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SOME REMARKS ON THE MAGNETIC FIELD OPERATORS $\nabla \pm iA$ AND ITS APPLICATIONS

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Abstract. In the present paper, we give some remarks on the magnetic field operators $\nabla \pm iA$. As its applications, we study the Schrödinger equation with a magnetic field

$$-\Delta u + |A(x)|^2 u + iA(x)\cdot \nabla u = \mu u + |u|^p u,\ x \in \mathbb{R}^N,$$

where u is a complex-valued function and $\mu \in \mathbb{R}$. When $N > 2$, for $2 < p+2 < \frac{2N}{N-2}$ or $N = 2$, for $2 < p+2 < +\infty$, the existence and nonexistence of minimizers of the corresponding minimization problem are given via constrained variational methods. As a by-product, the above equation admits a normalized solution. We point out that the condition $\mathrm{div} A(x) = 0$ plays a crucial role in our study.

Key words*:* magnetic field operators, minimizers, complex-valued functions.
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1. INTRODUCTION AND MAIN RESULTS

As pointed out in [9], in differential geometry it is often necessary to consider acting on complex value function u by the operators $\nabla \pm iA$, which are more complicated derivatives than ∇. See also [2]. Recently, in [4, 8], Guo et al. investigated the rotating Bose-Einstein condensates. For $u : \mathbb{R}^N \to \mathbb{C}$ ($N \geq 2$), the term

$$\frac{i(u\nabla \overline{u} - \overline{u}\nabla u)}{2} \tag{1}$$

appears in their papers. We can regard it as a part of $\nabla \pm iA$ (see below for details). So in this present paper, we are interested in the operators $\nabla \pm iA$, and we will give some remarks for the corresponding energy functional and provide some applications on the rotating Bose-Einstein condensates.

Let $A : \mathbb{R}^N \to \mathbb{R}^N$ be a magnetic potential. $u : \mathbb{R}^N \to \mathbb{C}$. At first, by a direct calculus, we get the following lemma.

LEMMA 1. *It holds that*

$$\begin{aligned} |(\nabla \pm iA)u|^2 =& |\nabla u|^2 \mp iA\cdot(\overline{u}\nabla u - \overline{u}\nabla u) + |Au|^2 \\ =& |\nabla u|^2 \mp 2\,\mathrm{Re}(iA\cdot \nabla u \overline{u}) + |Au|^2. \end{aligned} \tag{2}$$

Proof. By a direct calculus, we get

$$\begin{aligned} |(\nabla \pm iA)u|^2 =& (\nabla u \pm iAu)\cdot\overline{(\nabla u \pm iAu)} \\ =& (\nabla u \pm iAu)\cdot(\nabla \overline{u} \mp iA\overline{u}) \\ =& \nabla u\cdot\nabla\overline{u} \mp iA\cdot\nabla u\overline{u} \pm iAu\cdot\nabla\overline{u} - i^2 Au\cdot A\overline{u} \\ =& |\nabla u|^2 \mp iA\cdot(\overline{u}\nabla u - u\nabla\overline{u}) + |Au|^2. \end{aligned} \tag{3}$$

And it holds that

$$\begin{aligned} |(\nabla \pm iA)u|^2 =& |\nabla u|^2 \mp iA\cdot\nabla u\overline{u} \mp \overline{\overline{-i}A\overline{u}\cdot\nabla u} + |Au|^2 \\ =& |\nabla u|^2 \mp 2\,\mathrm{Re}(iA\cdot\nabla u\overline{u}) + |Au|^2. \end{aligned} \tag{4}$$

We finish the proof. □

Remark 1. For $x = (x_1, x_2) \in \mathbb{R}^2$, we denote $x^\perp := (-x_2, x_1)$, $A = \frac{\Omega}{2}x^\perp$, where $\Omega > 0$ is a real number. In rotating Bose-Einstein condensates, Ω describes the fixed rotational velocity (cf. [4] or [8]). Applying Lemma 1, we get the result as (2.4) in [8], i.e.,

$$|\nabla u|^2 - \Omega x^\perp \cdot \frac{i(u\nabla\overline{u} - \overline{u}\nabla u)}{2} = |(\nabla - iA)u|^2 - \frac{\Omega}{4}|x|^2|u|^2. \tag{5}$$

