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ON PERFECT 2-MATCHING UNIFORM GRAPHS

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Abstract. Let G be a graph. For a set $\mathscr{H}$ of connected graphs, an $\mathscr{H}$-factor of graph G is a spanning subgraph H of G such that every component of H is isomorphic to a member of $\mathscr{H}$. Denote $\mathscr{H}=\{P_2\}\cup\{C_i|i\geq 3\}$. We call $\mathscr{H}$-factor a perfect 2-matching of G, that is, a perfect 2-matching is a spanning subgraph of G such that each component of G is either an edge or a cycle. In this paper, we define the new concept of perfect 2-matching uniform graph, namely, a graph G is called a perfect 2-matching uniform graph if for arbitrary two distinct edges e_1 and e_2 of G, G contains a perfect 2-matching containing e_1 and avoiding e_2. In addition, we study the relationship between some graphic parameters and the existence of perfect 2-matching uniform graphs. The results obtained in this paper are sharp in some sense.

***Keywords**:* binding number, toughness, perfect 2-matching, perfect 2-matching uniform graph.
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1. INTRODUCTION

All graphs considered here are finite simple graphs. Let G be a graph, with vertex set $V(G)$ and edge set $E(G)$. For x in $V(G)$, the degree of x is denoted by $d_G(x)$. We write $N_G(x)$ for the vertices adjacent to x in G. If X is a subset of $V(G)$, $G[X]$ denotes the subgraph of G induced by X and $G-X$ denotes the induced subgraph of $V(G)\backslash X$. For convenience, we write $N_G(X)$ instead of $\cup_{x\in X}N_G(x)$. For $E'\subseteq E(G)$, $G-E'$ denotes the subgraph obtained from G by removing the edges in E'. We denote by $\omega(G)$, $i(G)$, $\alpha(G)$ and $\delta(G)$ the number of connected components, the number of isolated vertices, the independence number and the minimum degree of G, respectively. For two given graphs G_1 and G_2, $G_1\cup G_2$ we mean the graph with vertex set $V(G_1)\cup V(G_2)$ and edge set $E(G_1)\cup E(G_2)$, and $G_1\vee G_2$ we mean the graph with vertex set $V(G_1)\cup V(G_2)$ and edge set $E(G_1)\cup E(G_2)\cup\{e=xy:x\in V(G_1),y\in V(G_2)\}$. Let K_n and P_n denote the complete graph and the path of order n, respectively. We use $\kappa(G)$ to denote the vertex connectivity of G.

Woodall [21] introduced the concept of binding number. For a connected graph G, its *binding number*, was defined by

$$\mathrm{bind}(G)=\min\left\{\frac{|N_G(S)|}{|S|}:\emptyset\neq S\subseteq V(G),N_G(S)\neq V(G)\right\}.$$

Chvátal [5] introduced the graph parameter of *toughness* as follows: If G is a complete graph, then $t(G)=+\infty$; otherwise,

$$t(G)=\min\left\{\frac{|S|}{\omega(G-S)}:S\subseteq V(G),\omega(G-S)\geq 2\right\}.$$

Motivated by the idea of toughness, Yang et al. [24] defined a new graph parameter of *isolated toughness*, denoted by $I(G)$, i.e., if G is a complete graph, then $I(G) = +\infty$; otherwise,

$$I(G) = \min\left\{ \frac{|S|}{i(G-S)} : S \subseteq V(G), i(G-S) \geq 2 \right\}.$$

For a set $\{A_1, A_2, A_3, \cdots\}$ of graphs, an $\{A_1, A_2, A_3, \cdots\}$-factor of a graph G is a spanning subgraph of G such that each of its components is isomorphic to a member of $\{A_1, A_2, A_3, \cdots\}$. For a set $\mathscr{H}$ of connected graphs, an $\mathscr{H}$-factor of graph G is a spanning subgraph H of G such that every component of H is isomorphic to a member of $\mathscr{H}$. Denote $\mathscr{H} = \{P_2\} \cup \{C_i | i \geq 3\}$. Then a $\mathscr{H}$-factor is called a *perfect 2-matching* of G, that is, a perfect 2-matching is a spanning subgraph of G such that each component of G is either an edge or a cycle.

