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# A REALISTIC NOTION OF A RIGHT GYROGROUP ACTION

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**Abstract.** The main aim of this paper is to introduce a realistic notion of an action of a right gyrogroup. Moreover, we see that the associated isotropy set is an invariant right subgyrogroup. At the end, we pose a problem for future.

***Keywords**:* gyrogroup, right gyrogroup, group action.
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## 1. INTRODUCTION

In 1988, Ungar introduced a non-associative algebraic structure, *gyrogroup,* which is a generalization of groups. The concept of gyrogroups first came into the picture in the study of Einstein's relativistic velocity addition law [8, 11]. An elegant introduction of Möbius gyrogroups is presented in [9], and the interdisciplinarity of gyrogroups is indicated in [10].

In 2001, Foguel and Ungar [1, 2] gave examples of finite and infinite gyrogroups, both gyrocommutative and non-gyrocommutaive, in order to demonstrate that gyrogroups abound in group theory. The authors also introduced a weaker structure called a left (right) gyrogroup and showed that any given group can be turned into a left (right) gyrogroup. Further they proved that for any group, the associated left (right) gyrogroup is a gyrogroup if and only if the group is central by a 2-Engel group (see [2, Theorem 3.7]). In particular, if the given group is nilpotent group of class 3, then the associated left (right) gyrogroup is a gyrogroup. It is also shown that the associated right gyrogroup of a group is a group if and only if the group is nilpotent of class 2 (see [2, Theorem 3.6]).

In 2013, Lal et al. [4] studied topological right gyrogroups. In 2014, Lal et al. [3] worked on a classification problem for finite groups with isomorphic associated right gyrogroups. In [5], authors studied the properties of the associated right gyrogroup of a given group of nilpotency class 3. They proved that if 3 does not divide the order of the group, then the nilpotency class of the associated right gyrogroup is same as that of the group.

In this article, we introduce a more realistic notion of a right gyrogroup action/representation and try to develop the theory accordingly. We show that every group action is a right gyrogroup action for the group based right gyrogroup but converse need not be true. We give a sufficient condition under which a right gyrogroup action for the group based right gyrogroup is a group action. We also show that the associated isotropy set is an invariant right subgyrogroup. We introduce a relation associated with this action which turns out to be reflexive and symmetric. It is difficult to say anything about the transitivity of the relation which leads us to state a problem for future research.

Now, we recall the definition of a right gyrogroup and its properties which we will use further.

## 2. PRELIMINARIES

In this section, we give definition and some properties of right gyrogroup. We also state a result related to congruence in a right gyrogroup.

*Definition* 1 [4]. A groupoid $(G,\circ)$ is called a right gyrogroup if

1. there is an element $e$ in $G$ such that $a\circ e=a$, for all $a\in G$ (right identity);
2. for every $a\in G$, there exists an element $a'\in G$ such that $a\circ a'=e$ (right inverse);
3. for all $y,z\in G$, there exists an automorphism $\mathrm{gyr}[y,z]:G\longrightarrow G$ such that $(x\circ y)\circ z=\mathrm{gyr}[y,z](x)\circ(y\circ z)$, for all $x\in G$;
4. for all $y\in G$, $\mathrm{gyr}[y,y']=I_G$, where $y'$ is the right inverse of $y$.

Now, we give some properties of a right gyrogroup discussed in [4].

PROPOSITION 1. *Let $(G,\circ)$ be a right gyrogroup. Then*

1. *$a'$ is also the left inverse of $a$, for all $a\in G$.*
2. *$e$ is also the left identity of $G$.*
3. $\mathrm{gyr}[e,x]=I_G=\mathrm{gyr}[x,e]$, *for all* $x\in G$.
4. $\mathrm{gyr}[x,y]\mathrm{gyr}[x\circ y,z]=\mathrm{gyr}[y,z]\mathrm{gyr}[\mathrm{gyr}[y,z](x),y\circ z]$ *for all* $x,y,z\in G$.
5. $\mathrm{gyr}[x',x]=I_G$, *for all* $x\in G$.

*Note.* Now onwards, we denote $a'$ by $a^{-1}$. Also, by using identities (4) and (5) of Proposition 1, we have $\mathrm{gyr}^{-1}[x,y]=\mathrm{gyr}[\mathrm{gyr}[x,y](x^{-1}),x\circ y]$.

