in 1843, by Heine in 1847, and by other mathematicians (see [4, page 9]). Here and throughout the paper, the q-shifted factorial is defined by $(a;q)_0 = 1, (a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ for a positive integer n and $(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k)$. We refer the reader to [2, Section 2.2] for a simple proof of (1), and to $[1,6]$ for combinatorial proofs of (1).

Based on the work of Chapman [3], Guo [5] provided a new combinatorial proof of (1) through conjugation of 2-modular diagrams.

Motivated by the works due to Chapman [3] and Guo [5], we present a bijective proof of the following equivalent form of the q-binomial theorem (1).

THEOREM 1 (Equivalent form of (1)). *For positive integers m and r with $m \geq 2$ and $1 \leq r \leq m-1$, we have*

$$\sum_{k=0}^{\infty} \frac{(-xq^r;q^m)_k}{(q^m;q^m)_k} q^{mk} y^k = \frac{(-xyq^{m+r};q^m)_\infty}{(yq^m;q^m)_\infty}. \tag{2}$$

Note that letting $x \to -a/q^{r/m}, y \to z/q$ and $q \to \sqrt[m]{q}$ on both sides of (2) reduces to (1).

The main idea of the bijective proof is to extend Chapman's conjugation of 2-modular diagrams to conjugation of m-modular diagrams. The rest of the note is organized as follows. In Section 2, we translate (2) into its combinatorial interpretation. We provide a bijective proof of the combinatorial interpretation of (2) in Section 3.

2. COMBINATORIAL INTERPRETATION OF (2)

A partition λ of a positive integer n is a finite non-increasing sequence of positive integers $\lambda=(\lambda_1,\lambda_2,\cdots,\lambda_k)$ such that $\sum_{i=1}^{k}\lambda_i=n$. For a partition λ of n, we define the following three partition statistics:

$$\begin{aligned} l(\lambda) &= \text{the number of the parts of } \lambda, \\ s_{m,r}(\lambda) &= \text{the number of the parts of } \lambda \text{ which are congruent to } r \text{ modulo } m, \\ t_m(\lambda) &= \text{one } m\text{th of the largest part of } \lambda \text{ which is a multiple of } m. \end{aligned}$$

Let n,m and r be positive integers with $m\geq 2$ and $1\leq r\leq m-1$. Let $\mathscr{A}_{n,m,r}$ be the set of all partitions λ of n such that all parts of λ are congruent to 0 or r modulo m, the parts congruent to r modulo m are distinct and the largest part of λ is a multiple of m. Let $\mathscr{B}_{n,m,r}$ be the set of all partitions λ of n such that all parts of λ are congruent to 0 or r modulo m, the parts congruent to r modulo m are distinct and r is not a part of λ.

Note that (2) can be translated into the following combinatorial interpretation:

$$\sum_{n\geq 1} q^n \sum_{\lambda\in\mathscr{A}_{n,m,r}} x^{s_{m,r}(\lambda)}y^{t_m(\lambda)} = \sum_{n\geq 1} q^n \sum_{\lambda\in\mathscr{B}_{n,m,r}} x^{s_{m,r}(\lambda)}y^{l(\lambda)}. \tag{3}$$

In order to prove (3), it suffices to show that

$$\sum_{\lambda\in\mathscr{A}_{n,m,r}} x^{s_{m,r}(\lambda)}y^{t_m(\lambda)} = \sum_{\lambda\in\mathscr{B}_{n,m,r}} x^{s_{m,r}(\lambda)}y^{l(\lambda)}, \tag{4}$$

where n,m and r are positive integers with $m\geq 2$ and $1\leq r\leq m-1$.

In the next section, we shall establish a bijection $\varphi_{m,r}:\mathscr{A}_{n,m,r}\to\mathscr{B}_{n,m,r}$ such that $s_{m,r}\left(\varphi_{m,r}(\lambda)\right)=s_{m,r}(\lambda)$ and $l\left(\varphi_{m,r}(\lambda)\right)=t_m(\lambda)$, which implies (4).

