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# COMMUTATORS FOR THE FRACTIONAL MAXIMAL AND SHARP FUNCTIONS ON TOTAL MORREY SPACES

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**Abstract:** In this paper, we consider the commutators associated with the fractional maximal function and sharp maximal function when symbol function $b$ in Lipschitz spaces. We give some characterizations of the Lipschitz spaces via the boundedness of these commutators on total Morrey spaces.

***Keywords:*** fractional maximal function, Lipschitz space, commutator, total Morrey space, singular integral operator.
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## 1. INTRODUCTION AND MAIN RESULTS

Let $T$ a classical singular integral operator, the commutator $[T,b]$ generated by $T$ and a suitable function $b$ as follows

$$[T,b]f(x) = b(x)T(f)(x) - T(bf)(x). \tag{1}$$

In 1976, Coifman, Rochberg and Weiss [1] states that $[T,b]$ is bounded on $L^p(\mathbb{R}^n)(1<p<\infty)$ if and only if $b \in BMO(\mathbb{R}^n)$. Moreover, Janson [2] also gave some characterizations of Lipschitz spaces via the commutator. It was proved that $[T,b]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ if and only if $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$, where $0<\beta<1$, $1<p<n/\beta$, $\beta = n(1/p-1/q)$.

As usual, a cube $Q \subset \mathbb{R}^n$ always means its sides parallel to the coordinate axes. Denote by $|Q|$ the Lebesgue measure of $Q$ and $\chi_Q$ the characteristic function of $Q$. For a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, we write $f_Q = |Q|^{-1}\int_Q f(x)\mathrm{d}x$.

Let $0 \le \alpha < n$, for a locally integrable function $f$, the maximal and sharp functions are defined by

$$M_\alpha(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1-\alpha/n}} \int_Q \big|f(y)\big|\mathrm{d}y \quad \text{and} \quad M^\sharp(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q \big|f(y) - f_Q\big|\mathrm{d}y,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing $x$.

When $\alpha = 0$, $M_0$ is the classical Hardy-Littlewood maximal function denoted by $M$, and $M_\alpha$ is the classical fractional maximal function when $0<\alpha<n$.

The maximal commutator of $M_\alpha$ with a locally integrable function $b$ is defined by

$$M_{\alpha,b}(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1-\alpha/n}} \int_Q \big|b(x)f(y) - b(y)f(y)\big|\mathrm{d}y,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing $x$. $[M_\alpha, b]$ and $[M^\sharp, b]$ can refer to (1).

We would like to remark that operators $M_{\alpha,b}$ and $[M_\alpha,b]$ essentially differ from each other. For example, $M_{\alpha,b}$ is positive and sublinear, but $[M_\alpha,b]$ and $[M^\sharp,b]$ is neither positive nor sublinear.

In 2000, Bastero, Milman and Ruiz [3] gave the necessary and sufficient conditions of the boundedness of $[M,b]$ and $[M^\sharp,b]$ on Lebesgue spaces, where $b \in BMO(\mathbb{R}^n)$. In 2009, the authors [4] considered the same problem for the fractional maximal function. Then, the commutators theory of maximal functions have been studied intensively by many authors (see [5–14]). In 2017, Zhang [6] gave the necessary and sufficient conditions of the boundedness of $M_b$ and $[M,b]$ on Lebesgue spaces and Morrey spaces when the symbol $b$ belong to Lipschitz spaces, by which some new characterizations of Lipschitz functions are given. The results were extended to variable Lebesgue spaces in [11]. In 2018, Zhang, Wu and Sun [9] considered the boundedness of $M_{\alpha,b}$, $[M_\alpha,b]$ and $[M^\sharp,b]$ on Orlicz spaces. Recently, Guliyev [14] give necessary and sufficient conditions for the boundedness of $M_b$ and $[M,b]$ in total Morrey spaces when the function $b$ belongs to Lipschitz spaces, whereby some new characterizations of non-negative Lipschitz functions are obtained.

Inspired by above results, we want to study the boundedness of $M_{\alpha,b}$, $[M_\alpha,b]$ and $[M^\sharp,b]$ on total Morrey spaces. Some new characterizations of non-negative Lipschitz functions are obtained.

To state the results, we first give some definitions and notations.

