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HEAT TRANSFER WITH INTERFACIAL BARRIER IN A FINE SCALE MIXTURE OF TWO HIGHLY DIFFERENT CONDUCTIVE MATERIALS

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Abstract. The paper deals with the asymptotic behavior of the heat transfer in a bounded domain formed by two ε-periodically interwoven components, of highly different conductivities. Both components might be connected. At the interface, the heat flux is continuous and the temperature subjects to a first-order jump condition. The homogeneous Dirichlet condition is imposed on the exterior boundary. We determine the macroscopic law when the order of magnitude of the jump transmission coefficient is $\varepsilon^r, -1 < r \leq 1$, using the two-scale convergence technique of the homogenization theory.

Keywords: homogenization, two-scale convergence, heat conduction, first-order jump interface, jump transmission coefficient.

Mathematics Subject Classification (MSC2020): 35B27, 80M40, 76M50.

1. INTRODUCTION

In this paper we study the asymptotic behavior of the heat transfer problem in a bounded domain formed by two ε-periodically interwoven components, when ε goes to 0. Both components might be connected. The ambient component has conductivity of unity order and it is the only one which is reaching the boundary of the domain. The second component contains the core conductor and its conductivity is of ε^2-order. The homogeneous Dirichlet condition is imposed on the exterior boundary. The heat flux is continuous at the interface, where the temperature subjects to a first-order jump condition. We study the case when the jump transmission coefficient of the interface is of ε^r-order, with $-1 < r \leq 1$.

Similar macroscopic systems were studied in the context of miscible displacements, heat transfer, oil reservoir simulation, chromatography, adsorption and reaction of chemicals in heterogeneous porous media. The first were those of the regularized model of diffusion (see [3, 7]) which describes the fluid flow through a fractured porous media with a predominant storage of fluid in the porous matrix. They were followed by works which derived such microstructure models by homogenization of structures with isolated grains as core component (see [6–10, 13, 14, 18]). The system of parabolic equations in an ε-periodically bi-connected medium, subject to the homogeneous Neumann exterior condition, is treated in [15].

In the present paper, we use the properties of the realistic framework introduced by [12] which considered ε-periodic structures composed of two connected components. It corresponds to the geometry of many mixtures, porous media, filtering processes, radiators and so on. Various approaches to this framework can be found in [1, 5, 11, 16, 19].

In order to derive the macroscopic behavior we apply the two-scale convergence technique of the periodic homogenization theory (see [2, 4]). In each case we uncouple the local-periodic problems and determine the specific solutions which allow the identification of the macroscopic problems. Dealing with the Dirichlet conditions on the external boundary, we prove in detail the convergence results and present the effective behavior in the last two essentially different cases of the problem: $r = 1$ and $r \in (-1,1)$ (see [17]). The macroscopic problems turn out to be well-posed and hence, uniquely defining the asymptotic behavior of the temperature.

2. THE HEAT CONDUCTION PROBLEM

We use the ε-periodic structure already presented in [16]. Here it is.

Let Ω be an open connected bounded set in $\mathbb{R}^N$ ($N \geq 3$), locally located on one side of the boundary $\partial\Omega$, a Lipschitz manifold composed of a finite number of connected components.

Let Y_a be a Lipschitz open connected subset of the unit cube $Y = (0,1)^N$. We assume that $Y_b = Y \setminus \overline{Y}_a$ has a locally Lipschitz boundary and that the intersections of ∂Y_b with ∂Y are reproduced identically on the opposite faces of the cube, denoted for every $i \in \{1,2,\ldots,N\}$ by

$$\Sigma^{+i} = \{y \in \partial Y : \ y_i = 1\} \text{ şi } \Sigma^{-i} = \{y \in \partial Y : \ y_i = 0\}, \tag{1}$$

with the property that

$$\overline{Y}_b \cap \Sigma^{\pm i} \subset\subset \Sigma^{\pm i}, \quad \forall i \in \{1,2,...,N\}. \tag{2}$$

We assume that repeating Y by periodicity, the union of all the $\overline{Y}_a$ parts is a connected domain in $\mathbb{R}^N$ with a locally Lipschitz boundary; we denote it by $\mathbb{R}^N_a$ and further $\mathbb{R}^N_b = \mathbb{R}^N \setminus \mathbb{R}^N_a$ is also connected. Obviously, the origin of the coordinate system can be set such that there exists $R > 0$ with the property $B(0,R) \subseteq \mathbb{R}^N_a$.