A graph G is called *perfect 2-matching covered* if for each edge e of G, G admits a perfect 2-matching containing e. If for each edge e of G, G admits a perfect 2-matching excluding e, then graph G is called *perfect 2-matching deleted*. Perfect 2-matching uniform graphs with at least two edges are both perfect 2-matching covered and perfect 2-matching deleted. Namely, a graph G is *perfect 2-matching uniform* if for any two distinct edges e_1 and e_2 of G, G contains a perfect 2-matching including e_1 and excluding e_2. Trivially, the graph K_2 is perfect 2-matching uniform, but it is not perfect 2-matching deleted. Note that the converse of the statement that every perfect 2-matching uniform graph with at least two edges is both perfect 2-matching covered and perfect 2-matching deleted does not hold. For example, the 4-cycle is both perfect 2-matching covered and perfect 2-matching deleted but not perfect 2-matching uniform.

A *k-factor* is a spanning subgraph H of graph G such that $d_H(x) = k$ for every $x \in V(G)$. A *2-factor* is a special case of *k-factor* for $k = 2$. A *perfect matching* is also referred as 1-factor, which is a spanning subgraph H of graph G such that $d_H(x) = 1$ for every $x \in V(G)$. Since Tutte proposed the well known Tutte 1-factor theorem [17], there has been some research on graph factors [1, 16, 20, 38]. Brouwer and Haemers [3] presented an eigenvalue condition for a graph to have a perfect matching. O [14] put forward two sufficient conditions for graphs to admit perfect matchings. O and West [15] investigated the existence of a perfect matching in a regular graph. Zhang and Zhou [26] posed two criteria for graphs to be $P_{\geq 2}$-factor and $P_{\geq 3}$-factor covered graphs. Zhou, Wu and Bian [36] gave two sufficient conditions in terms of minimum degree for the existence of $P_{\geq 3}$-factor critical covered graphs or $P_{\geq 3}$-factor critical deleted graphs. Wang and Zhang [18, 19] got some sufficient conditions for the existence of $P_{\geq 3}$-factor critical covered graphs. Zhou and Zhang [37] established some sufficient conditions for graphs to be factor deleted graphs. Zhou [29] put forward some sufficient conditions on $P_{\geq 3}$-factor critical deleted graphs. Chen and Dai [4] improved Zhou, Bian and Pan's result.

Zhou, Sun and Bian [32], Gao and Wang [8] put forward an isolated toughness and a binding number condition for a graph being a $P_{\geq 3}$-factor uniform graph, respectively. Liu [11, 13] derived some sufficient conditions in terms of binding number and sun toughness for the existence of path-factor uniform graphs, respectively. Zhou, Sun and Liu [35] showed some sufficient conditions in terms of neighborhoods for path-factor uniform graphs. Zhou, Sun and Liu [34], Gao, Chen and Wang [7], Wu [23] obtained some results on $P_{\geq 3}$-factors of graphs with specific properties. Zhou [27] established some relationships between neighbourhood union and graph factors. Yuan and Hao [25] discussed the relationships between independence number and graph factors. Zhou, Zhang and Sun [39] put forward a sufficient condition on the existence of path factors in terms of spectral radii in graphs. For some other research on graph factors, see [9, 22, 28, 30, 31, 33].

In 1953, Tutte gave the following equivalent conditions for the existence of perfect 2-matchings in a graph.