*Definition* 1. *The Lipschitz space of order $\beta$, $0<\beta<1$, is the space of function $f$, such that*

$$\dot{\Lambda}_\beta = \{f : |f(x)-f(y)| \le C|x-y|^\beta\},$$

*and the smallest constant $C>0$ is the Lipschitz norm $\|\cdot\|_{\dot{\Lambda}_\beta}$.*

In 1938, the classical Morrey spaces were introduced by Morrey [15], he studied solutions of some quasi-linear elliptic partial differential equations. Then, the Morrey type spaces have been widely studied by many scholars [16–20]. In 2022, Guliyev [19] introduced a variant of Morrey spaces called total Morrey spaces $L^{p,\lambda,\mu}(\mathbb{R}^n)$, $0<p<\infty$, $\lambda\in\mathbb{R}$ and $\mu\in\mathbb{R}$. It was proved that necessary and sufficient conditions for the boundedness of the maximal commutator operator $M_b$ and the commutator of maximal operator $[M,b]$ on total Morrey spaces $L^{p,\lambda,\mu}(\mathbb{R}^n)$ when $b$ belongs to BMO($\mathbb{R}^n$) spaces.

We shall recall the definitions of the classical Morrey space, modified Morrey space and total Morrey space.

*Definition* 2. *Let $0<p<\infty$, $\lambda\in\mathbb{R}$, $\mu\in\mathbb{R}$ and $[t]_1=\min\{1,t\}(t>0)$. We denote by $L^{p,\lambda}(\mathbb{R}^n)$ the classical Morrey space, by $\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$ the modified Morrey space (see [20]), and by the total Morrey space $L^{p,\lambda,\mu}(\mathbb{R}^n)$ the set of all classes of locally integrable functions $f$ with the finite norms*

$$\|f\|_{L^{p,\lambda}} = \sup_{x\in\mathbb{R}^n,t>0} t^{-\lambda/p}\left(\int_Q |f(y)|^p \mathrm{d}y\right)^{1/p},\ \|f\|_{\widetilde{L}^{p,\lambda}} = \sup_{x\in\mathbb{R}^n,t>0} [t]_1^{-\lambda/p}\left(\int_Q |f(y)|^p \mathrm{d}y\right)^{1/p},$$

$$\|f\|_{L^{p,\lambda,\mu}} = \sup_{x\in\mathbb{R}^n,t>0} [t]_1^{-\lambda/p}[1/t]_1^{\mu/p}\left(\int_Q |f(y)|^p \mathrm{d}y\right)^{1/p},$$

*where $t$ is side length of cube $Q$.*

*Remark* 1. From [19] (see Lemma 2), when $0<p<\infty$, $0\le\lambda\le n$ and $0\le\mu\le n$, then

$$L^{p,\lambda,\mu}(\mathbb{R}^n) = L^{p,\min\{\lambda,\mu\}}(\mathbb{R}^n)\cap L^{p,\max\{\lambda,\mu\}}(\mathbb{R}^n)$$

and

$$\|f\|_{L^{p,\lambda,\mu}} = \max\{\|f\|_{L^{p,\min\{\lambda,\mu\}}}, \|f\|_{L^{p,\max\{\lambda,\mu\}}}\} = \|f\|_{L^{p,\mu,\lambda}}.$$

If $\lambda=\mu$ in $L^{p,\lambda,\mu}(\mathbb{R}^n)$, then $L^{p,\lambda,\mu}(\mathbb{R}^n)=L^{p,\lambda}(\mathbb{R}^n)$; if $\mu=0$ in $L^{p,\lambda,\mu}(\mathbb{R}^n)$, then $L^{p,\lambda,\mu}(\mathbb{R}^n)=\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$.

For a fixed cube $Q_0$, the maximal function with respect to $Q_0$ of a function $f$ is given by

$$M_{\alpha,Q_0}(f)(x) = \sup_{\substack{Q\ni x\\ Q\subset Q_0}} \frac{1}{|Q|^{1-\alpha/n}}\int_Q |f(y)|\mathrm{d}y,\ \ x\in Q_0.$$

where the supremum is taken over all cubes $Q$ with $Q \subset Q_0$ and $Q \ni x$. When $\alpha = 0$, $M_Q = M_{0,Q}$.

Our main results in this paper can be stated as follows.