Remark 2. We remark that when expanding the magnetic Laplacian $\Delta_A u := (\nabla + iA)^2 u$, we get

$$\Delta_A u = \Delta u - |A|^2 u + i\mathrm{div}(A)u + 2iA\cdot\nabla u. \tag{6}$$

It is different from the operator $|(\nabla \pm iA)u|^2$.

Based on the Lemma 1, we have

$$iA(x)\cdot\frac{\overline{u}\nabla u - u\nabla\overline{u}}{2} = \mathrm{Re}(iA\cdot\nabla u\overline{u}). \tag{7}$$

Therefore, we are interested in the following functional

$$J(u) := \frac{1}{2}\mathrm{Re}\int_{\mathbb{R}^N} iA\cdot\nabla u\overline{u}\,\mathrm{d}x. \tag{8}$$

And a natural question will be asked what the corresponding Euler-Lagrange equation of $J(u)$ looks like. The main goal of this paper is to answer this question. We introduce the space

$$X(\mathbb{R}^N,\mathbb{C}) := \{u : \mathbb{R}^N \to \mathbb{C} : \int_{\mathbb{R}^N} |\nabla u|^2\mathrm{d}x < \infty, \int_{\mathbb{R}^N} |A(x)|^2|u|^2\mathrm{d}x < \infty\} \tag{9}$$

equipped with the norm

$$\|u\| = (\int_{\mathbb{R}^N}(|\nabla u|^2 + |A(x)|^2|u|^2)\mathrm{d}x)^{\frac{1}{2}} \tag{10}$$

and inner product

$$(u,v)_X = \mathrm{Re}\int_{\mathbb{R}^N}\nabla u\nabla\overline{v}\,\mathrm{d}x + \mathrm{Re}\int_{\mathbb{R}^N}|A(x)|^2 u\overline{v}\,\mathrm{d}x \tag{11}$$

to make $J(u)$ well-defined. We remark that our space $X(\mathbb{R}^N,\mathbb{C})$ is different from the space H^1_A in [2] (see also [9]). In [2], the space H^1_A consists of all functions $f : \mathbb{R}^N \to \mathbb{C}$ such that $f \in L^2(\mathbb{R}^N)$ and $(\partial_j + iA_j)f \in L^2(\mathbb{R}^N)$ for $j = 1, 2, ..., N$.

In order to state our main results, we always assume that

(V) $A(x)$ is continuous and $\lim_{|x|\to\infty}|A(x)| = +\infty$, $\min_{x\in\mathbb{R}^N}|A(x)| = 0$.

And we have the following result.

LEMMA 2. *For $u \in X(\mathbb{R}^N, \mathbb{C})$, if* $\operatorname{div}A(x) = 0$*, it holds that*

$$\langle J'(u), \varphi \rangle = \operatorname{Re} \int_{\mathbb{R}^N} iA(x) \cdot \nabla u \overline{\varphi} \, dx, \ \forall \ \varphi \in X(\mathbb{R}^N, \mathbb{C}). \tag{12}$$

Proof. A direct calculation gives that

$$\langle J'(u), \varphi \rangle = \frac{1}{2} \operatorname{Re} \int_{\mathbb{R}^N} iA(x) \cdot (\nabla u \overline{\varphi} + \overline{u} \nabla \varphi) dx. \tag{13}$$

Note that

$$i\operatorname{div}(\overline{u}A(x)\varphi) = i\nabla \overline{u} \cdot A(x)\varphi + i\overline{u}\operatorname{div}(A(x))\varphi + i\overline{u}A(x) \cdot \nabla \varphi. \tag{14}$$

