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A VARIATION OF L^p LOCAL UNCERTAINTY PRINCIPLES FOR WEINSTEIN TRANSFORM

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Abstract. The main crux of this paper is to introduce L^p local uncertainty inequalities for the Weinstein transform, and we study L^p version of the Heisenberg-Pauli-Weyl uncertainty inequalities for this transform. Then, by using the L^p local uncertainty inequalities for the Weinstein transform and the tools of Donoho-Stark, we obtain uncertainty principles of concentration in the L^p theory, for all $1 < p \leq 2$.

Keywords: Weinstein operator, local uncertainty principles, Heisenberg-Pauli-Weyl-type inequality, Donoho-Stark-type inequality.

Mathematics Subject Classification (MSC2020): primary 43A32; secondary 44A15.

1. INTRODUCTION

The uncertainty principle appears in signal theory and harmonic analysis in various forms that include not only the signals φ and their Fourier transformations $\mathscr{F}(\varphi)$, but essentially all representations of signals in the time frequency space. These mathematical results limit the simultaneous concentration of signals and their Fourier transformations, and have a significant impact on quantum physics and signal analysis.

In the quantum physics, it is said that the speed and the position of particles cannot be measured with an infinite precision. On other hand, in the theory of signal analysis, it is said that if we observe signals only for a limited period of time, we lose the information about the frequency of the signals. Time-limited and band-limited functions are the basic tools for signal processing and image processing. For example, a simple form of uncertainty inequality tells us that signals cannot be limited simultaneously to time and bandwidth. This led to research into a set of functions that are almost time-limited and nearly band-limited, initially conducted by the works of Landau and Pollak [4, 5] and then by the work of Donoho-Stark [2].

In recent years, the behavior of Weinstein transform was investigated by many researchers, in relation to different problems already studied in classical Fourier transform. For instance, Wigner and Weyl transform [10, 15], wavelet transform [13, 14], reproducing kernels [1, 12], pseudo differential operators [17], inequalities and uncertainty principles [8, 9, 11]. Motivated by the works of f Faris [3], Price [6, 7] and Soltani [16] we prove an L^p local uncertainty inequalities in the Weinstein setting.

The layout of this manuscript is as follows. Section 2 is devoted to give a brief overview of the Weinstein operator that will play a significant role in the proofs of our main results. In section 3, we prove a local uncertainty principle for the Weinstein operator, and we establish for it an L^p version of the Heisenberg-Pauli-Weyl uncertainty inequality. In the last section, we show some uncertainty inequalities of concentration.

2. PRELIMINAIRES

The Weinstein operator or Laplace Bessel operator $\Delta^d_{W,\alpha}$ defined on $\mathbb{R}^{d+1}_+ = \mathbb{R}^d \times (0,\infty)$, by

$$\Delta^d_{W,\alpha} = \Delta_d + L_\alpha,\ \alpha > -1/2,$$

where Δ_d is the Laplacian operator on $\mathbb{R}^d$ and L_α is the Bessel operator for the last variable given on $(0,\infty)$ by

$$L_\alpha u = \frac{\partial^2 u}{\partial x_{d+1}^2} + \frac{2\alpha+1}{x_{d+1}}\frac{\partial u}{\partial x_{d+1}}.$$

For all $\lambda = (\lambda_1, ..., \lambda_{d+1}) \in \mathbb{C}^{d+1}$, the following system of equations

$$\frac{\partial^2 u}{\partial x_j^2}(x) = -\lambda_j^2 u(x), \quad \text{if } 1 \le j \le d$$
$$L_\alpha u(x) = -\lambda_{d+1}^2 u(x),$$
$$u(0) = 1, \quad \frac{\partial u}{\partial x_{d+1}}(0) = 0, \quad \frac{\partial u}{\partial x_j}(0) = -i\lambda_j, \quad \text{if } 1 \le j \le d$$

has a unique solution indicated by $\Lambda_\alpha^d(\lambda,.)$, and denoted by

$$\Lambda_\alpha^d(\lambda,x) = e^{-i<x',\lambda'>} j_\alpha(x_{d+1}\lambda_{d+1}) \tag{1}$$

where $\lambda = (\lambda', \lambda_{d+1})$, $x = (x', x_{d+1})$ and j_α is represent the normalized Bessel function defined by

$$j_\alpha(x) = \Gamma(\alpha+1)\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{2^k k!\Gamma(\alpha+k+1)}.$$