THEOREM 1. *Let* $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $0 < \beta < 1$, $0 \le \min\{\lambda,\mu\} \le \max\{\lambda,\mu\} < n - p(\alpha+\beta)$, $0 < \alpha < \frac{n-\max\{\lambda,\mu\}}{p}$ *and* $0 < \alpha+\beta < \frac{n-\max\{\lambda,\mu\}}{p}$. *Assume that* $1 < p < \frac{n-\max\{\lambda,\mu\}}{\alpha+\beta}$ *and* $\frac{\alpha+\beta}{n-\min\{\lambda,\mu\}} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha+\beta}{n-\max\{\lambda,\mu\}}$, *then the following assertions are equivalent:*

(1) $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$;

(2) $M_{\alpha,b}$ *is bounded from* $L^{p,\lambda,\mu}(\mathbb{R}^n)$ *to* $L^{q,\lambda,\mu}(\mathbb{R}^n)$;

(3) *There exists a constant* $C > 0$, *such that* $\sup_Q |Q|^{-\beta/n} \frac{\left\|(b-b_Q)\chi_Q\right\|_{L^{q,\lambda,\mu}}}{\|\chi_Q\|_{L^{q,\lambda,\mu}}} \le C$;

(4) *There exists a constant* $C > 0$, *such that* $\sup_Q |Q|^{-1-\beta/n} \int_Q |b(x) - b_Q| \mathrm{d}x \le C$.

*Remark* 2. For the case of $\alpha = 0$, the result of Theorem 1 was proved in [14] (see Theorem 3.5), and we aslo added two equivalent conditions. From Theorem 1 in the case $\lambda = \mu$ or $\mu = 0$, we can get similar results on classical Morrey spaces or modified Morrey spaces.

THEOREM 2. *Let* $1 < p,q < \infty$, $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ *and* $0 < \beta < 1$. *Let also* $0 \le \lambda,\mu < n$, $0 < \alpha < n$ *and* $0 < \alpha+\beta < n$. *If* $[M_\alpha, b]$ *is bounded from* $L^{p,\lambda,\mu}(\mathbb{R}^n)$ *to* $L^{q,\lambda,\mu}(\mathbb{R}^n)$, *then* $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ *and* $b \ge 0$.

*Remark* 3. For the case of $\alpha = 0$, the result of Theorem 2 was proved in [14] (see Theorem 4.6). From Theorem 2 in the case $\lambda = \mu$ or $\mu = 0$, we get similar results on classical Morrey spaces or modified Morrey spaces.

THEOREM 3. *Let* $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $0 < \beta < 1$ *and* $0 \le \min\{\lambda,\mu\} \le \max\{\lambda,\mu\} < n - p\beta$. *Assume that* $1 < p < \frac{n-\max\{\lambda,\mu\}}{\beta}$ *and* $\frac{\beta}{n-\min\{\lambda,\mu\}} \le \frac{1}{p} - \frac{1}{q} \le \frac{\beta}{n-\max\{\lambda,\mu\}}$, *then the following assertions are equivalent:*

(1) $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ *and* $b \ge 0$;

(2) $[M^\sharp, b]$ *is bounded from* $L^{p,\lambda,\mu}(\mathbb{R}^n)$ *to* $L^{q,\lambda,\mu}(\mathbb{R}^n)$;

(3) *There exists a constant* $C > 0$, *such that* $\sup_Q |Q|^{-\beta/n} \frac{\left\|\left(b-2M^\sharp(b\chi_Q)\right)\chi_Q\right\|_{L^{q,\lambda,\mu}}}{\|\chi_Q\|_{L^{q,\lambda,\mu}}} \le C$;

(4) *There exists a constant* $C > 0$, *such that* $\sup_Q |Q|^{-1-\beta/n} \int_Q |b(x) - 2M^\sharp(b\chi_Q)(x)| \mathrm{d}x \le C$.

*Remark* 4. We note that Theorem 3 extends the Corollary 1.2 when $\lambda = \mu = 0$ in [9]. From Theorem 3 in the case $\lambda = \mu$ or $\mu = 0$, we also may have similar results on classical Morrey spaces or modified Morrey spaces.

## 2. PRELIMINARIES AND LEMMAS

In this section, we recall some know preliminaries and lemmas. It is known that the Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}^n)$ coincides with some Morrey-Companato space (see [21] for example) and can be characterized by mean oscillation as follows, which is due to Janson, Taibleson and Weiss [21] and Paluszyński [22].

LEMMA 1. *Let* $0 < \beta < 1$ *and* $1 \le q < \infty$. *Define*

$$\dot{\Lambda}_{\beta,q}(\mathbb{R}^n) := \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \sup_Q \frac{1}{|Q|^{\beta/n}} \left( \frac{1}{|Q|} \int_Q |f(y) - f_Q|^q \mathrm{d}y \right)^{\frac{1}{q}} < \infty \right\}.$$

*Then, for all* $0 < \beta < 1$ *and* $1 \le p < \infty$, $\dot{\Lambda}_\beta(\mathbb{R}^n) = \dot{\Lambda}_{\beta,q}(\mathbb{R}^n)$ *with equivalent norms.*

From [11], we have the following characterization of non-negative Lipschitz functions.