In view of $\operatorname{div}A(x) = 0$, using the divergence theorem, we have

$$\begin{aligned} \int_{\mathbb{R}^N} iA(x) \cdot \overline{u} \nabla \varphi \, dx &= \int_{\mathbb{R}^N} -iA(x) \cdot \nabla \overline{u} \varphi \, dx \\ &= \int_{\mathbb{R}^N} \overline{iA(x) \cdot \nabla u \overline{\varphi}} \, dx. \end{aligned} \tag{15}$$

We have done the proof. □

It implies that, when $\operatorname{div}A(x) = 0$, the corresponding Euler-Lagrange equation of $J(u)$ is

$$iA(x) \cdot \nabla u. \tag{16}$$

According to the above reasons, we can investigate the following Schrödinger equation with a magnetic field

$$-\Delta u + |A(x)|^2 u + iA(x) \cdot \nabla u = \mu u + |u|^p u, \ x \in \mathbb{R}^N, \tag{17}$$

where u is a complex-valued function.

THEOREM 1. *For $u \in X(\mathbb{R}^N, \mathbb{C})$, when* $\operatorname{div}A(x) = 0$*, if $N = 2$, $2 < p+2 < +\infty$ or $N > 2$, $2 < p+2 \leq \frac{2N}{N-2}$ the corresponding functional of* (17) *is given by*

$$I_p(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + |A(x)|^2 |u|^2) dx + J(u) - \frac{1}{2} \mu \int_{\mathbb{R}^N} |u|^2 dx - \frac{1}{p+2} \int_{\mathbb{R}^N} |u|^{p+2} dx. \tag{18}$$

Proof. Obviously, by Sobolev embeddings, $I_p(u)$ is well-defined. It is easy to check that

$$\begin{aligned} \langle I_p'(u), \varphi \rangle =& \operatorname{Re} \int_{\mathbb{R}^N} \nabla u \nabla \overline{\varphi} \, dx + \operatorname{Re} \int_{\mathbb{R}^N} |A(x)|^2 u \overline{\varphi} \, dx + \operatorname{Re} \int_{\mathbb{R}^N} iA(x) \cdot \nabla u \overline{\varphi} \, dx \\ &- \mu \operatorname{Re} \int_{\mathbb{R}^N} u \overline{\varphi} \, dx - \operatorname{Re} \int_{\mathbb{R}^N} |u|^p u \overline{\varphi} \, dx, \ \forall \ \varphi \in X(\mathbb{R}^N, \mathbb{C}). \end{aligned} \tag{19}$$

Similar to [1, Definition 3.1], u is a weak solutions of (17) means that

$$\operatorname{Re} \int_{\mathbb{R}^N} \nabla u \nabla \overline{\varphi} \, dx + \operatorname{Re} \int_{\mathbb{R}^N} |A(x)|^2 u \overline{\varphi} \, dx + \operatorname{Re} \int_{\mathbb{R}^N} iA(x) \cdot \nabla u \overline{\varphi} \, dx = \mu \operatorname{Re} \int_{\mathbb{R}^N} u \overline{\varphi} \, dx + \operatorname{Re} \int_{\mathbb{R}^N} |u|^p u \overline{\varphi} \, dx. \tag{20}$$

That is $\langle I_p'(u), \varphi \rangle = 0$. □

According to the Lagrange multiplier theorem, we can focus on the following minimization problem

$$e_p(\rho) := \inf_{\{u \in X(\mathbb{R}^N, \mathbb{C}) : \|u\|_2^2 = \rho\}} E_p(u), \tag{21}$$

where

$$E_p(u) = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + |A(x)|^2|u|^2)\mathrm{d}x + J(u) - \frac{1}{p+2}\int_{\mathbb{R}^N}|u|^{p+2}\mathrm{d}x. \tag{22}$$