THEOREM 1.1 [17]. *Let G be a connected graph. Then the following conditions are equivalent:*
(a) *G has a perfect 2-matching;*
(b) *$i(G-S) \leq |S|$ for any subset $S \subset V(G)$;*
(c) *$|N_G(T)| \geq |T|$ for any independent subset $T \subset V(G)$;*
(d) *$|N_G(S)| \geq |S|$ for any subset $S \subset V(G)$.*

A graph G is said to be "regularizable" if by replacing some edges in G by multiple edges to obtain a regular graph. Berge [2] introduced and studied the perfect 2-matching covered graph, which is equivalent to a "regularizable" graph. A Tutte-type characterization of perfect 2-matching covered graph was given by Berge.

THEOREM 1.2 [2]. *Let G be a connected graph that is not a bipartite graph with partite sets of equal size, then G is perfect 2-matching covered if and only if* $i(G-S) < |S|$ *for any non-empty subset* $S \subset V(G)$.

The following toughness condition for the existence of 2-factor in a graph is due to Enomoto *et al.*, which was a conjecture of Chv*á*tal.

THEOREM 1.3 [6]. *Let G be a graph with at least 3 vertices. If* $t(G) \geq 2$*, then G has a 2-factor.*

Katerinis and Wang extended Theorem 1.3 by considering the existence of 2-factors in terms of toughness with inclusion or exclusion property involving two edges.

THEOREM 1.4 [10]. *Let G be a graph with at least 5 vertices. If* $t(G) \geq 2$*, then for any two arbitrarily given edges* e_1 *and* e_2 *of G, G has a 2-factor containing* e_1 *and avoiding* e_2.

Zhou and Sun obtained a binding number condition for the existence of $P_{\geq 2}$-factor uniform graph.

THEOREM 1.5 [32]. *Let G be a 2-edge-connected graph. If* $bind(G) > \frac{4}{3}$*, then G is a* $P_{\geq 2}$*-factor uniform graph, that is, for any two arbitrarily given edges* e_1 *and* e_2 *of G, G has a* $P_{\geq 2}$*-factor including* e_1 *and excluding* e_2.

Motivated by the above theorems, we discuss the following more general problem:

PROBLEM 1.6. *Is G a perfect 2-matching uniform graph, or for any two distinct edges* e_1 *and* e_2 *of G, does G contain a perfect 2-matching including* e_1 *and excluding* e_2*?*

In this article, we give sufficient conditions, expressed in terms of some graph parameters, for a graph to be a perfect 2-matching uniform graph. Several sufficient conditions on graphic parameters toughness and binding number for the existence of perfect 2-matching uniform graphs are presented in Theorem 2.2 and Theorem 3.1.

First, Theorem 2.1 gives a necessary and sufficient condition for a 2-connected non-bipartite graph to be a perfect 2-matching uniform graph, expressed in terms of a condition relating $|S|$ and the number of isolated vertices of $G-S$, for all non-empty subsets $S \subseteq V(G)$. The proof of Theorem 2.1 is based on the observation that a graph G is a perfect 2-matching uniform if and only if for every edge e of the graph, the graph $G-e$ is perfect 2-matching covered (that is, every edge is contained in a perfect 2-matching), and a Tutte-type characterization of perfect 2-matching covered graphs due to Berge. Then, Theorem 2.2 and Theorem 3.1 give sufficient conditions for a 2-connected non-bipartite graph to be a perfect 2-matching uniform graph, expressed in terms of lower bounds on two graph parameters, the toughness and the binding number. These two parameters were introduced half a century ago by Chv*á*tal and Woodall, respectively. Furthermore, we construct special counterexamples so as to show that Theorem 2.2 and Theorem 3.1 are sharp in some sense, respectively.

2. TOUGHNESS AND PERFECT 2-MATCHING UNIFORM GRAPH

Now, we first present the following result on the existence of perfect 2-matching uniform graphs.

