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FINITE GROUPS IN WHICH EVERY NON-NILPOTENT MAXIMAL INVARIANT SUBGROUP OF ORDER DIVISIBLE BY p IS A TI-SUBGROUP

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Abstract. Let A and G be finite groups such that A acts coprimely on G by automorphisms. We obtain a complete description of the structure of finite groups in which every non-nilpotent maximal A-invariant subgroup of order divisible by p is a TI-subgroup for any fixed prime divisor p of the order of groups.

Keywords: coprime action, non-nilpotent maximal A-invariant subgroup, TI-subgroup.
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1. INTRODUCTION

All groups are considered to be finite. Let G be a group and H a subgroup of G. If $H^g \cap H = 1$ or H for each $g \in G$, then H is called a TI-subgroup of G. In [10, Theorem 1.1] Shi and Zhang proved that every non-nilpotent subgroup of a group G is a TI-subgroup if and only if every non-nilpotent subgroup of G is normal. As a generalization, Shi and Li [7, Theorem 1] showed that every self-centralizing non-nilpotent subgroup of a group G is a TI-subgroup or a subnormal subgroup if and only if every non-nilpotent subgroup of G is subnormal. Consider the coprime action of groups, let A and G be groups such that A acts coprimely on G by automorphisms. Shao and Beltrán [5, Theorem B] obtained that every non-nilpotent A-invariant subgroup of G is a TI-subgroup if and only if every non-nilpotent A-invariant subgroup of G is normal. Moreover, Liu and Shi [2, Theorem 1.1] showed that every self-centralizing non-nilpotent A-invariant subgroup of G is a TI-subgroup or a subnormal subgroup if and only if every non-nilpotent A-invariant subgroup of G is subnormal.

Consider non-nilpotent maximal subgroups of a group, Lu, Pang and Zhong [3, Theorem 3.5] proved that every non-nilpotent maximal subgroup of a group G is a TI-subgroup if and only if every non-nilpotent maximal subgroup of G is normal. Combine the coprime action of groups, Shi and Liu [9, Theorem 1.8] obtained that every non-nilpotent maximal A-invariant subgroup of a group G is a TI-subgroup if and only if every non-nilpotent maximal A-invariant subgroup of G is normal.

Let G be a group and p any fixed prime divisor of $|G|$. In [8, Theorems 1.3, 1.4 and 1.6] Shi, Li and Shen provided a complete description of group G in which every non-nilpotent maximal subgroup of order divisible by p is normal. Moreover, Beltrán and Shao [1, Theorem C] indicated that G is solvable if every non-nilpotent maximal A-invariant subgroup of G of order divisible by p is normal.

As a generalization of above research, Shi [6, Theorem 1.1] had the following result.

THEOREM 1.1. [6, Theorem 1.1] *Let G be a group and p a fixed prime divisor of $|G|$. Then every maximal subgroup of G is nilpotent or a TI-subgroup or has p'-order if and only if one of the following statements holds:*

(1) *Every maximal subgroup of G is nilpotent or normal or has p'-order;*

(2) *$G = {Z_q}^m \rtimes (Z_p \rtimes H)$ is a Frobenius group with ${Z_q}^m$ being its Frobenius kernel and $Z_p \rtimes H$ being its Frobenius complement such that $Z_p \rtimes H$ acts irreducibly on ${Z_q}^m$, where $m > 1$, $Z_p \in \mathrm{Syl}_p(G)$, H is cyclic or a direct product of a generalized quaternion group and a cyclic group such that $[Z_p, H] \neq 1$.*

In this paper, as a further generalization of [6, Theorem 1.1], consider the coprime of groups, we obtain the following result, the proof of which is given in Section 3.

THEOREM 1.2. *Let A and G be groups such that A acts coprimely on G by automorphisms, let p be any fixed prime divisor of $|G|$. Then every non-nilpotent maximal A-invariant subgroup of G of order divisible by p is a TI-subgroup if and only if one of the following statements holds:*

(1) *every non-nilpotent maximal A-invariant subgroup of G of order divisible by p is normal;*

(2) *$G = {Z_q}^m \rtimes (Z_p \rtimes K)$ is a Frobenius group with kernel ${Z_q}^m$ and complement $Z_p \rtimes K$, where ${Z_q}^m$ is a minimal A-invariant normal subgroup of G and $m > 1$, Z_p is an A-invariant Sylow subgroup of G and K is an A-invariant cyclic group or an A-invariant direct product of a generalized quaternion group and a cyclic group of odd order with $[Z_p, K] \neq 1$.*

The following result is a direct corollary of Theorem 1.2.

COROLLARY 1.3. *Let A and G be groups such that A acts coprimely on G by automorphisms, let p be the smallest prime divisor of $|G|$. Then every non-nilpotent maximal A-invariant subgroup of G of order divisible by p is a TI-subgroup if and only if every non-nilpotent maximal A-invariant subgroup of G of order divisible by p is normal.*

2. TWO NECESSARY LEMMAS

LEMMA 2.1. [5, Lemma 2.3] *Suppose that a group A acts coprimely on a group G. If every maximal A-invariant subgroup of G is normal, then G is nilpotent.*

LEMMA 2.2. [9, Lemma 2.3] *Let A and G be groups such that A acts coprimely on G by automorphisms. Suppose that M is a maximal A-invariant subgroup of G, then M is either self-normalizing in G or normal in G.*

3. PROOF OF THEOREM 1.2

Proof. For the necessity part.