We first recall that

$$-\Delta\psi + (\frac{4}{pN} + \frac{2}{N} - 1)\psi = \frac{4}{qN}\psi^p\psi,\ \psi \in H^1(\mathbb{R}^N,\mathbb{R}) \tag{23}$$

has a unique positive solution (up to translations) which is radially symmetric, denoted by ϕ_p. Here if $N \geq 3$, $2 < p+2 < \frac{2N}{N-2}$; if $N = 2$, $2 < p+2 < +\infty$. According to Weinstein [13], ϕ_p or $c\phi_p$ ($c \neq 0$) is the unique (up to translations) optimizer of the sharp Gagliardo-Nirenberg inequality

$$\|f\|_{p+2}^{p+2} \leq \frac{p+2}{2a_p^*}\|\nabla f\|_2^{\frac{pN}{2}}\|f\|_2^{p+2-\frac{p}{2}N},\ f \in H^1(\mathbb{R}^N,\mathbb{R}), \tag{24}$$

where $a_p^* = \|\phi_p\|_2^p$ ($\|\cdot\|_s$ denotes the usual L^s norm in $\mathbb{R}^N$). Since our function is complex-valued, in view of [9, Theorem 7.8], we need to pay attention to the inequality

$$\int_{\mathbb{R}^N}|\nabla|u||^2\mathrm{d}x \leq \int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x,\ \text{for } u \in H^1(\mathbb{R}^N,\mathbb{C}), \tag{25}$$

and the strict inequality often holds. To be more precisely, according to the proof of [9, Theorem 7.8], we have the following proposition.

PROPOSITION 1. *Denote $u(x) := R(x) + iI(x)$, and it holds that*

(i) *For $R(x) \not\equiv 0$, $I(x) = 0$, i.e., it only has real part, we have*

$$\int_{\mathbb{R}^N}|\nabla|u||^2\mathrm{d}x = \int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x; \tag{26}$$

(ii) *For $R(x) = 0$, $I(x) \not\equiv 0$, i.e., it only has imaginary part, the above equality also holds;*

(iii) *If $R(x) \not\equiv 0$, $I(x) \not\equiv 0$, then the above equality* (26) *holds if and only if there exists a constant c such that $R(x) = cI(x)$.*

Obviously, combining (24) with (25), we have

$$\|f\|_{p+2}^{p+2} \leq \frac{p+2}{2a_p^*}\|\nabla f\|_2^{\frac{pN}{2}}\|f\|_2^{p+2-\frac{p}{2}N},\ f \in H^1(\mathbb{R}^N,\mathbb{C}), \tag{27}$$

and ϕ_p or $l\phi_p e^{i\theta}$ is the unique (up to translations) optimizer, where $l \in \mathbb{R}\setminus\ \{0\}$, $\theta \in [2k\pi, 2(k+1)\pi]$, $k = 0, \pm1, \pm2, \ldots$.

Similar to the paper [5] or [12], we divide it into three cases: (a) L^2-subcritical case $2 < p+2 < 2+\frac{4}{N}$; (b) L^2-critical case $p+2 = 2+\frac{4}{N}$; (c) L^2-supercritical case $2+\frac{4}{N} < p+2 < \frac{2N}{N-2}$. We have the following results.

THEOREM 2. *Suppose that $N > 2$ and (V) holds, then*

(i) *For $2+\frac{4}{N} < p+2 < \frac{2N}{N-2}$, it holds that $e_p(\rho) = -\infty$ for any $\rho > 0$;*

(ii) *For $2 < p+2 < 2+\frac{4}{N}$, $e_p(\rho)$ is well-defined and it admits minimizers which only have three froms $R_0(x)$, $iI_0(x)$ and $Q_{l,0}(x) + ilQ_{l,0}(x)$ with $\|Q_{l,0}\|_2^2 = \frac{\rho}{1+l^2}$ ($l \neq 0$) for any $\rho > 0$, the other form of third minimizer is $Q_{l,0}(x)\sqrt{1+l^2}e^{i(arccos\frac{1}{1+l^2})}$. In other words, one minimizer only has real part, the another one only has imaginary part, the third one both has real part and imaginary part but they are proportional ($\frac{0}{0}$ is reasonable).*

(iii) *When $p+2 = 2+\frac{4}{N}$, we have*

1) *If $0 < \rho < \rho_0 := (a_p^*)^{\frac{N}{2}}$, $e_p(\rho)$ is well-defined and it admits minimizers as (ii);*