LEMMA 2 [11]. *Let* $0 < \beta < 1$ *and* $b$ *be a locally integrable function. Then* $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ *and* $b \ge 0$ *if and only if there exists a constant* $C > 0$ *such that* $\sup_Q |Q|^{-1-\beta/n} \int_Q |b(x) - M_Q(b)(x)| \mathrm{d}x \le C$.

LEMMA 3 [14]. *Let* $1 < p < \frac{n-\max\{\lambda,\mu\}}{\alpha}$, $0 \le \min\{\lambda,\mu\} \le \max\{\lambda,\mu\} < n$ *and* $0 \le \alpha < \frac{n-\max\{\lambda,\mu\}}{p}$ . *Then the following assertions are equivalent:*

(1) $\frac{\alpha}{n-\min\{\lambda,\mu\}} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{n-\max\{\lambda,\mu\}}$ *;*

(2) $M_\alpha$ *is bounded from* $L^{p,\lambda,\mu}(\mathbb{R}^n)$ *to* $L^{q,\lambda,\mu}(\mathbb{R}^n)$.

LEMMA 4 [14, 19]. *Let* $1 \le p < \infty$, $0 \le \lambda,\mu < n$, $[r]_1 = \min\{1,r\}(r > 0)$. *For any cube* $Q$ *with side length* $r$*, then (see (3.12) in [14] or (11) in [19])*

$$\|\chi_Q\|_{L^{p,\lambda,\mu}} \approx r^{n/p}[r]_1^{-\lambda/p}[1/r]_1^{\mu/p}.$$

LEMMA 5 [3]. *For any cube* $Q \ni x$*, let* $E = \{x \in Q, b(x) \le b_Q\}$ *and* $F = \{x \in Q, b(x) > b_Q\}$*, then*

$$\int_F |b(x) - b_Q| \mathrm{d}x = \int_E |b(x) - b_Q| \mathrm{d}x.$$

LEMMA 6. *Let* $1 < p,q < \infty$, $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ *and* $0 < \beta < 1$. *Let also* $0 \le \lambda,\mu < n$, $0 < \alpha < n$ *and* $0 < \alpha + \beta < n$. *If* $[M_\alpha, b]$ *is bounded from* $L^{p,\lambda,\mu}(\mathbb{R}^n)$ *to* $L^{q,\lambda,\mu}(\mathbb{R}^n)$*, then there exists a constant* $C > 0$ *such that*

$$\sup_Q |Q|^{-\beta/n} \frac{\left\| (b - M_Q(b))\,\chi_Q \right\|_{L^{q,\lambda,\mu}}}{\|\chi_Q\|_{L^{q,\lambda,\mu}}} \le C.$$

*Proof.* For any fixed cube $Q$,

$$\begin{aligned}
\frac{1}{|Q|^{\beta/n}} \frac{\|(b - M_Q(b))\,\chi_Q\|_{L^{q,\lambda,\mu}}}{\|\chi_Q\|_{L^{q,\lambda,\mu}}} &\le \frac{1}{|Q|^{\beta/n}} \frac{\left\| b - |Q|^{-\alpha/n} M_{\alpha,Q}(b) \right\|_{L^{q,\lambda,\mu}}}{\|\chi_Q\|_{L^{q,\lambda,\mu}}} \\
&\quad + \frac{1}{|Q|^{\beta/n}} \frac{\left\| |Q|^{-\alpha/n} M_{\alpha,Q}(b) - M_Q(b) \right\|_{L^{q,\lambda,\mu}}}{\|\chi_Q\|_{L^{q,\lambda,\mu}}} \\
&:= I_1 + I_2.
\end{aligned}$$

For $I_1$, from the definition of $M_{\alpha,Q}$, it is not difficult to check that

$$M_{\alpha,Q}(\chi_Q)(x) = |Q|^{\alpha/n} \text{ for all } x \in Q.$$

For any fixed $Q \subset \mathbb{R}^n$ and $x \in Q$, we have (see (2.4) in [4])

$$M_\alpha(\chi_Q)(x) = M_{\alpha,Q}(\chi_Q)(x) = |Q|^{\alpha/n} \text{ and } M_\alpha(b\chi_Q)(x) = M_{\alpha,Q}(b)(x).$$