When every non-nilpotent maximal A-invariant subgroup of G of order divisible by p is normal, it is obvious that every non-nilpotent maximal A-invariant subgroup of G of order divisible by p is a TI-subgroup.

When G has non-normal non-nilpotent maximal A-invariant subgroups of order divisible by p. Suppose that M is a non-normal non-nilpotent maximal A-invariant subgroup of G of order divisible by p. It is easily seen that $N_G(M)$ is also a non-nilpotent A-invariant subgroup of G of order divisible by p and $N_G(M) < G$. Therefore, $M = N_G(M)$ by the maximality of M. Since M is a TI-subgroup of G by the hypothesis, it follows that G is a Frobenius group with complement M. Let N be the kernel. Then $G = N \rtimes M$.

Note that both N and M are Hall-subgroups of G. One has that N is a characteristic subgroup of G, which implies that N is also an A-invariant subgroup of G. By the maximality of M, N is a minimal A-invariant normal subgroup of G. It follows that N is a characteristic simple group which is a direct product of some isomorphic simple groups. Since N is nilpotent by [4, Theorem 10.5.6], N is an elementary abelian group. Let $N = {Z_q}^m$, where $q \neq p$, $m > 1$ since M is non-nilpotent.

Let P be an A-invariant Sylow p-subgroup of M. We will show that P is normal in M. Assume that P is not normal in M. Then $N_M(P) < M$. Note that $N_M(P)$ is also an A-invariant subgroup of M. It follows that there is a maximal A-invariant subgroup M_0 of M such that $N_M(P) \leq M_0$. Since $N \rtimes M_0$ is a non-nilpotent maximal A-invariant subgroup of G of order divisible by p, $N \rtimes M_0$ is a TI-subgroup of G. However, $(N \rtimes M_0)^x \cap (N \rtimes$

$M_0) = (N \rtimes M_0{}^x) \cap (N \rtimes M_0) \geq N \neq 1$ for each $x \in G$, which implies that $N \rtimes M_0$ is normal in $G = N \rtimes M$. It follows that M_0 is normal in M. By Frattini's argument, one has $M = N_M(P)M_0 = M_0$, a contradiction. Therefore, P is normal in M.

If every maximal A-invariant subgroup of M is normal, then M is nilpotent by Lemma 2.1, a contradiction. Thus M has non-normal maximal A-invariant subgroups. Let K be a non-normal maximal A-invariant subgroup of M. By above argument, one has that $|K|$ is not divisible by p. Then $P \cap K = 1$. It follows that $M = P \rtimes K$. By the maximality of K, P is a minimal A-invariant normal subgroup of M, which implies that P is an elementary abelian subgroup. Then P is a cyclic group of order p by [4, Theorem 10.5.6]. Let $P = Z_p$. Then $M = Z_p \rtimes K$. Suppose that K_0 is any maximal A-invariant subgroup of K. Then $N \rtimes (Z_p \rtimes K_0)$ is a non-nilpotent maximal A-invariant subgroup of G of order divisible by p. Arguing as above, one has that K_0 is normal in K, which implies that K is nilpotent. Then K is cyclic or a direct product of a generalized quaternion group and a cyclic group of odd order by [4, Theorem 10.5.6] and $[Z_p, K] \neq 1$ since M is non-nilpotent.

For the sufficiency part.

We only need to verify group G that belongs to case (2). Let L be any non-nilpotent maximal A-invariant subgroup of G of order divisible by p. Note that ${Z_q}^m$ is a minimal A-invariant normal subgroup of G.

When the case that ${Z_q}^m \nleq L$. Then $G = {Z_q}^m L$. It is easy to see that ${Z_q}^m \cap L$ is an A-invariant normal subgroup of G and ${Z_q}^m \cap L < {Z_q}^m$, by the minimality of ${Z_q}^m$, one has ${Z_q}^m \cap L = 1$. It follows that $G = {Z_q}^m \rtimes L$. Note that ${Z_q}^m$ is a normal Hall-subgroup of G. By Schur-Zassenhaus theorem, one has that L and $Z_p \rtimes K$ are conjugate in G. It follows that L is also a Frobenius complement of G and then L is a TI-subgroup of G.

When the case that ${Z_q}^m < L$. Then $L = {Z_q}^m \rtimes (L \cap (Z_p \rtimes K)) = {Z_q}^m \rtimes (Z_p \rtimes (L \cap K))$ since $|L|$ is divisible by p. Note that $L \cap K$ is a maximal A-invariant subgroup of K and K is nilpotent, one has that $L \cap K$ is normal in K by Lemma 2.2, which implies that $L = {Z_q}^m \rtimes (Z_p \rtimes (L \cap K))$ is normal in G.

□

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