For any $\varepsilon \in (0,1)$ we denote

$$\mathbb{Z}_\varepsilon = \{k \in \mathbb{Z}^N : \ \varepsilon k + \varepsilon Y \subseteq \Omega\}, \tag{3}$$

$$I_\varepsilon = \{k \in \mathbb{Z}_\varepsilon : \ \varepsilon k \pm \varepsilon e_i + \varepsilon Y \subseteq \Omega, \ \forall i \in \{1,...,N\}\}, \tag{4}$$

where e_i are the unit vectors of the canonical basis in $\mathbb{R}^N$.

The core component of our structure is defined by

$$\Omega_{\varepsilon b} = \operatorname{int}\left(\bigcup_{k \in I_\varepsilon} (\varepsilon k + \varepsilon \overline{Y}_b)\right) \tag{5}$$

and the ambient conductor and the interface between the two components by

$$\Omega_{\varepsilon a} = \Omega \setminus \overline{\Omega}_{\varepsilon b}, \qquad \Gamma_\varepsilon = \partial\Omega_{\varepsilon a} \cap \partial\Omega_{\varepsilon b}. \tag{6}$$

Let us remark here that all the boundaries are at least locally Lipschitz and that both $\Omega_{\varepsilon a}$ and $\Omega_{\varepsilon b}$ are connected.

We introduce the Hilbert space

$$H_\varepsilon = \left\{ v \in L^2(\Omega) : v\Big|_{\Omega_{\varepsilon a}} \in H^1(\Omega_{\varepsilon a}), \ v\Big|_{\Omega_{\varepsilon b}} \in H^1(\Omega_{\varepsilon b}), \ v = 0 \text{ on } \partial\Omega \right\} \tag{7}$$

endowed with the scalar product

$$(u,v)_{H_\varepsilon} = \int_{\Omega_{\varepsilon a}} \nabla u \nabla v + \varepsilon^2 \int_{\Omega_{\varepsilon b}} \nabla u \nabla v + \varepsilon \int_{\Gamma_\varepsilon} [u][v], \tag{8}$$

where $[u] = \gamma_{\varepsilon b} u - \gamma_{\varepsilon a} u$ and $\gamma_{\varepsilon a} u$, $\gamma_{\varepsilon b} u$ are the traces of u on Γ_ε defined in $H^1(\Omega_{\varepsilon a})$ and $H^1(\Omega_{\varepsilon b})$, respectively.

Let us denote $\Gamma := \partial Y_a \cap \partial Y_b$. Obviously, if ν is the normal on Γ(exterior to Y_a) and $x \in (\varepsilon k + \varepsilon\Gamma)$ for some $k \in \mathbb{Z}_\varepsilon$ then $\nu^\varepsilon(x) = \nu(\{x/\varepsilon\})$, where $\{x/\varepsilon\}$ is the vector formed by the fractional parts of the components of $\varepsilon^{-1}x$.

Our domain has the following properties (see [11] and [16]):

For any $v \in H_\varepsilon$ there exists $C > 0$, independent of ε, such that

$$|v|_{L^2(\Omega_{\varepsilon a})} \le C\,|\nabla v|_{L^2(\Omega_{\varepsilon a})}, \tag{9}$$

$$\varepsilon^{1/2}\,|\gamma_{\varepsilon a} v|_{L^2(\Gamma_\varepsilon)} \le C\left(|v|_{L^2(\Omega_{\varepsilon a})} + \varepsilon\,|\nabla v|_{L^2(\Omega_{\varepsilon a})}\right), \tag{10}$$

$$|v|_{L^2(\Omega_{\varepsilon b})} \le C\left(\varepsilon^{1/2}\,|\gamma_{\varepsilon b} v|_{L^2(\Gamma_\varepsilon)} + \varepsilon\,|\nabla v|_{L^2(\Omega_{\varepsilon b})}\right). \tag{11}$$

Now, we can present our problem:

For any $\varepsilon \in (0,1)$, $f \in L^2(\Omega)$ and $r \in (-1,1]$, we search for the temperature u^ε which verifies the heat conduction equations at the microscopic level:

$$-\frac{\partial}{\partial x_i}\left(a_{ij}^\varepsilon \frac{\partial u^\varepsilon}{\partial x_j}\right) = f \text{ in } \Omega_{\varepsilon a}, \tag{12}$$

$$-\varepsilon^2 \frac{\partial}{\partial x_i}\left(b_{ij}^\varepsilon \frac{\partial u^\varepsilon}{\partial x_j}\right) = f \text{ in } \Omega_{\varepsilon b}, \tag{13}$$

where, for any $\varepsilon \in (0,1)$, we introduce the transmission factor $h^\varepsilon(x) = h(x/\varepsilon)$ and the conductivities $a_{ij}^\varepsilon(x) = a_{ij}(x/\varepsilon)$, $b_{ij}^\varepsilon(x) = b_{ij}(x/\varepsilon)$ which have the property that there exists $\delta > 0$ such that

$$h, a_{ij} = a_{ji} \text{ and } b_{ij} = b_{ji} \in L^\infty_{per}(Y)\ \ \forall\imath, j \in \{1, 2, ..., N\}, \tag{14}$$

$$h(y) \ge \delta, \tag{15}$$

$$a_{ij}(y)\xi_i\xi_j \ge \delta\ \xi_i\xi_i \text{ and } b_{ij}(y)\xi_i\xi_j \ge \delta\ \xi_i\xi_i,\quad \forall y \in Y,\ \forall \xi \in \mathbb{R}^N. \tag{16}$$

Reminding that the core component is entirely surrounded by the ambient conductor, we impose to u^ε the following transmission and boundary conditions:

$$a_{ij}^\varepsilon \frac{\partial(\gamma_{\varepsilon a} u^\varepsilon)}{\partial x_j}\nu_i^\varepsilon = \varepsilon^2 b_{ij}^\varepsilon \frac{\partial(\gamma_{\varepsilon b} u^\varepsilon)}{\partial x_j}\nu_i^\varepsilon = \varepsilon^r h_\varepsilon\left(\gamma_{\varepsilon b} u^\varepsilon - \gamma_{\varepsilon a} u^\varepsilon\right) \text{ on } \Gamma_\varepsilon, \tag{17}$$

$$u_\varepsilon = 0 \text{ on } \partial\Omega. \tag{18}$$

The variational formulation of the problem (12)–(18) is the following:

Find $u^\varepsilon \in H_\varepsilon$ such that

$$a_\varepsilon(u^\varepsilon, v) := \int_{\Omega_{\varepsilon a}} a_{ij}^\varepsilon \frac{\partial u^\varepsilon}{\partial x_i}\frac{\partial v}{\partial x_j} + \varepsilon^2 \int_{\Omega_{\varepsilon b}} b_{ij}^\varepsilon \frac{\partial u^\varepsilon}{\partial x_i}\frac{\partial v}{\partial x_j} + \varepsilon^r \int_{\Gamma_\varepsilon} h^\varepsilon [u^\varepsilon][v] =$$

$$= \int_\Omega f v, \quad \forall v \in H_\varepsilon. \tag{19}$$

The variational problem (19) is well-posed:

THEOREM 1. *For any* $\varepsilon \in (0,1)$ *there exists a unique* $u^\varepsilon \in H_\varepsilon$, *solution of the problem (19).*

Proof. Using (9)–(11) the theorem is proved by applying the Lax-Milgram Theorem. □

3. A PRIORI ESTIMATES AND COMPACTNESS RESULTS

We begin by giving the a priori estimates of u^ε, solution of (19).

Setting $v = u^\varepsilon$ in (2.19) and using the coerciveness of $a_\varepsilon(\cdot,\cdot)$, we find that $\{u^\varepsilon\}_\varepsilon$ is bounded in H_ε from:

$$\delta\,|u^\varepsilon|^2_{H_\varepsilon} \le a(u^\varepsilon,u^\varepsilon) = \int_{\Omega_{\varepsilon a}} a^\varepsilon_{ij}\frac{\partial u^\varepsilon}{\partial x_i}\frac{\partial u^\varepsilon}{\partial x_j}\mathrm{d}x + \varepsilon^2\int_{\Omega_{\varepsilon b}} b^\varepsilon_{ij}\frac{\partial u^\varepsilon}{\partial x_i}\frac{\partial u^\varepsilon}{\partial x_j}\mathrm{d}x +$$

$$+\varepsilon^r\int_{\Gamma_\varepsilon} h^\varepsilon[u^\varepsilon]^2 = \int_\Omega f u^\varepsilon \le C\,|u^\varepsilon|_{H_\varepsilon}, \tag{20}$$