According to $[M_\alpha, b]$ is bounded from $L^{p,\lambda,\mu}(\mathbb{R}^n)$ to $L^{q,\lambda,\mu}(\mathbb{R}^n)$ and Lemma 4, then

$$\begin{aligned}
I_1 \le \frac{C}{r^{(\alpha+\beta)}} \frac{\left\|\chi_Q\right\|_{L^{p,\lambda,\mu}}}{\|\chi_Q\|_{L^{q,\lambda,\mu}}} &\le C r^{-\alpha-\beta+\frac{n}{p}-\frac{n}{q}} [r]_1^{\frac{\lambda}{q}-\frac{\lambda}{p}} [1/r]_1^{\frac{\mu}{p}-\frac{\mu}{q}} \\
&\approx C[r]_1^{-\alpha-\beta+\frac{n-\lambda}{p}-\frac{n-\lambda}{q}} [1/r]_1^{\alpha+\beta-\frac{n-\mu}{p}+\frac{n-\mu}{q}} \le C,
\end{aligned}$$

where $r$ is side length of cube $Q$.

Next, we estimate $I_2$. Noting that (see the proof Proposition 4.1 in [3])

$$M(\chi_Q)(x) = \chi_Q(x) \ \text{ and } \ M(b\chi_Q)(x) = M_Q(b)(x), \ x \in Q.$$

Then, for any $x \in Q$ we have (for details see [9], page 8)

$$\left| |Q|^{-\alpha/n} M_{\alpha,Q}(b)(x) - M_Q(b)(x) \right| \le \left| |Q|^{-\alpha/n} [M_\alpha, |b|] \chi_Q(x) \right| + \left| [M, |b|] \chi_Q(x) \right|.$$

For any fixed cube $Q$ with side length $r$, by Hölder's inequality, Lemma 4 and estimation of $I_1$, it is easy to see that

$$\begin{aligned}\frac{1}{|Q|^{1+\beta/n}}\int_Q |b(x)-|Q|^{-\alpha/n}M_{\alpha,Q}(b)(x)|\mathrm{d}x &\le \frac{C}{|Q|^{1+\beta/n}}\left\|b-|Q|^{-\alpha/n}M_{\alpha,Q}(b)\right\|_{L^p(Q)}|Q|^{\frac{1}{p'}}\\ &\le C|Q|^{-\frac{1}{q}-\frac{\beta}{n}}[r]_1^{\frac{\lambda}{q}}[1/r]_1^{-\frac{\mu}{q}}\left\|b-|Q|^{-\alpha/n}M_{\alpha,Q}(b)\right\|_{L^{q,\lambda,\mu}}\\ &\le Cr^{-\frac{n}{q}}[r]_1^{\frac{\lambda}{q}}[1/r]_1^{-\frac{\mu}{q}}\|\chi_Q\|_{L^{q,\lambda,\mu}}\\ &\le Cr^{-\frac{n}{q}}[r]_1^{\frac{\lambda}{q}}[1/r]_1^{-\frac{\mu}{q}}r^{\frac{n}{q}}[r]_1^{-\frac{\lambda}{q}}[1/r]_1^{\frac{\mu}{q}}\le C.\end{aligned}$$

For any cube $Q\subset\mathbb{R}^n$, noticing the obvious estimate $|b_Q|\le|Q|^{-\alpha/n}M_{\alpha,Q}(b)(x),\ x\in Q$. From Lemma 5, for any $x\in E$, $b(x)\le b_Q\le|b_Q|\le|Q|^{-\alpha/n}M_{\alpha,Q}(b)(x)$, we have

$$\begin{aligned}\frac{1}{|Q|^{1+\beta/n}}\int_Q|b(x)-b_Q|\mathrm{d}x &= \frac{2}{|Q|^{1+\beta/n}}\int_E|b(x)-b_Q|\mathrm{d}x\\ &\le \frac{2}{|Q|^{1+\beta/n}}\int_Q|b(x)-|Q|^{-\alpha/n}M_{\alpha,Q}(b)(x)|\mathrm{d}x\le C,\end{aligned}$$

thus, $b\in\dot{\Lambda}_\beta(\mathbb{R}^n)$, which implies $|b|\in\dot{\Lambda}_\beta(\mathbb{R}^n)$.