THEOREM 2.1. *A 2-connected non-bipartite graph G is a perfect 2-matching uniform graph if and only if*

$i(G-S) \le |S| - \varepsilon(S)$ for any non-empty subset $S \subseteq V(G)$, in which $\varepsilon(S)$ is represented by

$$\varepsilon(S) = \begin{cases} 3, & if\ (a):\ there\ is\ a\ component\ of\ G-S\ containing\ exactly\ two\ vertices; \\ 2, & if\ (a)\ does\ not\ hold\ and\ there\ is\ a\ component\ C\ of\ G-S\ with\ pendant \\ & edges\ and\ |V(C)| \ge 3; \\ 1, & otherwise. \end{cases}$$

Proof of Theorem 2.1. We first prove the necessity. Suppose that G is a perfect 2-matching uniform graph. Clearly, G is a perfect 2-matching covered graph. Then, by Theorem 1.2, $i(G-S) \le |S| - 1$ for any non-empty subset $S \subset V(G)$. In what follows, we shall consider three cases to prove the necessity.

Case 1. $i(G-S) \le |S| - 3$.

If $i(G-S) \le |S| - 3$, then according to the definition of $\varepsilon(S)$, we have $i(G-S) \le |S| - \varepsilon(S)$.

Case 2. $i(G-S) = |S| - 2$.

In this case, we infer that $\varepsilon(S) \le 2$. Otherwise $\varepsilon(S) = 3$, and thus there is a component of $G-S$ containing exactly two vertices, denoted by u and v. Let edge $e = uv$, it follows that $i(G-e-S) = i(G-S) + 2 = |S|$. If $G-e$ satisfies the hypotheses of Theorem 1.2, then by Theorem 1.2, $G-e$ is not a perfect 2-matching covered graph. Hence, G is not a perfect 2-matching uniform graph, a contradiction. Otherwise $G-e$ is a bipartite graph with partite sets of equal size. Note that G is a 2-connected non-bipartite graph, and thus G is not a perfect 2-matching covered graph. Hence, G is not a perfect 2-matching uniform graph, which is a contradiction. Thus, $\varepsilon(S) \le 2$. It follows that $i(G-S) = |S| - 2 \le |S| - \varepsilon(S)$.

Case 3. $i(G-S) = |S| - 1$.

In this case, we infer that $\varepsilon(S) = 1$. Otherwise $\varepsilon(S) = 2$ or $\varepsilon(S) = 3$, which implies $G-S$ has a component with pendant edges. It follows that $i(G-e-S) \ge i(G-S) + 1 = |S|$. If $G-e$ satisfies the hypotheses of Theorem 1.2, then by Theorem 1.2, $G-e$ is not a perfect 2-matching covered graph. Hence, G is not a perfect 2-matching uniform graph, which is a contradiction. Otherwise $G-e$ is a bipartite graph with partite sets of equal size. Note that G is a 2-connected non-bipartite graph, and thus G is not a perfect 2-matching covered graph. Hence, G is not a perfect 2-matching uniform graph, also a contradiction. Therefore $\varepsilon(S) = 1$ and $i(G-S) = |S| - 1 \le |S| - \varepsilon(S)$. This completes the proof of the necessity of Theorem 2.1.

We next prove the sufficiency of Theorem 2.1. Assume that for any non-empty subset $S \subset V(G)$, $i(G-S) \le |S| - \varepsilon(S)$. Let $H = G-e$ for any $e = xy \in E(G)$. In order to verify that G is a perfect 2-matching uniform graph, it suffices to show that H is a perfect 2-matching covered graph. If e belongs to some component of $G-S$ containing exactly two vertices, then $\varepsilon(S) = 3$. Thus we have $i(G-e-S) = i(G-S) + 2 \le |S| - \varepsilon(S) + 2 = |S| - 1$. By Theorem 1.2 $H = G-e$ is a perfect 2-matching covered graph, and hence G is a perfect 2-matching uniform graph. If e is a pendant edge belongs to a component C of $G-S$ such that $|C| \ge 3$, then $\varepsilon(S) = 2$, and thus $i(G-e-S) = i(G-S) + 1 \le |S| - \varepsilon(S) + 1 \le |S| - 1$. By Theorem 1.2 $H = G-e$ is a perfect 2-matching covered graph, and hence G is a perfect 2-matching uniform graph. Otherwise, $i(G-e-S) = i(G-S)$, and hence $i(G-e-S) = i(G-S) \le |S| - \varepsilon(S) \le |S| - 1$. By Theorem 1.2 $H = G-e$ is a perfect 2-matching covered graph, and hence G is a perfect 2-matching uniform graph. We complete the proof of Theorem 2.1. □