$(\lambda,x) \mapsto \Lambda_\alpha^d(\lambda,x)$ is named the Weinstein kernel and satisfies for all $(\lambda,x) \in \mathbb{R}^{d+1}\times\mathbb{R}^{d+1}$

$$\left|\Lambda_\alpha^d(\lambda,x)\right| \le 1. \tag{2}$$

In the following, we note by $L_\alpha^p(\mathbb{R}_+^{d+1})$, $1 \le p \le \infty$, the space of measurable functions f on $\mathbb{R}_+^{d+1}$ satisfying

$$\|f\|_{\alpha,p} = \left(\int_{\mathbb{R}_+^{d+1}} |f(x)|^p \mathrm{d}\mu_\alpha(x)\right)^{1/p} < \infty,\ p \in [1,\infty), \quad \text{and} \quad \|f\|_{\alpha,\infty} = \operatorname{ess}\sup_{x\in\mathbb{R}_+^{d+1}} |f(x)| < \infty,$$

where $\mathrm{d}\mu_\alpha(x)$ denote measure on $\mathbb{R}_+^{d+1} = \mathbb{R}^d \times (0,\infty)$ defined by

$$\mathrm{d}\mu_\alpha(x) = \frac{x_{d+1}^{2\alpha+1}}{(2\pi)^{\frac{d}{2}}2^\alpha\Gamma^2(\alpha+1)}\mathrm{d}x.$$

If $\varphi \in L_\alpha^1(\mathbb{R}_+^{d+1})$ is radial function then $\tilde{\varphi}$ defined on $\mathbb{R}_+$ by $\varphi(x) = \tilde{\varphi}(|x|)$, for all $x \in \mathbb{R}_+^{d+1}$, is integrable function with respect to $r^{2\alpha+d+1}dr$, and we have the equality

$$a_\alpha \int_0^\infty \tilde{\varphi}(r) r^{2\alpha+d+1}\mathrm{d}r = \int_{\mathbb{R}_+^{d+1}} \varphi(x)\mathrm{d}\mu_\alpha(x), \tag{3}$$

where a_α is a constant given by

$$a_\alpha = \frac{1}{2^{\alpha+\frac{d}{2}}\Gamma(\alpha+\frac{d}{2}+1)}.$$

The Weinstein (Laplace Bessel) Fourier transform is a hybrid integral transform defined for $\varphi \in L_\alpha^1(\mathbb{R}_+^{d+1})$ by

$$\forall \lambda \in \mathbb{R}_+^{d+1}, \quad \mathscr{F}_{W,\alpha}(\varphi)(\lambda) = \int_{\mathbb{R}_+^{d+1}} \varphi(x)\Lambda_\alpha^d(x,\lambda)\mathrm{d}\mu_\alpha(x).$$

From [11], we list the next properties which are useful in the rest of this paper:

- If $\varphi \in L_\alpha^1(\mathbb{R}_+^{d+1})$, then $\mathscr{F}_{W,\alpha}(\varphi)$ is continuous on $\mathbb{R}_+^{d+1}$ such that

$$\|\mathscr{F}_{W,\alpha}\varphi\|_{\alpha,\infty} \le \|\varphi\|_{\alpha,1}. \tag{4}$$

- For all $\varphi \in L^2_\alpha(\mathbb{R}^{d+1}_+)$, we have
$$\|\mathscr{F}_{W,\alpha}\varphi\|_{\alpha,2} = \|\varphi\|_{\alpha,2}\,. \tag{5}$$

- For all $\varphi \in L^p_\alpha(\mathbb{R}^{d+1}_+)$, $1 \le p \le 2$, $\mathscr{F}_{W,\alpha}(\varphi)$ belongs to $L^q_\alpha(\mathbb{R}^{d+1}_+)$, where $q = p/(p-1)$, and we have
$$\|\mathscr{F}_{W,\alpha}\varphi\|_{\alpha,q} \le \|\varphi\|_{\alpha,p}\,. \tag{6}$$