By the definitions of $[M_\alpha,b]$ and $M_\alpha$, we have, for any $x\in Q$,

$$|[M_\alpha,|b|](\chi_Q)(x)|\le\|b\|_{\dot{\Lambda}_\beta}|Q|^{(\alpha+\beta)/n}\chi_Q(x).$$

Similarly, we have

$$\left|[M,|b|](\chi_Q)(x)\right|\le\|b\|_{\dot{\Lambda}_\beta}|Q|^{\beta/n}\chi_Q(x)\ \text{ for any }\ x\in Q.$$

So, we obtain, for any $x\in Q$,

$$\left||Q|^{-\alpha/n}M_{\alpha,Q}(b)(x)-M_Q(b)(x)\right|\le C\|b\|_{\dot{\Lambda}_\beta}|Q|^{\beta/n}\chi_Q(x).$$

Then, by Lemma 4 have

$$I_2\le C\frac{\|\chi_Q\|_{L^{q,\lambda,\mu}}}{\|\chi_Q\|_{L^{q,\lambda,\mu}}}\le C.$$

Combining $I_1$ and $I_2$, we may get

$$\frac{1}{|Q|^{\beta/n}}\frac{\|(b-M_Q(b))\chi_Q\|_{L^{q,\lambda,\mu}}}{\|\chi_Q\|_{L^{q,\lambda,\mu}}}\le C.$$

Since the cube $Q\subset\mathbb{R}^n$ is arbitrary, then the poof of Lemma 6 is completed. □

## 3. PROOF OF MAIN RESULTS

*Proof of Theorem* 1. (1)⇒(2) Assume $b\in\dot{\Lambda}_\beta(\mathbb{R}^n)$. For any fixed cube $Q\subset\mathbb{R}^n$, we have ( [5], Lemma 4.3)

$$M_{\alpha,b}(f)(x)\le C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}M_{\alpha+\beta}f(x).$$

This, together with Lemma 3, shows that $M_{\alpha,b}$ is bounded from $L^{p,\lambda,\mu}(\mathbb{R}^n)$ to $L^{q,\lambda,\mu}(\mathbb{R}^n)$.

(2)⇒(3) For any fixed cube $Q$ with side length $r$, we have for all $x \in Q$,

$$|b(x) - b_Q| \le \frac{1}{|Q|}\int_Q |b(x) - b(y)|\mathrm{d}y \le |Q|^{-\alpha/n} M_{\alpha,b}(\chi_Q)(x).$$

Since $M_{\alpha,b}$ is bounded from $L^{p,\lambda,\mu}(\mathbb{R}^n)$ to $L^{q,\lambda,\mu}(\mathbb{R}^n)$, then by Lemma 4 we obtain that

$$\begin{aligned}\frac{1}{|Q|^{\beta/n}} \frac{\|(b-b_Q)\chi_Q\|_{L^{q,\lambda,\mu}}}{\|\chi_Q\|_{L^{q,\lambda,\mu}}} &\le \frac{C}{r^{\alpha+\beta}} \frac{\|\chi_Q\|_{L^{p,\lambda,\mu}}}{\|\chi_Q\|_{L^{q,\lambda,\mu}}} \\ &\le C r^{-\alpha-\beta+\frac{n}{p}-\frac{n}{q}} [r]_1^{\frac{\lambda}{q}-\frac{\lambda}{p}} [1/r]_1^{\frac{\mu}{p}-\frac{\mu}{q}} \\ &\approx C[r]_1^{-\alpha-\beta+\frac{n-\lambda}{p}-\frac{n-\lambda}{q}} [1/r]_1^{\alpha+\beta-\frac{n-\mu}{p}+\frac{n-\mu}{q}} \le C,\end{aligned}$$

which implies (3) since the cube $Q \subset \mathbb{R}^n$ is arbitrary.

(3)⇒(4) By using Lemma 5, Hölder's inequality and condition (3), we have

$$\begin{aligned}\frac{1}{|Q|^{1+\beta/n}}\int_Q |b(y)-b_Q|\,\mathrm{d}y &\le \frac{2}{|Q|^{1+\beta/n}} \|b-b_Q\|_{L^q(Q)} |Q|^{\frac{1}{q'}} \\ &\le 2|Q|^{-\frac{1}{q}-\frac{\beta}{n}} [r]_1^{\frac{\lambda}{q}} [1/r]_1^{-\frac{\mu}{q}} \|(b-b_Q)\chi_Q\|_{L^{q,\lambda,\mu}} \\ &\le C r^{-\frac{n}{q}} [r]_1^{\frac{\lambda}{q}} [1/r]_1^{-\frac{\mu}{q}} \|\chi_Q\|_{L^{q,\lambda,\mu}} \\ &\le C r^{-\frac{n}{q}} [r]_1^{\frac{\lambda}{q}} [1/r]_1^{-\frac{\mu}{q}} r^{\frac{n}{q}} [r]_1^{-\frac{\lambda}{q}} [1/r]_1^{\frac{\mu}{q}} \le C.\end{aligned}$$

which implies (4) since the cube $Q \subset \mathbb{R}^n$ is arbitrary.