In order to find a priori estimates of u^ε we notice that (20) implies

$$|\nabla u^\varepsilon|^2_{L^2(\Omega_{\varepsilon a})} + \varepsilon^2\,|\nabla u^\varepsilon|^2_{L^2(\Omega_{\varepsilon b})} + \varepsilon^r\,|[u^\varepsilon]|^2_{L^2(\Gamma_\varepsilon)} \le C\,|u^\varepsilon|_{H_\varepsilon} \le C$$

It follows

$$|\nabla u^\varepsilon|_{L^2(\Omega_{\varepsilon a})} \le C,\quad \varepsilon\,|\nabla u^\varepsilon|_{L^2(\Omega_{\varepsilon b})} \le C,\quad \varepsilon^{r/2}\,|[u^\varepsilon]|_{L^2(\Gamma_\varepsilon)} \le C. \tag{21}$$

Using (9)–(11) we immediately get

$$|u^\varepsilon|_{L^2(\Omega_{\varepsilon a})} \le C,\quad |\nabla u^\varepsilon|_{L^2(\Omega_{\varepsilon a})} \le C,\quad \varepsilon\,|\nabla u^\varepsilon|_{L^2(\Omega_{\varepsilon b})} \le C, \tag{22}$$

where $C > 0$ is independent of ε. Let us introduce the following Hilbert spaces:

$$H^1_{per}(Y_a) = \left\{\varphi \in H^1_{loc}\left(\mathbb{R}^N_a\right) :\ \varphi \text{ is } Y\text{-periodic}\right\} \tag{23}$$

$$\widetilde{H}^1_{per}(Y_a) = \left\{\varphi \in H^1_{loc}\left(\mathbb{R}^N_a\right) :\ \int_{Y_a}\varphi = 0 \ \text{ and } \ \varphi \text{ is } Y\text{-periodic}\right\}. \tag{24}$$

For any $\alpha \in \{a,b\}$, we denote χ_α the indicator function of Y_α in Y. Also, we define:

$$\widehat{u}^\varepsilon_\alpha = \begin{cases} u^\varepsilon_\alpha & \text{in } \Omega_{\varepsilon\alpha} \\ 0 & \text{in } \Omega - \Omega_{\varepsilon\alpha}\end{cases} \qquad \widehat{\nabla u}^\varepsilon_\alpha = \begin{cases} \nabla u^\varepsilon_\alpha & \text{in } \Omega_{\varepsilon\alpha} \\ 0 & \text{in } \Omega - \Omega_{\varepsilon\alpha}\end{cases} \tag{25}$$

Now, with a proof similar to that of the Theorem 3.1 [16], we have the main compactness result:

THEOREM 2. *For every $-1 < r \le 1$ there exist $\eta_a \in L^2\left(\Omega;\widetilde{H}^1_{per}(Y_a)\right)$ and $u_a \in H^1_0(\Omega)$ such that (on some subsequence) we have the convergences*

$$\widehat{u}^\varepsilon_a \stackrel{2s}{\rightharpoonup} \chi_a(y)u_a \tag{26}$$

$$\widehat{\nabla u}^\varepsilon_a \stackrel{2s}{\rightharpoonup} \chi_a(y)\left(\nabla_x u_a + \nabla_y \eta_a(\cdot,y)\right). \tag{27}$$

Besides (26)–(27), here are the two-scale convergences corresponding to the second component:

THEOREM 3. *There exists a function $u_b(x,y) \in L^2(\Omega;H^1_{per}(Y_b))$ such that*

$$\widehat{u}^\varepsilon_b \stackrel{2s}{\rightharpoonup} \chi_b(y)u_b(x,y) \tag{28}$$

$$\varepsilon\widehat{\nabla u}^\varepsilon_b \stackrel{2s}{\rightharpoonup} \chi_b(y)\nabla_y u_b(x,y). \tag{29}$$