Through this paper, we need to give the precise assumptions on the data of the results:

- (A1) Ω be a measurable subset of $\mathbb{R}^{d+1}_+$ satisfying $\mu_\alpha(\Omega) < \infty$.
- (A2) Σ be a measurable subset of $\mathbb{R}^{d+1}_+$ such that $\mu_\alpha(\Sigma) < \infty$.
- (A3) $\varphi \in L^p_\alpha(\mathbb{R}^{d+1}_+)$ with $1 < p \le 2$ and $q = p/(p-1)$.

3. L^p-LOCAL UNCERTAINTY PRINCIPLE

In practice, the uncertainty principle is often discussed in the context of specific measurements made on individual particles or within localized regions of a quantum system and both the supports of time and frequency are often limited. In such case, the infinite support fails to hold true. Therefore, there has a great need to discuss the uncertainty principles in finite support domains. Local uncertainty principles for the classical Fourier transform were firstly attained by Faris [3] and they were subsequently refined and generalized by Price [6, 7]. Motivated by the above works, we extend the uncertainty inequality studied in [11].

THEOREM 1. *Let $s > 0$ and $\mu_\alpha(\Sigma) > 0$. Under assumptions (A1) and (A3), we have*
$$\|\chi_\Sigma \mathscr{F}_{W,\alpha}(\varphi)\|_{\alpha,q} \le \begin{cases} C_1(s)(\mu_\alpha(\Sigma))^{\frac{s}{\beta}}\, \||x|^s\varphi\|_{\alpha,p}\,, & sq < \beta;\\[2mm] C_2(s)(\mu_\alpha(\Sigma))^{\frac{1}{q}}\, \|\varphi\|_{\alpha,p}^{1-\frac{\beta}{sq}}\, \||x|^s\varphi\|_{\alpha,p}^{\frac{\beta}{sq}}\,, & sq > \beta;\\[2mm] 2C_1(\frac{s}{2})(\mu_\alpha(\Sigma))^{\frac{1}{2q}}\, \|\varphi\|_{\alpha,p}^{\frac12}\, \||x|^s\varphi\|_{\alpha,p}^{\frac12}\,, & sq = \beta, \end{cases}$$
where $\beta = 2\alpha + d + 2$, C_1 and C_2 are constants depend on s given by
$$C_1(s) = \frac{\beta}{\beta - sq}\left(\frac{(\beta - sq)^{q-1}}{2^{\frac{\beta}{2}-1}\Gamma(\frac{\beta}{2})(sq)^q}\right)^{\frac{s}{\beta}}, \quad C_2(s) = \frac{sq}{sq-\beta}\left(\frac{sq}{\beta} - 1\right)^{\frac{\beta}{spq}}\left(\frac{(sq-\beta)\Gamma\left(\frac{sq-\beta}{sp}\right)\Gamma\left(\frac{\beta}{sp}\right)}{2^{\frac{\beta}{2}-1}pqs^2\Gamma(\frac{\beta}{2})\Gamma\left(\frac{q}{p}\right)}\right)^{\frac{1}{q}}.$$