(4)⇒(1) For any fixed cube $Q$, we assume that

$$\sup_Q \frac{1}{|Q|^{1+\beta/n}} \int_Q |b(y)-b_Q|\,\mathrm{d}y \le C.$$

By using Lemma 1, we have $b \in \dot{\Lambda}_{\beta,1}(\mathbb{R}^n)$. For all $1 \le q < \infty$, because of $\dot{\Lambda}_\beta(\mathbb{R}^n) = \dot{\Lambda}_{\beta,q}(\mathbb{R}^n)$ with equivalent norms, thus $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$.

*Proof of Theorem* 2. Assume that $[M_\alpha, b]$ is bounded from $L^{p,\lambda,\mu}(\mathbb{R}^n)$ to $L^{q,\lambda,\mu}(\mathbb{R}^n)$. For any fixed cube $Q$ with side length $r$, by Hölder's inequality, Lemma 4 and Lemma 6, we have

$$\begin{aligned}\frac{1}{|Q|^{1+\beta/n}}\int_Q |b(x)-M_Q(b)(x)|\mathrm{d}x &\le \frac{1}{|Q|^{1+\beta/n}} \|b-M_Q(b)\|_{L^q(Q)} |Q|^{\frac{1}{q'}} \\ &\le |Q|^{-\frac{1}{q}-\frac{\beta}{n}} [r]_1^{\frac{\lambda}{q}} [1/r]_1^{-\frac{\mu}{q}} \|(b-M_Q(b))\chi_Q\|_{L^{q,\lambda,\mu}} \\ &\le C r^{-\frac{n}{q}} [r]_1^{\frac{\lambda}{q}} [1/r]_1^{-\frac{\mu}{q}} r^{\frac{n}{q}} [r]_1^{-\frac{\lambda}{q}} [1/r]_1^{\frac{\mu}{q}} \le C.\end{aligned}$$

Thus, by using Lemma 2, we get $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ and $b \ge 0$.

*Proof of Theorem* 3. (1)⇒(2) Assume $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$, $b \ge 0$. For any $f \in L^{p,\lambda,\mu}(\mathbb{R}^n)$, by using Lemma 3 (case of $\alpha = 0$) and $M^\sharp f(x) \le 2Mf(x)$, we have $M^\sharp f(x) < \infty$ for a.e. $x \in \mathbb{R}^n$. Then the following estimate was given in [10] (see page 1422):

$$|[M^\sharp, b]f(x)| \le \|b\|_{\dot{\Lambda}_\beta} M_\beta f(x) \quad \text{for a.e. } x \in \mathbb{R}^n.$$

Thus, it follows from Lemma 3 that $[M^\sharp, b]$ is bounded from $L^{p,\lambda,\mu}(\mathbb{R}^n)$ to $L^{q,\lambda,\mu}(\mathbb{R}^n)$.

(2)⇒(3) Assume $[M^\sharp, b]$ is bounded from $L^{p,\lambda,\mu}(\mathbb{R}^n)$ to $L^{q,\lambda,\mu}(\mathbb{R}^n)$. For any fixed cube $Q$, we have (see [3], page 3333)

$$M^{\sharp}(\chi_Q)(x) = 1/2 \quad \text{for } \ x \in Q.$$

Then, by applying Lemma 4 we get

$$\begin{aligned}
\frac{1}{|Q|^{\beta/n}} \frac{\|\left(b-2M^{\sharp}(b\chi_Q)\right)\chi_Q\|_{L^{q,\lambda,\mu}}}{\|\chi_Q\|_{L^{q,\lambda,\mu}}} &\leq \frac{C}{r^{\beta}} \frac{\|\chi_Q\|_{L^{p,\lambda,\mu}}}{\|\chi_Q\|_{L^{q,\lambda,\mu}}} \\
&\leq Cr^{-\beta+\frac{n}{p}-\frac{n}{q}} [r]_1^{\frac{\lambda}{q}-\frac{\lambda}{p}} [1/r]_1^{\frac{\mu}{p}-\frac{\mu}{q}} \\
&\approx C[r]_1^{-\beta+\frac{n-\lambda}{p}-\frac{n-\lambda}{q}} [1/r]_1^{\beta-\frac{n-\mu}{p}+\frac{n-\mu}{q}} \leq C,
\end{aligned}$$

which implies (3) since the cube $Q \subset \mathbb{R}^n$ is arbitrary.