Proof. The estimations (22) imply that $\widehat{u}^\varepsilon_b$ and $\varepsilon\widehat{\nabla u}^\varepsilon_b$ are bounded sequences in $L^2(\Omega)$ and consequently we find that they have two-scale limits on some subsequences (see the main compactness theorem of [4] or [2]). Using standard methods (see Proposition 1.14 of [4]) the form of these limits is identified as being that specified by (28)–(29). □

4. THE HOMOGENIZATION PROCESS FOR $r = 1$

In this section, defining

$$V := H_0^1(\Omega) \times L^2(\Omega; H^1_{per}(Y_b)) \times L^2(\Omega, \widetilde{H}^1_{per}(Y_a)),$$

which is a Hilbert space endowed with the scalar product

$$\langle (u_a, u_b, \eta_a), (\Phi, \varphi_b, \varphi_a) \rangle_V = \int_\Omega \nabla u_a \nabla \Phi + \int_{\Omega \times Y_b} \nabla_y u_b \nabla_y \varphi_b +$$

$$+ \int_{\Omega \times \Gamma} (u_b - u_a)(\varphi_b - \Phi) + \int_{\Omega \times Y_a} \nabla_y \eta_a \nabla_y \varphi_a, \tag{30}$$

we can present the two-scale system verified by the limits:

THEOREM 4. *If u^ε is the solution of the problem (19) then, the convergences (26)–(27) and (28)–(29) hold on the whole sequence and the limit $(u_a, u_b, \eta_a) \in V$ is the unique solution of the following problem:*

$$\int_{\Omega \times Y_a} a_{ij} \left(\frac{\partial u_a}{\partial x_i} + \frac{\partial \eta_a}{\partial y_i} \right) \left(\frac{\partial \Phi}{\partial x_j} + \frac{\partial \varphi_a}{\partial y_j} \right) + \int_{\Omega \times Y_b} b_{ij} \frac{\partial u_b}{\partial y_i} \frac{\partial \varphi_b}{\partial y_j} +$$

$$+ \int_{\Omega \times \Gamma} h (u_b - u_a)(\varphi_b - \Phi) = \int_{\Omega \times Y_a} f \Phi + \int_{\Omega \times Y_b} f \varphi_b, \tag{31}$$

for any $(\Phi, \varphi_b, \varphi_a) \in V$.

Proof. In order to prove (31), for $\Phi \in \mathscr{D}(\Omega)$ and $\varphi_\alpha \in \mathscr{D}\left(\Omega; C^\infty_{per}(Y_\alpha)\right)$, with $\alpha \in \{a, b\}$, we set $v \in H_\varepsilon$ in (19) as follows:

$$v(x) = \left(\Phi(x) + \varepsilon \varphi_a \left(x, \frac{x}{\varepsilon} \right), \varphi_b \left(x, \frac{x}{\varepsilon} \right) \right), \quad x \in \Omega. \tag{32}$$

Then, passing to the limit($\varepsilon \to 0$) and taking into account the two-scale convergences (26), (27), (28), (29) and the a priori estimates (22), we find like in the Theorem 4.2[16] that (31) is valid for any $(\Phi, \varphi_b, \varphi_a) \in \mathscr{D}(\Omega) \times \mathscr{D}\left(\Omega; C^\infty_{per}(Y_b)\right) \times \mathscr{D}\left(\Omega; C^\infty_{per}(Y_a)\right)$. The proof is completed by density arguments. □

Remark 1. It is easy to verify that (31) is a well-posed problem in V. Hence, the convergences (26)–(27) and (28)–(29) hold on the whole sequence.

Here are the local-periodic problems and their solutions which allow the identification of (u_a, u_b, η_a).

For any $k \in \{1, 2, ..., N\}$ there exists $\eta_{ak} \in \widetilde{H}^1_{per}(Y_a)$ which is the unique solution of the problem

$$-\frac{\partial}{\partial y_i} \left(a_{ij} \frac{\partial (\eta_{ak} + y_k)}{\partial y_j} \right) = 0 \quad \text{in } Y_a, \tag{33}$$

$$a_{ij} \frac{\partial (\eta_{ak} + y_k)}{\partial y_j} \nu_i = 0 \quad \text{on } \Gamma. \tag{34}$$

The effective conductivity A, symmetric and positively defined, is given by

$$A_{ij} = \int_{Y_a} (a_{ij} + a_{ik} \frac{\partial \eta_{aj}}{\partial y_k}) \mathrm{d}y, \quad \forall i, j \in \{1, 2, ..., N\}. \tag{35}$$