Proof. (a) The first inequality is trivial if the norm $\||x|^s\varphi\|_{\alpha,p}$ is infinite. Now, suppose that $\||x|^s\varphi\|_{\alpha,p} < \infty$ and we take $B_\rho = \{x \in \mathbb{R}^{d+1}_+ : |x| < \rho\}$ and $B^c_\rho = \mathbb{R}^{d+1}_+ \backslash B_\rho$, where $\rho > 0$. Denote by χ_Σ, χ_{B_ρ} and $\chi_{B^c_\rho}$ the indicator functions. Let $\varphi \in L^p_\alpha(\mathbb{R}^{d+1}_+)$, $1 < p \le 2$ with $q = p/(p-1)$. According to inequality of Minkowski, we obtain for all $\rho > 0$
$$\begin{aligned}\|\chi_\Sigma \mathscr{F}_{W,\alpha}(\varphi)\|_{\alpha,q} &\le \left\|\chi_\Sigma \mathscr{F}_{W,\alpha}(\chi_{B_\rho}\varphi)\right\|_{\alpha,q} + \left\|\chi_\Sigma \mathscr{F}_{W,\alpha}(\chi_{B^c_\rho}\varphi)\right\|_{\alpha,q}\\ &\le (\mu_\alpha(\Sigma))^{\frac{1}{q}}\left\|\mathscr{F}_{W,\alpha}(\chi_{B_\rho}\varphi)\right\|_{\alpha,\infty} + \left\|\mathscr{F}_{W,\alpha}(\chi_{B^c_\rho}\varphi)\right\|_{\alpha,q},\end{aligned}$$
hence, according to the inequalities (4) and (6), it comes that
$$\|\chi_\Sigma \mathscr{F}_{W,\alpha}(\varphi)\|_{\alpha,q} \le (\mu_\alpha(\Sigma))^{\frac{1}{q}}\left\|\chi_{B_\rho}\varphi\right\|_{\alpha,1} + \left\|\chi_{B^c_\rho}\varphi\right\|_{\alpha,p}. \tag{7}$$
Furthermore, using Hölder's inequality, we obtain
$$\left\|\chi_{B_\rho}\varphi\right\|_{\alpha,1} \le \left\||x|^{-s}\chi_{B_\rho}\right\|_{\alpha,q}\, \||x|^s\varphi\|_{\alpha,p}\,.$$

Since $sq<\beta$, then according to relation (3) we get

$$\left\||x|^{-s}\chi_{B_\rho}\right\|_{\alpha,q}=C_\alpha\rho^{-s+\beta/q},\quad \text{with}\quad C_\alpha=\left((\beta-sq)2^{\frac{\beta}{2}-1}\Gamma(\frac{\beta}{2})\right)^{-\frac{1}{q}},$$

thus,

$$\left\|\chi_{B_\rho}\varphi\right\|_{\alpha,1}\leq C_\alpha\rho^{-s+\beta/q}\,\||x|^s\varphi\|_{\alpha,p}. \tag{8}$$

Moreover, we have

$$\left\|\chi_{B_\rho^c}\varphi\right\|_{\alpha,p}\leq\left\||x|^{-s}\chi_{B_\rho^c}\right\|_{\alpha,\infty}\||x|^s\varphi\|_{\alpha,p}\leq\rho^{-s}\,\||x|^s\varphi\|_{\alpha,p}. \tag{9}$$

Next, combining the inequalities (7), (8) and (9), we conclude that

$$\|\chi_\Sigma\mathscr{F}_{W,\alpha}(\varphi)\|_{\alpha,q}\leq\left[\rho^{-s}+C_\alpha\left(\mu_\alpha(\Sigma)\right)^{\frac{1}{q}}\rho^{-s+\beta/q}\right]\||x|^s\varphi\|_{\alpha,p}.$$

Then, the first inequality holds by choosing

$$r=\left(\frac{sq}{(\beta-sq)C_\alpha}\right)^{\frac{q}{\beta}}\left(\mu_\alpha(\Sigma)\right)^{-\frac{1}{\beta}}.$$

(b) The second inequality is true if the norm $\|\varphi\|_{\alpha,p}$ or $\||x|^s\varphi\|_{\alpha,p}$ is infinite. Now, assume that the sum $\|\varphi\|_{\alpha,p}+\||x|^s\varphi\|_{\alpha,p}$ is finite. From the assumption $sq>\beta$ we conclude that $x\longrightarrow(1+|x|^{sp})^{-1/p}$ is in $L^q_\alpha(\mathbb{R}^{d+1}_+)$, then using Hölder's inequality, we get

$$\begin{aligned}\|\varphi\|^p_{\alpha,1}&=\left(\int_{\mathbb{R}^{d+1}_+}(1+|x|^{sp})^{1/p}|\varphi(x)|(1+|x|^{sp})^{-1/p}\mathrm{d}\mu_\alpha(x)\right)^p\\&\leq\left(\|\varphi\|^p_{\alpha,p}+\||x|^s\varphi\|^p_{\alpha,p}\right)\left(\int_{\mathbb{R}^{d+1}_+}\frac{\mathrm{d}\mu_\alpha(x)}{(1+|x|^{sp})^{q/p}}\right)^{p/q}.\end{aligned}$$