PROPOSITION 2. *Let $(G,\cdot)$ be a group. Define a binary operation $\circ$ on $G$ by $a\circ b=b^{-1}ab^2$. Then, $(G,\circ)$ is a right gyrogroup with gyro automorphism $\mathrm{gyr}[a,b]:G\longrightarrow G$ defined by $\mathrm{gyr}[a,b](c)=[b^{-1},a]c[a,b^{-1}]$, where $[a,b]$ denotes the commutator of $a,b$ in $G$. We call $(G,\circ)$ the $G$-based right gyrogroup.*

*Definition* 2 [6]. An equivalence relation $R$ on a right gyrogroup $G$ is called a congruence in $G$ if $R$ is a right subgyrogroup of $G\times G$.

The following result is a particular case of [6, Theorem 2.7].

THEOREM 1. *Let $R$ be a congruence on a right gyrogroup $G$ and $R_e$ the equivalence class determined by the identity $e$. Then*

1. *$R_e$ is a right subgyrogroup of $G$.*
2. $\mathrm{gyr}[y,z](x)\in R_e$ *and* $\mathrm{gyr}^{-1}[y,z](x)\in R_e$ *for all* $x\in R_e$ *and* $y,z\in G$.
3. $(y\circ(x\circ z))\circ(y\circ z)^{-1}\in R_e$ *for all* $x\in R_e$ *and* $y,z\in G$.
4. $\mathrm{gyr}[x,y](z\circ x^{-1})\circ z^{-1}\in R_e$ *for all* $x\in R_e$ *and* $y,z\in G$.
5. $(y,z)\in R$ *and* $y\circ z^{-1}\in R_e$.
6. $R=\{(a,b)\in G\times G\mid a=x\circ b$ *for some* $x\in R_e\}$

*Conversely, let $H\neq\{e\}$ be a sub right gyrogroup of $G$ satisfying $(2)-(4)$, then there is a unique congruence $R$ on $G$ such that $R_e=H$.*

## 3. MAIN RESULTS

Now, we give a more realistic action of a right gyrogroup. Further, we show that every right gyrogroup action gives a right gyrogroup homomorphism and vice-versa.

We use notations 1 or $e$ for the identity element.

*Definition* 3. Let $(G,\circ)$ be a right gyrogroup and $X$ be a non empty set. An action of $G$ on $X$ is a map $*: G\times X \longrightarrow X$ such that

1. $1*x=x$
2. $(a\circ b)*x=b^{-1}*(a*(b*(b*x)))$, for all $a,b\in G$ and $x\in X$.

*Definition* 4. Let $(G,\circ)$ be a right gyrogroup and $(G',\odot)$ be a group. A map $\phi : G\to G'$ is called a right gyrogroup homomorphism if

$$\phi(a\circ b)=(\phi(b))^{-1}\odot\phi(a)\odot(\phi(b))^2.$$

THEOREM 2. *Let $(G,\circ)$ be a right gyrogroup and $*$ be an action of G on a set X. For each $a\in G$, define a map $f_a : X\longrightarrow X$, by $f_a(x)=a*x$. Then*

(1) *$f_a\in \mathrm{Sym}(X)$, $f_{a^{-1}}=f_a^{-1}$ and $f_{a^2}=f_a^2$.*

(2) *Define $\phi : G\to \mathrm{Sym}(X)$ by $\phi(a)=f_a$. Then the map $\phi$ is a right gyrogroup homomorphism from G to $\mathrm{Sym}(X)$ with $\ker\phi=\{a\in G : a*x=x\,\forall x\in X\}$.*

(3) *Let $\hat{\phi} : G\to \mathrm{Sym}(X)$ be a right gyrogroup homomorphism. The map $\hat{*}$ defined by $a\hat{*}x=\hat{\phi}(a)(x)$, $a\in G, x\in X$ is a right gyrogroup action of G on X.*

*Proof.* Suppose $f_a(x)=f_a(y)$ for some $x,y\in X$. Then

$$\begin{aligned}
f_a(x)=f_a(y) &\Rightarrow a*x=a*y\\
&\Rightarrow a^{-1}*(a^{-1}*(a*(a*x)))=a^{-1}*(a^{-1}*(a*(a*y)))\\
&\Rightarrow (a\circ a^{-1})*x=(a\circ a^{-1})*y\\
&\Rightarrow 1*x=1*y\Rightarrow x=y
\end{aligned}$$

Hence, $f_a$ is one one. Now let $x\in X$. Then $f_a(a*(a^{-1}*(a^{-1}*x)))=a*(a*(a^{-1}*(a^{-1}*x)))=(a^{-1}\circ a)*x=x$. Therefore, $f_a$ is onto as well so that $f_a\in\mathrm{Sym}(X)$.