3. A BIJECTION BETWEEN $\mathscr{A}_{n,m,r}$ AND $\mathscr{B}_{n,m,r}$

THEOREM 2. *There exists a bijection $\varphi_{m,r}:\mathscr{A}_{n,m,r}\to\mathscr{B}_{n,m,r}$ such that $s_{m,r}\left(\varphi_{m,r}(\lambda)\right)=s_{m,r}(\lambda)$ and $l\left(\varphi_{m,r}(\lambda)\right)=t_m(\lambda)$.*

The map $\varphi_{m,r}$ from $\mathscr{A}_{n,m,r}$ to $\mathscr{B}_{n,m,r}$: For $\lambda=(\lambda_1,\lambda_2,\cdots,\lambda_k)\in\mathscr{A}_{n,m,r}$, we split the part λ_i into some copies of m for $\lambda_i\equiv 0 \pmod m$ and into some copies of m and one r for $\lambda_i\equiv r \pmod m$, and put them in the ith row of a matrix in a non-increasing order $(i=1,\cdots,k)$. Summing up the entries in each column of the matrix, we obtain a partition $\varphi_{m,r}(\lambda)$ of n. Since no part $\equiv r \pmod m$ of λ is repeated, the r's can only occur at the bottom of columns, and so the parts $\equiv r \pmod m$ of $\varphi_{m,r}(\lambda)$ are distinct. Since the largest part of λ is a multiple of m, the first row of the matrix is made up of m's, and so r is not a part of $\varphi_{m,r}(\lambda)$. It follows that $\varphi_{m,r}(\lambda)\in\mathscr{B}_{n,m,r}$, $s_{m,r}\left(\varphi_{m,r}(\lambda)\right)=s_{m,r}(\lambda)$ and $l\left(\varphi_{m,r}(\lambda)\right)=t_m(\lambda)$.

For example,

$$\varphi_{3,1}:\quad (12,10,9,9,4,1)\to\begin{pmatrix}12\\10\\9\\9\\4\\1\end{pmatrix}\to\begin{pmatrix}3&3&3&3\\3&3&3&1\\3&3&3&\\3&3&3&\\3&1&&\\1&&&\end{pmatrix}\to(16,13,12,4).$$

The map $\varphi_{m,r}^{-1}$ from $\mathscr{B}_{n,m,r}$ to $\mathscr{A}_{n,m,r}$: For $\lambda=(\lambda_1,\lambda_2,\cdots,\lambda_k)\in\mathscr{B}_{n,m,r}$, we split the part λ_i into some copies of m for $\lambda_i\equiv 0 \pmod m$ and into some copies of m and one r for $\lambda_i\equiv r \pmod m$, and put them in the ith column of a matrix in a non-increasing order ($i = 1, \cdots, k$). Summing up the entries in each row of the matrix, we obtain a partition $\varphi_{m,r}^{-1}(\lambda)$ of n. Since no part $\equiv r \pmod{m}$ of λ is repeated, the r's can only occur at the rightmost of rows, and so the parts $\equiv r \pmod{m}$ of $\varphi_{m,r}^{-1}(\lambda)$ are distinct. Since r is not a part of λ, the first row of the matrix is made up of m's, and so the largest part of $\varphi_{m,r}^{-1}(\lambda)$ is a multiple of m. It follows that $\varphi_{m,r}^{-1}(\lambda) \in \mathscr{A}_{n,m,r}$.

For example,

$$\varphi_{3,1}^{-1}: \quad (16,13,12,4) \to \begin{pmatrix} 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 1 \\ 3 & 3 & 3 & \\ 3 & 3 & 3 & \\ 3 & 1 & & \\ 1 & & & \end{pmatrix} \to \begin{pmatrix} 12 \\ 10 \\ 9 \\ 9 \\ 4 \\ 1 \end{pmatrix} \to (12,10,9,9,4,1).$$

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