2) *If* $\rho > \rho_0$, *it holds that* $e_p(\rho) = -\infty$;

3) *If* $\rho = \rho_0$, $e_p(\rho) = 0$ *but there is no minimizer.*

Proof. Choosing $v \in C_0^\infty(\mathbb{R}^N, \mathbb{R}) \subset X(\mathbb{R}^N, \mathbb{C})$ satisfying $\|v\|_2^2 = \rho$, it yields that $J(v) = 0$. $v^t := t^{\frac{N}{2}} v(tx)$, then $\|v^t\|_2^2 = \rho$. And we have

$$E_p(v^t) \to -\infty \text{ as } t \to +\infty. \tag{28}$$

The conclusion (i) follows.

For the second conclusion (ii), for $u \in X(\mathbb{R}^N, \mathbb{C})$ with $\|u\|_2^2 = \rho$, using diamagnetic inequality [9, Theorem 7.21], it is easy to check that

$$\begin{aligned} E_p(u) =& \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + |A(x)|^2|u|^2)\mathrm{d}x + J(u) - \frac{1}{p+2}\int_{\mathbb{R}^N} |u|^{p+2}\mathrm{d}x \\ \geq& \frac{1}{2}\int_{\mathbb{R}^N} \left|\left(\nabla - i\frac{A(x)}{2}\right)u\right|^2 \mathrm{d}x - \frac{1}{p+2}\int_{\mathbb{R}^N} |u|^{p+2}\mathrm{d}x \\ \geq& \frac{1}{2}\int_{\mathbb{R}^N} |\nabla|u||^2\mathrm{d}x - \frac{1}{p+2}\int_{\mathbb{R}^N} |u|^{p+2}\mathrm{d}x. \end{aligned} \tag{29}$$

Let $R(x) := |u(x)|$, it follows form (24) that

$$E_p(u) \geq \frac{1}{2}\|\nabla R\|_2^2 - \frac{\rho^{1+\frac{p}{2}-\frac{p}{4}N}}{2a_p^*}\|\nabla R\|_2^{\frac{pN}{2}}. \tag{30}$$

$p < \frac{4}{N}$ gives that $e_p(\rho) > -\infty$. Let $\{u_n\}$ be minimization sequence. Using Proposition 1, we can divide it into three cases.

Case (a): $\{u_n\}$ only have real part, and denote them $\{R_n(x)\}$;

Case (b): $\{u_n\}$ only have imaginary part, and denote them $\{iI_n(x)\}$;

Case (c): $\{u_n\}$ have real part and imaginary part, and denote this sequence as $w_{l,n}(x) := Q_{l,n}(x) + ilQ_{l,n}(x)$ (for all $l \neq 0$).

For the case (a) and (b), we only deal with case (a) since the other case is similar. It yields that $J(R_n) = 0$. Clearly,

$$E_p(R_n) \geq \frac{1}{2}\|\nabla R_n\|_2^2 - \frac{\rho^{1+\frac{p}{2}-\frac{p}{4}N}}{2a_p^*}\|\nabla R_n\|_2^{\frac{pN}{2}}. \tag{31}$$

We have $\{\nabla R_n\}$ is bounded in L^2, So $\{R_n\}$ is bounded in L^{p+2}. Moreover, $\{\int_{\mathbb{R}^N} |A(x)|^2|R_n|^2dx\}$ is bounded. By Sobolev compact embedding, we get $e_p(\rho)$ has a minimizer R_0.

For the case (c), using Proposition 1 again, one has

$$E_p(w_{l,n}) \geq \frac{1}{2}\|\nabla|w_{l,n}|\|_2^2 - \frac{\rho^{1+\frac{p}{2}-\frac{p}{4}N}}{2a_p^*}\|\nabla|w_{l,n}|\|_2^{\frac{pN}{2}}. \tag{32}$$

It gives that $\{\nabla w_{l,n}\}$ is bounded in L^2. Similar arguments as above give that $e_p(\rho)$ admits a minimizer like $w_{l,0} = Q_{l,0}(x) + ilQ_{l,0}(x)$ with $\|Q_{l,0}\|_2^2 = \frac{\rho}{1+l^2}$ for all $l \neq 0$.