For $a\in G$, we have $a=(a\circ a^{-1})\circ a$. Thus

$$\begin{aligned}
a*x&=((a\circ a^{-1})\circ a)*x\\
&=a^{-1}*((a\circ a^{-1})*(a*(a*x)))\\
&=a^{-1}*(a*(a*(a^{-1}*(a^{-1}*(a*(a*x))))))\\
&=a^{-1}*(a*(a*((a^{-1}\circ a)*x)))\\
&=a^{-1}*(a*(a*x))
\end{aligned}$$

Therefore, $a^{-1}*(a*x)=a^{-1}*(a^{-1}*(a*(a*x)))=(a^{-1}\circ a)*x=x$. Thus $f_{a^{-1}}=f_a^{-1}$. It is easy to see that $f_{a^2}=f_a^2$.

Let $a,b\in G$. Then $\phi(a\circ b)=f_{a\circ b}$ and $f_{a\circ b}(x)=(f_{b^{-1}}\circ f_a\circ f_b^2)(x)$ (here $\circ$ is composition of maps). Thus, $f_{a\circ b}(x)=((f_b)^{-1}\circ f_a\circ f_b^2)(x)\Rightarrow\phi(a\circ b)=(\phi(b))^{-1}\phi(a)(\phi(b))^2$, i.e., $\phi$ is a right gyrogroup homomorphism. Then

$$\begin{aligned}
\ker\phi&=\{a\in G\mid \phi(a)=Id_X\}\\
&=\{a\in G\mid f_a=Id_X\}\\
&=\{a\in G\mid a*x=x\,\forall x\in X\}.
\end{aligned}$$

To prove the third statement, note that $e\hat{*}x = \hat{\phi}(e)(x) = Id_X(x) = x$ for all $x \in X$. For $a,b \in G$ and $x \in X$, $b^{-1}\hat{*}(a\hat{*}(b\hat{*}(b\hat{*}x))) = (\hat{\phi}(b^{-1}) \circ \hat{\phi}(a) \circ \hat{\phi}(b) \circ \hat{\phi}(b))(x) = \hat{\phi}(a \circ b)(x) = (a \circ b)\hat{*}x$. □

PROPOSITION 3. *Let $G$ be group and $(G,\circ)$ be $G$-based right gyrogroup. Then every group action of $G$ on a set $X$ is a right gyrogroup action of $(G,\circ)$ on $X$.*

*Proof.* Let $*$ be an action of $G$ on $X$. Then $e * x = x$ and $(a \circ b) * x = (b^{-1}ab^2) * x = b^{-1} * (a * (b * (b * x)))$. This shows that $*$ is a right gyrogroup action of $(G,\circ)$ on the set $X$. □

The following example shows that a right gyrogroup action of $(G,\circ)$ need not be a group action of $G$.

*Example* 1. Consider the non-abelian group $G = \langle a,b | a^3 = b^3 = e = [a,b]^3 = [a,[a,b]] = [b,[a,b]]\rangle$ of order 27. Then the $G$-based right gyrogroup $(G,\circ)$ is an abelian group.

Let $*$ be the left action of the group $(G,\circ)$ on $G$, that is, $g * x = g \circ x$. Then, it is easy to see that

$$\begin{aligned}(g_1 \circ g_2) * x &= (g_1 \circ g_2) \circ x\\ &= (g_1 \circ g_2) \circ x\\ &= g_2^{-1} \circ (g_1 \circ (g_2 \circ (g_2 \circ x))) \text{ (by using commutative property of } \circ)\\ &= g_2^{-1} * (g_1 * (g_2 * (g_2 * x)))\end{aligned}$$

This shows that $*$ is a right gyrogroup action of $(G,\circ)$. We can also conclude it by the following argument:

Since $(G,\circ)$ is a group and $*$ be an action of $(G,\circ)$, by Proposition 2, $*$ is also a right gyrogroup action of $(G,\circ_1)$, where $(G,\circ_1)$ is $(G,\circ)$-based right gyrogroup. Since $(G,\circ)$ is an abelian group, $g_1 \circ_1 g_2 = g_2^{-1} \circ g_1 \circ g_2^2 = g_1 \circ g_2$. Hence, $*$ is a right gyrogroup action of $(G,\circ)$.