Due to the existence of the first-order jump interface Γ_ε, we have to introduce the function q_b which is the only solutions in $H^1_{per}(Y_b)$ of the following problem:

$$-\frac{\partial}{\partial y_i} \left(b_{ij} \frac{\partial q_b}{\partial y_j} \right) = 1 \text{ in } Y_b, \qquad -b_{ij} \frac{\partial q_b}{\partial y_j} \nu_i + h q_b = 0 \text{ on } \Gamma. \tag{36}$$

Remark 2. The above local-periodic problems were obtained by the heuristic method of two-scale expansion (ansatz). They will prove to be correct.

Now we can summarize the homogenization results of this case:

THEOREM 5. *If u^ε is the solution of (19) then*

$$\widehat{u}_a^\varepsilon \stackrel{2s}{\rightharpoonup} \chi_a(y)u_a \tag{37}$$

$$\widehat{\nabla u}_a^\varepsilon \stackrel{2s}{\rightharpoonup} \chi_a(y)\left(\nabla_x u_a + \frac{\partial u_a}{\partial x_k}\nabla_y \eta_{a_k}(\cdot,y)\right), \tag{38}$$

$$\widehat{u}_b^\varepsilon \stackrel{2s}{\rightharpoonup} \chi_b(y)\left(u_a + f q_b(y)\right), \tag{39}$$

$$\varepsilon\widehat{\nabla u}_b^\varepsilon \stackrel{2s}{\rightharpoonup} \chi_b(y) f \nabla_y q_b(y), \tag{40}$$

where $u_a \in H_0^1(\Omega)$ is the unique solution of the Dirichlet problem

$$-\mathrm{div}\,(A\nabla u_a) = f \text{ in } \Omega \tag{41}$$

$$u = 0 \text{ on } \partial\Omega. \tag{42}$$

Proof. If $u_a \in H_0^1(\Omega)$ is the solution of (41)–(42) then it is easy to verify that the only solution of the problem (31) is

$$u_b(x,y) = u_a(x) + q_b(y)f(x), \quad x\in\Omega,\ y\in Y_b, \tag{43}$$

$$\eta_a(x,y) = \eta_{a_k}(y)\frac{\partial u_a}{\partial x_k}(x), \quad x\in\Omega,\ y\in Y_a. \tag{44}$$

The proof is completed by Theorem 2 and Theorem 3. □

5. THE HOMOGENIZATION PROCESS FOR $r \in (-1,1)$

In this case the preliminary results are given by:

LEMMA 1. *The limit u satisfies:*

$$u_a = u_b \text{ on } \Omega\times\Gamma. \tag{45}$$

Moreover, for any $\Phi \in \mathscr{D}(\Omega)$ and $\varphi_\alpha \in \mathscr{D}(\Omega; C_{per}^\infty(Y_\alpha))$, $\alpha \in \{a,b\}$ such that

$$\varphi_b(x,y) = \Phi(x),\ \forall (x,y)\in\Omega\times\Gamma, \tag{46}$$

we have:

$$\int_{\Omega\times Y_a} a_{ij}\left(\frac{\partial u_a}{\partial x_j} + \frac{\partial \eta_a}{\partial y_j}\right)\left(\frac{\partial \Phi}{\partial x_i} + \frac{\partial \varphi_a}{\partial y_i}\right) + \int_{\Omega\times Y_b} b_{ij}\frac{\partial u_b}{\partial y_j}\frac{\partial \varphi_b}{\partial y_i} =$$

$$= \int_{\Omega\times Y_a} f\Phi + \int_{\Omega\times Y_b} f\varphi_b. \tag{47}$$

Proof. We begin by proving (45). We use the same ideas as in the proof of the Theorem 4. We multiply the variational problem (19) with ε^{1-r} and then we take the test function (32) with $\Phi \in \mathscr{D}(\Omega)$, $\varphi_a \in \mathscr{D}(\Omega; C_{per}^\infty(Y_a))$ and $\varphi_b \in \mathscr{D}(\Omega; C_{per}^\infty(Y_b))$. Passing to the limit with $\varepsilon \to 0$, we get

$$\int_{\Omega\times\Gamma} h(y)\left(u_b(x,y) - u_a(x)\right)\left(\varphi_b(x,y) - \Phi(x)\right) = 0,$$

$$\forall\Phi \in \mathscr{D}(\Omega),\ \varphi_b \in \mathscr{D}(\Omega; C^\infty_{per}(Y_b)), \tag{48}$$

which obviously imply (45). In order to obtain the homogenized equation (47) we take in (19) the same test function (32) with the supplementary condition (46).