As (29) and (30), for $u \in X(\mathbb{R}^N, \mathbb{C})$ with $\|u\|_2^2 = \rho$, when $\rho < (a_p^*)^{\frac{N}{2}}$, it holds that

$$E_p(u) \geq \frac{1}{2}\int_{\mathbb{R}^N} |\nabla|u||^2\mathrm{d}x(1 - \frac{\rho^{\frac{2}{N}}}{a_p^*}) \geq 0. \tag{33}$$

So $e_p(\rho)$ is well-defined. Repeating the arguments of (ii), it admits minimizers like $R_0(x)$, $iI_0(x)$ and $Q_{l,0}(x) + ilQ_{l,0}(x)$ with $\|Q_{l,0}\|_2^2 = \frac{\rho}{1+l^2}$ $(l \neq 0)$. The desired conclusions 1) of (iii) follows.

$X(\mathbb{R}^N,\mathbb{R})$ means that all elements in $X(\mathbb{R}^N,\mathbb{C})$ only have real part. For $v\in X(\mathbb{R}^N,\mathbb{R})$, it holds that $J(v)=0$. So

$$\begin{aligned} e_p(\rho) =& \inf_{\{u\in X(\mathbb{R}^N,\mathbb{C}):\|u\|_2^2=\rho\}} E_p(u) \\ \leq& \inf_{\{u\in X(\mathbb{R}^N,\mathbb{R}):\|u\|_2^2=\rho\}} \left\{\frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2+|A(x)|^2|u|^2)\mathrm{d}x - \frac{1}{p+2}\int_{\mathbb{R}^N}|u|^{p+2}\mathrm{d}x\right\}. \end{aligned} \tag{34}$$

When $\rho>(a_p^*)^{\frac{N}{2}}$, in the spirit of [3], we can draw the conclusion 2) of (iii) safely.

For $\rho=(a_p^*)^{\frac{N}{2}}$, jointly with (34), like [3], one has

$$\begin{aligned} e_p(\rho) =& \inf_{\{u\in X(\mathbb{R}^N,\mathbb{C}):\|u\|_2^2=\rho\}} E_p(u) \\ \leq& \inf_{\{u\in X(\mathbb{R}^N,\mathbb{R}):\|u\|_2^2=\rho\}} \left\{\frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2+|A(x)|^2|u|^2)\mathrm{d}x - \frac{1}{p+2}\int_{\mathbb{R}^N}|u|^{p+2}\mathrm{d}x\right\} \\ =&0. \end{aligned} \tag{35}$$

Here we use that for a real function v, $J(v)=0$. Similar to (33), for $u\in X(\mathbb{R}^N,\mathbb{C})$ with $\|u\|_2^2=(a_p^*)^{\frac{N}{2}}$, it gives that

$$E_p(u)\geq \frac{1}{2}\int_{\mathbb{R}^N}|\nabla|u||^2\mathrm{d}x(1-\frac{a_p^*}{a_p^*})\geq 0. \tag{36}$$

Thus, in this case $e_p(\rho)=0$ but there is no minimizer. If not, let u_0 is a minimizer. Similar to (29) but with more elaborate, with Gagliardo-Nirenberg inequality (27) in hand, we can verify that