Now finally we show that $*$ is not a group action of $G$. It is enough to show that

$$(ab^{-1}) * a \neq a * (b^{-1} * a).$$

Suppose, $(ab^{-1}) * a = a * (b^{-1} * a)$
$\Rightarrow a^{-1}(ab^{-1})a^2 = a * (a^{-1}b^{-1}a^2)$
$\Rightarrow b^{-1}a^2 = (a^{-2}ba)(a)(a^{-1}b^{-1}a^2)(a^{-1}b^{-1}a^2)$
$\Rightarrow b^{-1}a^2 = abab^{-1}ab^{-1}a^2$
$\Rightarrow e = a[b,a]a^2$
$\Rightarrow ab = ba$, which is a contradiction. Therefore, $*$ is not an action of $G$.

In the next proposition, we give a sufficient condition such that every group based right gyrogroup action is a group action.

THEOREM 3. *Let $G$ be a finite group and $(G,\circ)$ be the $G$-based right gyrogroup. If $3 \nmid |G|$, then every action of $(G,\circ)$ is also an action of $G$.*

*Proof.* The theorem follows from Theorem 3 of [7]. But for the convenience of readers, we would like to give the proof on the same line of Theorem 3 of [7]. To prove the theorem, it suffices to prove that every right gyrogroup homomorphism from $(G,\circ)$ to $\mathrm{Sym}(X)$ is a group homomorphism from $G$ to $\mathrm{Sym}(X)$.

Let $\phi : (G,\circ) \longrightarrow \mathrm{Sym}(X)$ be a right gyrogroup homomorphism, that is,

$$\phi(b^{-1}ab^2) = \phi(b)^{-1}\phi(a)\phi(b)^2, \text{ for all } a,b \in G. \tag{1}$$

CLAIM. $\phi(ab) = \phi(a)\phi(b)$, for all $a,b \in G$.

It is easy to see that $\phi(e) = e, \phi(a^{-1}) = \phi(a)^{-1}$ and $\phi(a^2) = \phi(a)^2$, for all $a \in G$. By replacing $a$ by $bab^{-1}$ in Equation (1), we have

$$\phi(ab) = \phi(b)^{-1}\phi(bab^{-1})\phi(b)^2, \text{ for all } a,b \in G. \tag{2}$$

By replacing $a$ by $a^{-1}b^{-1}$ in Equation (1), we have $\phi(b^{-1}a^{-1}b) = \phi(b)^{-1}\phi(a^{-1}b^{-1})\phi(b)^2$. Since $\phi(a^{-1}) = \phi(a)^{-1}$, we have $\phi(b^{-1}ab) = \phi(b)^{-2}\phi(ba)\phi(b)$. Hence,

$$\phi(bab^{-1}) = \phi(b)^2\phi(b^{-1}a)\phi(b)^{-1}.$$

By putting the value of $\phi(bab^{-1})$ in Equation (2), we have

$$\phi(ab) = \phi(b)\phi(b^{-1}a)\phi(b), \text{ for all } a,b \in G. \tag{3}$$

By replacing $a$ by $ba$ in Equation (3), we have

$$\phi(bab) = \phi(b)\phi(a)\phi(b), \text{ for all } a,b \in G. \tag{4}$$

By replacing $a$ by $b^{-1}ab^2$ in Equation (4), we have

$$\phi(ab^3) = \phi(a)\phi(b)^3, \text{ for all } a,b \in G. \tag{5}$$

By taking $a = e$ in Equation (5), we have

$$\phi(b^3) = \phi(b)^3, \text{ for all } b \in G. \tag{6}$$

Since $3 \nmid |G|$, $3 \nmid o(b)$, where $o(b)$ denotes the order of $b$. Thus $gcd(3, o(b)) = 1$, that is, $1 = 3x + o(b)y$ for some integers $x$ and $y$. Therefore, we have $b = b^{3x}$.

Then from equation (5) and (6),

$$\phi(ab) = \phi(ab^{3x}) = \phi(a(b^x)^3) = \phi(a)(\phi(b^x))^3 = \phi(a)\phi(b^{3x}) = \phi(a)\phi(b) \text{ for all } a,b \in G.$$

□

Next, we discuss an example of a right gyrogroup for which every right gyrogroup action is trivial.