THEOREM 2.2. *Let G be a 2-connected non-bipartite graph with order $n \ge 5$. If $t(G) > 1$, then G is a perfect* 2*-matching uniform graph.*

Proof of Theorem 2.2. We verify the theorem by contradiction. Suppose that G is not a perfect 2-matching uniform graph. Then by Theorem 2.1, there exists some non-empty subset $S \subseteq V(G)$ such that

$$i(G-S) \ge |S| - \varepsilon(S) + 1. \tag{1}$$

CLAIM 1. *G has no pendant edges.*

Proof. Assume that G has a pendant edge $e = xy$ such that $d_G(y) = 1$, then $t(G) \leq \frac{|\{x\}|}{\omega(G-x)} \leq \frac{1}{2}$, which is a contradiction. Claim 1 is verified. □

Next, we shall consider three cases and derive a contradiction in each case.

Case 1. $\varepsilon(S) = 1$.

If $\varepsilon(S) = 1$, then by (1), we have $i(G-S) \geq |S| - \varepsilon(S) + 1 = |S|$. If $|S| \geq 2$, in view of the definition of $t(G)$, we obtain $t(G) \leq \frac{|S|}{\omega(G-S)} \leq \frac{|S|}{i(G-S)} \leq \frac{|S|}{|S|} = 1$, a contradiction. Otherwise $|S| = 1$. Note that G is a 2-connected non-bipartite graph. By $\varepsilon(S) = 1$ and Claim 1, we infer that there is nontrivial component of $G-S$. According to (1), we obtain $\omega(G-S) \geq i(G-S) + 1 \geq |S| - \varepsilon(S) + 2 = |S| + 1 \geq 2$, and thus $t(G) \leq \frac{|S|}{\omega(G-S)} \leq \frac{|S|}{i(G-S)+1} \leq \frac{|S|}{|S|+1} < 1$, which is a contradiction.

Case 2. $\varepsilon(S) = 2$.

If $\varepsilon(S) = 2$, then there is nontrivial component of $G-S$ with pendant edge zw such that $d_{G-S}(w) = 1$. By (1), we have $i(G-S) \geq |S| - \varepsilon(S) + 1 = |S| - 1$. If $|S| \geq 2$, then $\omega(G-S) \geq i(G-S) + 1 \geq |S| - 1 + 1 = |S| \geq 2$. In view of the definition of $t(G)$, we obtain $t(G) \leq \frac{|S|}{\omega(G-S)} \leq \frac{|S|}{i(G-S)+1} \leq \frac{|S|}{|S|} = 1$, a contradiction. Otherwise $|S| = 1$. Note that G is a connected non-bipartite graph. According to (1) and Claim 1, we obtain $t(G) \leq \frac{|S \cup \{z\}|}{\omega(G-S\cup\{z\})} \leq \frac{2}{2} = 1$, also a contradiction.

Case 3. $\varepsilon(S) = 3$.

If $\varepsilon(S) = 3$, then there is a component of $G-S$ containing exactly two vertices, denoted by u and v. By (1), we have

$$i(G-S) \geq |S| - \varepsilon(S) + 1 = |S| - 2, \tag{2}$$

and thus $\omega(G-S) \geq i(G-S) + 1 \geq |S| - 2 + 1 = |S| - 1$.

Subcase 3.1. $i(G-S) \geq |S| - 1$.