$$\begin{aligned} E_p(u_0) =& \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u_0|^2+|A(x)|^2|u_0|^2)\mathrm{d}x + J(u_0) - \frac{1}{p+2}\int_{\mathbb{R}^N}|u_0|^{p+2}\mathrm{d}x \\ \geq& \frac{1}{2}\int_{\mathbb{R}^N}\left|\left(\nabla - i\frac{A(x)}{2}\right)u\right|^2\mathrm{d}x - \frac{1}{p+2}\int_{\mathbb{R}^N}|u|^{p+2}\mathrm{d}x \\ \geq& \frac{1}{2}\int_{\mathbb{R}^N}|\nabla|u||^2\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^N}|A(x)|^2|u_0|^2\mathrm{d}x - \frac{1}{p+2}\int_{\mathbb{R}^N}|u|^{p+2}\mathrm{d}x \\ \geq& \frac{1}{4}\int_{\mathbb{R}^N}|A(x)|^2|u_0|^2\mathrm{d}x, \end{aligned} \tag{37}$$

which implies that

$$\int_{\mathbb{R}^N}|A(x)|^2|u_0|^2\mathrm{d}x=0. \tag{38}$$

The results (ii) imply that u_0 possesses only three forms: a) only has real part; b) only has imaginary part; c) both has real part and imaginary part but they are proportional. Without loss of generality, we consider the case c). $u_0:=R(x)+ilR(x)$ $(l\neq 0)$ satisfying

$$\|R\|_2^2=\frac{(a_p^*)^{\frac{N}{2}}}{1+l^2},\ \text{and}\ \int_{\mathbb{R}^N}|A(x)|^2|R|^2\mathrm{d}x=0. \tag{39}$$

On the one hand, in light of u_0 is a minimizer, the following Remark 3 gives that

$$\begin{cases} -\Delta R+|A(x)|^2R=\mu R+(1+l^2)^{\frac{p}{2}}|R|^pR,\ x\in\mathbb{R}^N, \\ A(x)\cdot\nabla R=0. \end{cases} \tag{40}$$

It follows that

$$\int_{\mathbb{R}^N}|\nabla R|^2\mathrm{d}x=\mu\int_{\mathbb{R}^N}|R|^2\mathrm{d}x+(1+l^2)^{\frac{p}{2}}\int_{\mathbb{R}^N}|R|^{p+2}\mathrm{d}x. \tag{41}$$

By elliptic regularity theory (cf. [11, Appendix B]), one has $R \in C^2(\mathbb{R}^N)$. Thus, the Pohozaev identity in [14]

$$\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla R|^2 \mathrm{d}x = \frac{N\mu}{2}\int_{\mathbb{R}^N} |R|^2 \mathrm{d}x + \frac{N(1+l^2)^{\frac{p}{2}}}{p+2}\int_{\mathbb{R}^N} |R|^{p+2} \mathrm{d}x \tag{42}$$

holds. By (41) and (42), we see that

$$\|R\|_{p+2}^{p+2} = \frac{p+2}{2a_p^*}\|\nabla R\|_2^2 \|R\|_2^p. \tag{43}$$

Theorem B in [13] provides that $R(x) > 0$ for all $x \in \mathbb{R}^N$.

On the other hand, since the assumption (V), we derive that that R must have compact support. Indeed, taking into account that $\lim_{|x|\to\infty} |A(x)| = +\infty$, for any $M > 0$, there exists $r_0 > 0$ such that $|A(x)|^2 > M$, $\forall\ |x| > r_0$. If R does not have compact support, then for $r_0 > 0$, there exists $x_0 \in B^c_{r_0+1}(0)$ such that $R(x_0) > \delta > 0$. So there exists $1 > \varepsilon > 0$ small such that

$$\int_{\mathbb{R}^N} |A(x)|^2 |R|^2 \mathrm{d}x > M\left(\frac{\delta}{2}\right)^2 meas(B_\varepsilon(x_0)). \tag{44}$$