(3)⇒(4) For any fixed cube $Q$ with side length $r$, by Hölder's inequality, condition (3) and Lemma 4, it is easy to see that

$$\begin{aligned}
\frac{1}{|Q|^{1+\beta/n}} \int_Q |b(x) - 2M^{\sharp}(b\chi_Q)(x)| \mathrm{d}x &\leq \frac{1}{|Q|^{1+\beta/n}} \left\|b - 2M^{\sharp}(b\chi_Q)\right\|_{L^p(Q)} |Q|^{\frac{1}{p'}} \\
&\leq C|Q|^{-\frac{1}{q}-\frac{\beta}{n}} [r]_1^{\frac{\lambda}{q}} [1/r]_1^{-\frac{\mu}{q}} \left\| \left(b - 2M^{\sharp}(b\chi_Q)\right)\chi_Q \right\|_{L^{q,\lambda,\mu}} \\
&\leq Cr^{-\frac{n}{q}} [r]_1^{\frac{\lambda}{q}} [1/r]_1^{-\frac{\mu}{q}} \|\chi_Q\|_{L^{q,\lambda,\mu}} \\
&\leq Cr^{-\frac{n}{q}} [r]_1^{\frac{\lambda}{q}} [1/r]_1^{-\frac{\mu}{q}} r^{\frac{n}{q}} [r]_1^{-\frac{\lambda}{q}} [1/r]_1^{\frac{\mu}{q}} \leq C,
\end{aligned}$$

which implies (4) since the cube $Q \subset \mathbb{R}^n$ is arbitrary.

(4)⇒(1) For any fixed cube $Q$, Bastero, Milman and Ruiz [3] given the following equality

$$|b_Q| \leq 2M^{\sharp}(b\chi_Q)(x), \ \ x \in Q.$$

From Lemma 5, condition (4) and $b(x) \leq b_Q \leq |b_Q| \leq 2M^{\sharp}(b\chi_Q)(x)$, we have

$$\frac{1}{|Q|^{1+\beta/n}} \int_Q |b(x) - b_Q| \mathrm{d}x \leq \frac{2}{|Q|^{1+\beta/n}} \int_Q |b(x) - 2M^{\sharp}(b\chi_Q)(x)| \mathrm{d}x \leq C.$$

Thus, we may obtain $b \in \dot{\Lambda}_{\beta}(\mathbb{R}^n)$.

Next, we will prove $b \geq 0$, it suffices to show $b^- = 0$, where $b^- = -\min\{b, 0\}$ and let $b^+ = |b| - b^-$. For any fixed cube $Q$, observe that (for details see [3,9])

$$2M^{\sharp}(b\chi_Q)(x) - b(x) \geq |b_Q| - b(x) = |b_Q| - b^+(x) + b^-(x) \text{ for } x \in Q.$$

Hence, there exists a constant $C > 0$ such that for any cube $Q$

$$\begin{aligned}
C &\geq \frac{1}{|Q|^{1+\beta/n}} \int_Q \left|2M^{\sharp}(b\chi_Q)(x) - b(x)\right| \mathrm{d}x \\
&\geq \frac{1}{|Q|^{1+\beta/n}} \int_Q \left(|b_Q| - b^+(x) + b^-(x)\right) \mathrm{d}x \\
&= \frac{1}{|Q|^{\beta/n}} \left(|b_Q| - \frac{1}{|Q|} \int_Q b^+(x) \mathrm{d}x + \frac{1}{|Q|} \int_Q b^-(x) \mathrm{d}x\right).
\end{aligned}$$

This gives

$$|b_Q| - \frac{1}{|Q|} \int_Q b^+(x) \mathrm{d}x + \frac{1}{|Q|} \int_Q b^-(x) \mathrm{d}x \leq C|Q|^{\beta/n}.$$

Thus, $b^- = 0$ follows from Lebesgue differentiation theorem.

So far, the proofs of Theorem 1, 2 and 3 are completed.

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