If $|S| \geq 2$, in view of the definition of $t(G)$, we obtain $t(G) \leq \frac{|S|}{\omega(G-S)} \leq \frac{|S|}{i(G-S)+1} \leq \frac{|S|}{|S|} = 1$, a contradiction. Otherwise $|S| = 1$. Note that G is a 2-connected non-bipartite graph. By $\varepsilon(S) = 3$, $n \geq 5$ and Claim 1, we infer that $\omega(G-S) \geq 2$. Then $t(G) \leq \frac{|S|}{\omega(G-S)} \leq \frac{1}{2} < 1$, also a contradiction.

Subcase 3.2. $i(G-S) = |S| - 2$.

Note that $\omega(G-S) \geq i(G-S) + 1$. If $\omega(G-S) \geq i(G-S) + 2 = |S|$, then $t(G) \leq \frac{|S|}{\omega(G-S)} \leq \frac{|S|}{|S|} = 1$, which is a contradiction. Otherwise $\omega(G-S) = i(G-S) + 1$. If $i(G-S) \geq 1$, then $t(G) \leq \frac{|\{u\}|}{\omega(G-\{u\})} \leq 1$, a contradiction. If $i(G-S) = 0$, by (2) we obtain $|S| = 1$ or 2. Note that G is a 2-connected non-bipartite graph. By $\varepsilon(S) = 3$, $n \geq 5$ and Claim 1, we infer that $\omega(G-S) \geq 2$. Then $t(G) \leq \frac{|S|}{\omega(G-S)} \leq \frac{2}{2} = 1$. This is a contradiction. We complete the proof of Theorem 2.2. □

Remark 2.1. The condition $t(G) > 1$ in Theorem 2.2 is tight, it cannot be substitute for $t(G) \geq 1$.

We construct a graph $G = C_4 \vee 4K_1$, where C_4 is a cycle with order 4. Obviously, $t(G) = 1$. We choose $S = V(C_4)$. Applying Theorem 1.2, it is easy to observe that graph $G = C_4 \vee 4K_1$ is not perfect 2-matching covered, and thus it is not a perfect 2-matching uniform graph. The example could also be simplified by taking the graph $G = C_3 \vee 3K_1$ by observing that no edge connecting two vertices of degree 4 belongs to a perfect 2-matching.

3. BINDING NUMBER AND PERFECT 2-MATCHING UNIFORM GRAPH

THEOREM 3.1. *Let G be a 2-connected non-bipartite graph with order $n \geq 5$. If $bind(G) > 2$, then G is a perfect 2-matching uniform graph.*

Proof of Theorem 3.1. We verify the theorem by contradiction. Suppose that G is not a perfect 2-matching uniform graph. Then by Theorem 2.1, there exists some non-empty subset $S \subseteq V(G)$ such that

$$i(G-S) \geq |S| - \varepsilon(S) + 1. \tag{3}$$

We infer that G has no pendant edges. We prove it by contradiction. Assume that G has a pendant edge $e = xy$ such that $d_G(x) = 1$, then $bind(G) \leq \frac{|N_G(x)|}{|\{x\}|} = 1 < 2$, which is a contradiction.

Next, we shall consider three cases and derive a contradiction in each case.

Case 1. $\varepsilon(S) = 1$.

If $\varepsilon(S) = 1$, then by (3), we have $i(G-S) \geq |S| - \varepsilon(S) + 1 = |S|$. It follows that $I(G-S) \neq \phi$ and $N_G(I(G-S)) \neq V(G)$. We choose $X = I(G-S)$. Then in view of the definition of binding number, we get $bind(G) \leq \frac{|S|}{i(G-S)} \leq \frac{|S|}{|S|} = 1 < 2$, which is a contradiction.

Case 2. $\varepsilon(S) = 2$.