*Example* 2 [4, Corollary 5.13]. Let $G$ be a group and $H$ be a subgroup of $G$. Let $S$ be a right transversal of $H$ in $G$. Then $(S,\circ)$ is a groupoid where $\circ$ is defined by

$$x \circ y = Hxy \cap S.$$

Now, consider $G = \mathrm{Sym}(n)$, $H = \mathrm{Sym}(n-1)$ and $S = \{I, (1\ n), (2\ n), \cdots (n-1\ n)\}$. For $1 \le i \ne j \le n-1$, we have

$H(i\ n)(j\ n) = H(i\ n\ j) = H(i\ j)(i\ n) = H(i\ n)$.

Hence, $(i\ n) \circ (jn) = (i\ n)$, for $i \ne j$.

Then $(S,\circ)$ is a right gyrogroup. In this case, $\mathrm{gyr}[a,b] : S \longrightarrow S$ is given by

$$\mathrm{gyr}[a,b](c) = \begin{cases} c, \text{ if } c = b \\ b, \text{ if } a = c \\ c, \text{ if } a = b \\ c, \text{ if } a \ne b \ne c \end{cases}$$

Let $*$ be an action of $S$ on a non empty set $X$. Then we have a homomorphism $\phi : S \longrightarrow \mathrm{Sym}(X)$ by

$\phi(x \circ y) = \phi(y)^{-1}\phi(x)\phi(y)^2$

$\Rightarrow \phi((i\ n) \circ (j\ n)) = \phi((j\ n))^{-1}\phi(i\ n)\phi((j\ n))^2$

$\Rightarrow \phi((i\ n)) = \phi((j\ n))\phi((i\ n))\phi((j\ n))^2$

$\Rightarrow \phi((i\ n)) = \phi((j\ n))\phi((i\ n))$

$\Rightarrow \phi((j\ n)) = I$, for all $1 \le j \le n-1$.

Hence $\phi$ is a trivial homomorphism, which implies $*$ is a trivial right gyrogroup action.

Here, we use the fact that a right gyrogroup is a particular case of right loops and adopt certain notions and terminologies from [6] to define an invariant right subgyrogroup. Further, we show that the isotropy subgyrogroup of a right gyrogroup action is an invariant subgyrogroup.

*Definition* 5. A right subgyrogroup of a right gyrogroup $G$ will be called an invariant right subgyrogroup if it satisfies the conditions $(2),(3)$ and $(4)$ of Theorem 1.

PROPOSITION 4. *Let $G$ be a right gyrogroup and $*$ be an action of $G$ on a non empty set $X$. Then the isotropy set $G_X = \{a \in G \mid a*x = x \; \forall x \in X\}$ is an invariant right subgyrogroup of $G$.*

*Proof.* First, we show that $G_X$ is a right subgyrogroup. Let $a \in G_X$. Then $a^{-1}*x = a^{-1}*(e*(a*(a*x))) = (e\circ a)*x = a*x = x \; \forall x \in X$. Let $a,b \in G_X$. Then $(a\circ b^{-1})*x = b*(a*(b^{-1}*(b^{-1}*x))) = x \; \forall x \in X$. Thus, $G_X$ is a right subgyrogroup.

For $x \in X$, $a \in G_X$ and $b,c \in G$,

$$\begin{aligned} ((a\circ b)\circ c)*x &= c^{-1}*((a\circ b)*(c*(c*x))) \\ &= c^{-1}*(b^{-1}*(a*(b*(b*(c*(c*x)))))) \\ &= c^{-1}*(b^{-1}*(b*(b*(c*(c*x))))) \\ &= c^{-1}*((e\circ b)*(c*(c*x))) \\ &= c^{-1}*(b*(c*(c*x))) \\ &= (b\circ c)*x. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathrm{gyr}[b,c](a)*x &= ((a\circ b)\circ c)\circ(boc)^{-1})*x \\ &= (b\circ c)*((a\circ b)\circ c))*(b\circ c)^{-1}*(b\circ c)^{-1}*x \\ &= (b\circ c)*(b\circ c))*(b\circ c)^{-1}*(b\circ c)^{-1}*x \\ &= x. \end{aligned}$$

Thus, $\mathrm{gyr}[b,c](a) \in G_X$ for $a \in G_X$ and $b,c \in G$. Note that $\mathrm{gyr}^{-1}[b,c](a) = \mathrm{gyr}[\mathrm{gyr}[b,c](b^{-1}),(b\circ c)](a)$ and hence, $\mathrm{gyr}^{-1}[b,c](a)*x = x$. This proves the identity (2) of Theorem 1.