Therefore, $\varphi\in L^1_\alpha(\mathbb{R}^{d+1}_+)$, and by replacing $\varphi(x)$ by $\varphi(\rho x)$ in the above inequality, we obtain

$$\|\varphi\|^p_{\alpha,1}\leq\left(\rho^{\beta(p-1)}\|\varphi\|^p_{\alpha,p}+\rho^{\beta(p-1)-sp}\||x|^s\varphi\|^p_{\alpha,p}\right)\left(\int_{\mathbb{R}^{d+1}_+}\frac{\mathrm{d}\mu_\alpha(x)}{(1+|x|^{sp})^{q/p}}\right)^{p/q}.$$

By taking

$$\rho=\left(\frac{sq}{\beta}-1\right)^{\frac{1}{sp}}\left(\frac{\||x|^s\varphi\|_{\alpha,p}}{\|\varphi\|_{\alpha,p}}\right)^{\frac{1}{s}},$$

and the fact that

$$\int_{\mathbb{R}^{d+1}_+}\frac{\mathrm{d}\mu_\alpha(x)}{(1+|x|^{sp})^{q/p}}=\frac{1}{2^{\frac{\beta}{2}-1}\Gamma(\frac{\beta}{2})}\int_0^\infty\frac{\rho^{\beta-1}\mathrm{d}\rho}{(1+\rho^{sp})^{q/p}}=\frac{\Gamma(\frac{sq-\beta}{sp})\Gamma(\frac{\beta}{sp})}{2^{\frac{\beta}{2}-1}sp\Gamma(\frac{\beta}{2})\Gamma(\frac{q}{p})},$$

we conclude that

$$\|\varphi\|_{\alpha,1}\leq C_2(s)\,\|\varphi\|^{1-\frac{\beta}{sq}}_{\alpha,p}\,\||x|^s\varphi\|^{\frac{\beta}{sq}}_{\alpha,p}.$$

Afterwards

$$\|\chi_\Sigma\mathscr{F}_{W,\alpha}(\varphi)\|_{\alpha,q}\leq(\mu_\alpha(\Sigma))^{\frac{1}{q}}\|\mathscr{F}_{W,\alpha}(\varphi)\|_{\alpha,\infty}\leq(\mu_\alpha(\Sigma))^{\frac{1}{q}}\|\mathscr{F}_{W,\alpha}(\varphi)\|_{\alpha,1}\leq C_2(s)\,(\mu_\alpha(\Sigma))^{\frac{1}{q}}\|\varphi\|^{1-\frac{\beta}{sq}}_{\alpha,p}\||x|^s\varphi\|^{\frac{\beta}{sq}}_{\alpha,p},$$

which gives the second inequality.

(c) Putting $\rho>0$. Taking into account the inequality $\left(\frac{|x|}{\rho}\right)^{\frac{\beta}{2q}}\leq 1+\left(\frac{|x|}{\rho}\right)^{\frac{\beta}{q}}$, it follows that

$$\left\||x|^{\frac{\beta}{2q}}\varphi\right\|_{\alpha,p}\leq\rho^{\frac{\beta}{2q}}\|\varphi\|_{\alpha,p}+\rho^{-\frac{\beta}{2q}}\left\||x|^{\frac{\beta}{q}}\varphi\right\|_{\alpha,p}.$$

Optimizing in ρ, we obtain

$$\left\| |x|^{\frac{\beta}{2q}} \varphi \right\|_{\alpha,p} \leq 2 \|\varphi\|_{\alpha,p}^{\frac{1}{2}} \left\| |x|^{\frac{\beta}{q}} \varphi \right\|_{\alpha,p}^{\frac{1}{2}}.$$