If $\varepsilon(S) = 2$, then there is nontrivial component C of $G-S$ with pendant edge $e = xy$ such that $d_G(y) = 1$ and $|V(C)| \geq 3$; By (3), we have

$$i(G-S) \geq |S| - \varepsilon(S) + 1 = |S| - 1. \tag{4}$$

Subcase 2.1. $|S| \geq 2$.

If $|S| \geq 2$, we choose $Y = I(G-S) \cup V(C)$, then in view of the definition of binding number and (4), we get $bind(G) \leq \frac{|N_G(Y)|}{|Y|} \leq \frac{|S \cup V(C)|}{i(G-S)+|C|} \leq \frac{|S|+|C|}{|S|-1+|C|} \leq 1 + \frac{1}{|S|+|C|-1} \leq \frac{5}{4} < 2$, which is a contradiction.

Subcase 2.2. $|S| = 1$.

If $|S| = 1$, note that G is a connected non-bipartite graph and has no pendant edges, then $I(G-S) = \phi$. Thus according to (4) and $n \geq 5$, we obtain $bind(G) \leq \frac{|N_G(\{y\})|}{|\{y\}|} \leq 2$, a contradiction.

Case 3. $\varepsilon(S) = 3$.

If $\varepsilon(S) = 3$, then there is a component C of $G-S$ containing exactly two vertices. By (3), we have

$$i(G-S) \geq |S| - \varepsilon(S) + 1 = |S| - 2. \tag{5}$$

Subcase 3.1. $|S| \geq 2$.

If $|S| \geq 2$, we choose $Y = I(G-S) \cup V(C)$, then in view of the definition of binding number and (5), we get $bind(G) \leq \frac{|N_G(Y)|}{|Y|} \leq \frac{|S \cup V(C)|}{i(G-S)+|C|} \leq \frac{|S|+2}{|S|} = 1 + \frac{2}{|S|} \leq 2$, , which is a contradiction.

Subcase 3.2. $|S| = 1$.

If $|S| = 1$, note that G is a 2-connected non-bipartite graph and has no pendant edges, then $I(G-S) = \phi$. Thus according to (5) and $n \geq 5$, we obtain $bind(G) \leq \frac{|N_G(C)|}{|C|} \leq \frac{|S \cup V(C)|}{|C|} \leq \frac{|S|+|C|}{|C|} \leq 1 + \frac{1}{|C|} = \frac{3}{2} < 2$. This is a contradiction. We complete the proof of Theorem 3.1. □

Remark 3.1. The condition $bind(G) > 2$ in Theorem 3.1 is sharp in the following sense.

We construct a graph $G = K_4 \vee (2K_1 \cup K_2)$. Obviously, $bind(G) = \frac{3}{2}$. We choose $S = V(K_4)$, then we obtain $i(G-S) = 2 > |S| - \varepsilon(S) = 4 - 3 = 1$. By Theorem 2.1, G is not a perfect 2-matching uniform graph. Nevertheless, perhaps there exists some real number $c \in (\frac{3}{2}, 2]$ such that every 2-connected non-bipartite graph G with order at least 5 and $bind(G) \geq c$ is a perfect 2-matching uniform graph.

4. CONCLUSIONS

Graph factors have wide range of applications in data transmission networks, network flows, network design, and other fields in computer networks [7–9]. The work on the existence of perfect 2-matching uniform graph is a branch of graph factor theory. Computer experts often use graph parameters (such as toughness, isolated toughness, binding number, neighborhood union, neighbor set and minimum degree) to measure the robustness and vulnerability of network attacks. It is crucial for network security to consider graph parameters from the perspective of the topology of a graph.

In this article, we first define the new concept of perfect 2-matching uniform graph. And next we acquired a necessary and sufficient condition to characterize a perfect 2-matching uniform graph. In addition, several sufficient conditions on toughness and binding number for the existence of perfect 2-matching uniform graphs are presented. It is pointed out that these conditions are best possible in some sense. The relationship between other graph parameters and perfect 2-matching uniform graphs can be studied further, which may also help scientists to design and construct networks with high data transmission.

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