Observe, for $a \in G_X$ and $b \in G$, $(a\circ b)*x = b*x$ and $(a\circ b)^{-1}*x = b^{-1}*x$. Now suppose $a \in G_X$ and $b,c \in G$.

$$\begin{aligned} (b\circ(a\circ c))*x &= (a\circ c)^{-1}*(b*((a\circ c)*((a\circ c)*x))) \\ &= c^{-1}*(b*((a\circ c)*((a\circ c)*x))) \\ &= c^{-1}*(b*((c*(c*x))) \\ &= (b\circ c)*x. \end{aligned}$$

Therefore,

$$\begin{aligned} ((b\circ(a\circ c))\circ(b\circ c)^{-1})*x &= (b\circ c)*((b\circ(a\circ c))*((b\circ c)^{-1}*((b\circ c)^{-1}*x))) \\ &= (b\circ c)*((b\circ c)*((b\circ c)^{-1}*((b\circ c)^{-1}*x))) \\ &= ((b\circ c)\circ(b\circ c)^{-1})*x \\ &= x. \end{aligned}$$

This proves the identity (3) of Theorem 1.

To prove the identity (4) of Theorem 1, we first observe that $\mathrm{gyr}[a,b](c\circ a^{-1})\circ c^{-1} = ((c\circ b)\circ(a\circ b)^{-1})\circ c^{-1}$.

Therefore,

$$\begin{aligned}(((c\circ b)\circ(a\circ b)^{-1})\circ c^{-1})*x &= c*(((c\circ b)\circ(a\circ b)^{-1})*(c^{-1}*(c^{-1}*x)))\\ &= c*((b*((c\circ b)*(b^{-1}*(b^{-1}*((c^{-1}*(c^{-1}*x))))\\ &= c*((c\circ b)\circ b^{-1})*((c^{-1}*(c^{-1}*x))))\\ &= c*((c*((c^{-1}*(c^{-1}*x))))\\ &= x\end{aligned}$$

Thus, $G_X$ is an invariant right subgyrogroup of $G$. □

*Definition* 6. Let $H$ be an invariant right subgyrogroup of a right gyrogroup $G$. The set $G/H = \{H\circ g \mid g \in G\}$ is a right gyrogroup under the following operation

$$(H\circ a)\circ(H\circ b) = H\circ(a\circ b),$$

where $H\circ a, H\circ b \in G/H$. The right gyrogroup $(G/H,\circ)$ is called the quotient right gyrogroup with gyro automorphism $\mathrm{gyr}[(H\circ a),(H\circ b)](H\circ c) = H\circ\mathrm{gyr}[a,b](c)$.

PROPOSITION 5. *Let $G$ be a right gyrogroup and $*$ be an action of $G$ on a non empty set $X$. Let $G_X = \{a\in G \mid a*x = x\ \forall x\in X\}$. Then $G/G_X$ is isomorphic to the right subgyrogroup $\phi(G)$ of* $\mathrm{Sym}(X)$*, where $\phi$ is the corresponding representation.*

*Remark* 1. Let $G$ be a right gyrogroup and $*$ be an action of $G$ on a non empty set $X$. The stabilizer of $x$ in G, denoted by stab $x$, is defined as stab $x = \{a\in G : a*x = x\}$. Then stab $x$ is a right subgyrogroup of $G$.

PROBLEM. Let $*$ be an action of a right gyrogroup $G$ on a non empty set $X$. Define a relation $\sim$ on $X$ by the condition $x\sim y \Leftrightarrow y = a*x$ for some $a\in G$. Then the relation $\sim$ is reflexive and symmetric. But we are not able to say anything about transitivity. Also we are not able to get any example of an action such that the relation defined above is not transitive. So it will be interesting to see the problem ”Whether $\sim$ is a transitive relation”. If answer is yes, then one can develop the theory of this action analogous to theory of natural action. If answer is no, then one can develop the theory of this action analogous to theory of natural action by taking the transitive closure of the relation $\sim$ .

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