Therefore,we deduce that

$$\|\chi_\Sigma \mathscr{F}_{W,\alpha}(\varphi)\|_{\alpha,q} \leq C_1(\frac{\beta}{2q})\,(\mu_\alpha(\Sigma))^{\frac{1}{2q}} \left\| |x|^{\frac{\beta}{2q}} \varphi \right\|_{\alpha,p} \leq 2C_1(\frac{\beta}{2q}) \leq (\mu_\alpha(\Sigma))^{\frac{1}{2q}} \|\varphi\|_{\alpha,p}^{\frac{1}{2}} \left\| |x|^{\frac{\beta}{q}} \varphi \right\|_{\alpha,p}^{\frac{1}{2}},$$

which gives the result for $s = \beta/q$. □

THEOREM 2. *Let $s,t > 0$, under assumption (A3), we have*

$$\|\mathscr{F}_{W,\alpha}(\varphi)\|_{\alpha,q} \leq \begin{cases} C_1(s,t)\, \||x|^s\varphi\|_{\alpha,p}^{\frac{t}{s+t}} \,\||y|^t \mathscr{F}_{W,\alpha}(\varphi)\|_{\alpha,q}^{\frac{s}{s+t}}, & sq < \beta; \\ C_2(s,t)\, \|\varphi\|_{\alpha,p}^{\frac{t(sq-\beta)}{s(qt+\beta)}} \,\||x|^s\varphi\|_{\alpha,p}^{\frac{t\beta}{s(\beta+qt)}} \,\||y|^t \mathscr{F}_{W,\alpha}(\varphi)\|_{\alpha,q}^{\frac{\beta}{\beta+qt}}, & sq > \beta; \\ C_3(s,t)\, \|\varphi\|_{\alpha,p}^{\frac{t}{s+2t}} \,\||x|^s\varphi\|_{\alpha,p}^{\frac{t}{s+2t}} \,\||y|^t \mathscr{F}_{W,\alpha}(\varphi)\|_{\alpha,q}^{\frac{s}{s+2t}}, & sq = \beta, \end{cases}$$

where

$$C_1(s,t) = \frac{\left(\frac{t}{s}\right)^{\frac{s-t}{(s+t)q}}}{\left(2^{\frac{\beta}{2}}\Gamma\left(\frac{\beta}{2}+1\right)\right)^{\frac{st}{\beta(s+t)}}} (C_1(s))^{\frac{t}{s+t}}, \quad C_2(s,t) = \frac{\left(\frac{qt}{\beta}\right)^{\frac{\beta-qt}{(\beta+qt)q}}}{\left(2^{\frac{\beta}{2}}\Gamma\left(\frac{\beta}{2}+1\right)\right)^{\frac{t}{\beta+qt}}} (C_2(s))^{\frac{qt}{\beta+qt}},$$

and

$$C_3(s,t) = \frac{\left(\frac{2t}{s}\right)^{\frac{s-2t}{(s+2t)q}}}{\left(2^{\frac{\beta}{2}}\Gamma\left(\frac{\beta}{2}+1\right)\right)^{\frac{t}{\beta+2qt}}} \left(2C_1\left(\frac{s}{2}\right)\right)^{\frac{2t}{s+2t}}.$$

Proof. (a) Let $s,t > 0$, $1 < p \leq 2$, with $q = p/(p-1)$ and $\varphi \in L^p_\alpha(\mathbb{R}^{d+1}_+)$. Then, we have

$$\|\mathscr{F}_{W,\alpha}(\varphi)\|_{\alpha,q}^q = \left\|\chi_{B_\rho}\mathscr{F}_{W,\alpha}(\varphi)\right\|_{\alpha,q}^q + \left\|\chi_{B_\rho^c}\mathscr{F}_{W,\alpha}(\varphi)\right\|_{\alpha,q}^q. \tag{10}$$

On the other hand, we have

$$\left\|\chi_{B_\rho^c}\mathscr{F}_{W,\alpha}(\varphi)\right\|_{\alpha,q}^q \leq \rho^{-qt} \,\||y|^t \mathscr{F}_{W,\alpha}(\varphi)\|_{\alpha,q}^q. \tag{11}$$