This contradicts (39). □

Remark 3. When we obtain a minimizer $u : \mathbb{R}^N \to \mathbb{C}$ in the Theorem 2, Lagrange multiplier theorem implies that there is a $\mu \in \mathbb{R}$ such that

$$-\Delta u + |A(x)|^2 u + iA(x)\cdot\nabla u = \mu u + |u|^p u,\ x \in \mathbb{R}^N, \tag{45}$$

in the weak sense. In other words, the minimizer u satisfies

$$\mathrm{Re}\int_{\mathbb{R}^N} \nabla u \nabla\overline{\varphi}\,\mathrm{d}x + \mathrm{Re}\int_{\mathbb{R}^N} |A(x)|^2 u\overline{\varphi}\,\mathrm{d}x + \mathrm{Re}\int_{\mathbb{R}^N} iA(x)\cdot\nabla u\overline{\varphi}\,\mathrm{d}x = \mu\,\mathrm{Re}\int_{\mathbb{R}^N} u\overline{\varphi}\,\mathrm{d}x + \mathrm{Re}\int_{\mathbb{R}^N} |u|^p u\overline{\varphi}\,\mathrm{d}x, \tag{46}$$

for all $\varphi \in X(\mathbb{R}^N, \mathbb{C})$. In the future work, we are going to prove $\mu \neq 0$ and find solutions which satisfy

$$A(x)\cdot\nabla u \neq 0. \tag{47}$$

Remark 4. $N = 2$, $\frac{2N}{N-2}$ replaced by $+\infty$, the similar results like the Theorem 2 are also valid. Compared with [5], our results are extremely useful as a supplemental text since Guo and Seiringer do not consider the function is a complex-valued function. As we know, there is no result on the problem (46) which contains the term $iA(x)\cdot\nabla u$ for general potential. In this regard, we generalize the special potential $x^\perp := (-x_2, x_1)$ in [8] to general potential A. After submitting our paper, we learned that many results about rotating Bose-Einstein condensates appear in [6, 7, 10].

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REFERENCES

[1] Ambrosio V, d'Avenia P. Nonlinear fractional magnetic Schrödinger equation: Existence and multiplicity. J. Differential Equations 2018;264:3336–3368.

[2] Esteban MJ, Lions PL. Stationary solutions of nonlinear Schrödinger equations with an external magnetic field, Partial differential equations and the calculus of variations. In: Partial Differential Equations and the Calculus of Variations, vol. I. Boston, MA: Birkhäuser; 1989, pp. 401–449.

[3] Gu L, Zeng X, Zhou H. Eigenvalue problems for p-Laplacian equation with trapping potentials. Nonlinear Anal. 2017;148:212–227.

[4] Guo Y, The nonexistence of vortices for rotating Bose-Einstein condensates in non-radially symmetric traps. J. Math. Pures Appl. 2023;170:1–32.

[5] Guo Y, Seiringer. On the mass concentration for Bose-Einstein condensates with attractive interactions. Lett. Math. Phys. 2014;104:141–156.

[6] Guo Y, Li Y, Luo Y, Peng S. Axial symmetry of normalized solutions for magnetic Gross-Pitaevskii equations with anharmonic potentials. arXiv preprint 2023; https://doi.org/10.48550/arXiv.2310.00556.

[7] Guo Y, Luo Y, Peng S. Existence and asymptotic behavior of ground States for rotating Bose-Einstein condensates. SIAM J. Math. Anal. 2023;55:773–804.

[8] Guo Y, Luo Y, Yang W. The nonexistence of vortices for rotating Bose-Einstein condensates with attractive interactions. Arch. Rational Mech. Anal. 2020;238:1231–1281.

[9] Lieb E, Loss M. Analysis. Graduate studies in mathematics. Providence, Rhode Island: AMS; 2001.

[10] Luo X, Yang T. Multiplicity, asymptotics and stability of standing wavesfor nonlinear Schrödinger equation with rotation. J. Differential Equations 2021;304:326–347.

[11] Struwe M. Variational methods. Berlin: Springer; 2008.

[12] Wang W, Zhang W, Li Y. Minimizers of the planar Schrödinger-Newton equations. Complex Variables and Elliptic Equations 2022;67(1):151–161.

[13] Weinstein M. Nonlinear Schrödinger equations and sharp interpolation estimates, Commun. Math. Phy. 1983;87:567–576.

[14] Willem M. Minimax theorems. In: Progress in Nonlinear Diferential Equations and their Applications, vol. 24. Boston, MA: Birkhäuser; 1996.

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