The proof is completed again in a straightforward manner, the term corresponding to the integral on Γ_ε being of order $\varepsilon^{(1+r)/2}$. □

In the light of the previous result, we introduce the spaces

$$H^1_{0,per}(Y_b) := \left\{\varphi \in H^1_{per}(Y_b),\ \varphi = 0 \text{ on } \Gamma\right\}. \tag{49}$$

$$W := H^1_0(\Omega) \times L^2(\Omega; H^1_{0,per}(Y_b)) \times L^2(\Omega; \widetilde{H}^1_{per}(Y_a)). \tag{50}$$

Remark 3. With density arguments in Lemma 1 and setting there

$$v_b = u_b - u_a \quad \text{and} \quad \psi_b = \varphi - \Phi$$

we find that $(u_a, v_b, \eta_a) \in W$ is solution of the following problem:

To find $(u_a, v_b, \eta_a) \in W$ satisfying

$$\int_{\Omega\times Y_a} a_{ij}\left(\frac{\partial u_a}{\partial x_j} + \frac{\partial \eta_a}{\partial y_j}\right)\left(\frac{\partial \Phi}{\partial x_i} + \frac{\partial \varphi_a}{\partial y_i}\right) + \int_{\Omega\times Y_b} b_{ij}\frac{\partial v_b}{\partial y_j}\frac{\partial \psi_b}{\partial y_i} =$$

$$= \int_{\Omega\times Y} f\Phi + \int_{\Omega\times Y_b} f\psi_b, \quad \forall(\Phi, \psi_b, \varphi_a) \in W. \tag{51}$$

It is easy to verify that (51) is a well-posed problem in the Hilbert space W, endowed with the scalar product:

$$\langle (u_a, v_b, \eta_a), (\Phi, \psi_b, \varphi_a)\rangle_W = \int_\Omega \nabla u_a \nabla\Phi + \int_{\Omega\times Y_b} \nabla_y v_b \nabla_y \psi_b + \int_{\Omega\times Y_a} \nabla_y \varphi_a \nabla_y \eta_a.$$

Let us denote $w_b \in H^1_{0,per}(Y_b)$, the only solution of the problem:

$$-\frac{\partial}{\partial y_i}\left(b_{ij}\frac{\partial w_b}{\partial y_j}\right) = 1 \text{ in } Y_b, \qquad w_b = 0 \text{ on } \Gamma. \tag{52}$$

Now we can summarize the results of the homogenization process in the present case:

THEOREM 6. *If u^ε is the solution of (19) then*

$$\widehat{u}^\varepsilon_a \overset{2s}{\rightharpoonup} \chi_a(y) u_a, \tag{53}$$

$$\widehat{\nabla u}^\varepsilon_a \overset{2s}{\rightharpoonup} \chi_a(y)\left(\nabla_x u_a + \frac{\partial u_a}{\partial x_k}\nabla_y \eta_{a_k}(\cdot, y)\right), \tag{54}$$

$$\widehat{u}^\varepsilon_b \overset{2s}{\rightharpoonup} \chi_b(y)\left(u_a + f w_b(y)\right), \tag{55}$$

$$\varepsilon\widehat{\nabla u}^\varepsilon_b \overset{2s}{\rightharpoonup} \chi_b(y) f \nabla_y w_b(y), \tag{56}$$

where $u_a \in H^1_0(\Omega)$ is the unique solution of the Dirichlet problem (41)–(42).

Proof. As $u_a \in H^1_0(\Omega)$ is the solution of (41)–(42), it remains to verify that the only solution of the problem (51) is given by:

$$u_b(x,y) = u_a(x) + f(x) w_b(y), \quad x \in \Omega,\ y \in Y_b, \tag{57}$$

$$\eta_a(x,y) = \eta_{a_k}(y)\frac{\partial u_a}{\partial x_k}(x), \quad x \in \Omega,\ y \in Y_a. \tag{58}$$

The proof is completed by Theorem 2 and Theorem 3. □

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