According to the relation (3) and Theorem 1, we get

$$\left\|\chi_{B_\rho}\mathscr{F}_{W,\alpha}(\varphi)\right\|_{\alpha,q}^q \leq C_1\rho^{qs} \,\||x|^s\varphi\|_{\alpha,q}^q, \quad \text{with} \quad C_1 = (C_1(s))^q \left(2^\beta\Gamma(\frac{\beta}{2}+1)\right)^{-\frac{sq}{\beta}}. \tag{12}$$

By Combining the above relations (10), (11) and (12), we get

$$\|\mathscr{F}_{W,\alpha}(\varphi)\|_{\alpha,q}^q \leq C_1\rho^{qs} \,\||x|^s\varphi\|_{\alpha,q}^q + \rho^{-qt} \,\left\||y|^t \mathscr{F}_{W,\alpha}(\varphi)\right\|_{\alpha,q}^q.$$

Hence, the first inequality holds by choosing

$$\rho = \left(\frac{t}{sC_1}\right)^{\frac{1}{q(s+t)}} \left(\frac{\||y|^t \mathscr{F}_{W,\alpha}(\varphi)\|_{\alpha,q}}{\||x|^s\varphi\|_{\alpha,q}}\right)^{\frac{1}{s+t}}.$$

(b) We assume that $sq > \beta$, $t > 0$ and $\rho > 0$. Then, according to the relation (3) and Theorem 1, we get

$$\left\|\chi_{B_\rho}\mathscr{F}_{W,\alpha}(\varphi)\right\|_{\alpha,q}^q \leq C_2\rho^\beta \,\|\varphi\|_{\alpha,p}^{q-\frac{\beta}{s}} \,\||x|^s\varphi\|_{\alpha,p}^{\frac{\beta}{s}}, \quad \text{with} \quad C_2 = (C_2(s))^q \left(2^\beta\Gamma(\frac{\beta}{2}+1)\right)^{-1}. \tag{13}$$

Next, by combining the relations (10), (11) and (13), we get

$$\left\|\chi_{B_\rho}\mathscr{F}_{W,\alpha}(\varphi)\right\|_{\alpha,q}^q \le C_2\rho^\beta \|\varphi\|_{\alpha,p}^{q-\frac{\beta}{s}} \||x|^s\varphi\|_{\alpha,p}^{\frac{\beta}{s}} + \rho^{-qt}\left\||y|^t\mathscr{F}_{W,\alpha}(\varphi)\right\|_{\alpha,q}^q.$$

We obtain the second inequality, by choosing

$$\rho = \left(\frac{qt}{\beta C_2}\right)^{\frac{1}{\beta+qt}} \left(\frac{\||y|^t\mathscr{F}_{W,\alpha}(\varphi)\|_{\alpha,q}^q}{\|\varphi\|_{\alpha,p}^{q-\frac{\beta}{s}} \||x|^s\varphi\|_{\alpha,p}^{\frac{\beta}{s}}}\right)^{\frac{1}{\beta+qt}}.$$

(c) We assume that $sq=\beta$, $t>0$ and $\rho>0$. Then, according to Theorem 1, we get

$$\int_{B_\rho}|\mathscr{F}_{W,\alpha}(\varphi)(y)|^q \mathrm{d}\mu_\alpha(y) \le C_3\rho^{\frac{\beta}{2}}\|\varphi\|_{\alpha,p}^{\frac{q}{2}}\left\||x|^{\frac{\beta}{q}}\varphi\right\|_{\alpha,p}^{\frac{q}{2}}, \quad \text{with} \quad C_3 = \left(C_1(\frac{\beta}{2q})\right)^q\left(2^\beta\Gamma(\frac{\beta}{2}+1)\right)^{-\frac{1}{2}}.$$

Thus, we get

$$\left\|\chi_{B_\rho}\mathscr{F}_{W,\alpha}(\varphi)\right\|_{\alpha,q}^q \le C_3\rho^{\frac{\beta}{2}}\|\varphi\|_{\alpha,p}^{\frac{q}{2}}\left\||x|^{\frac{\beta}{q}}\varphi\right\|_{\alpha,p}^{\frac{q}{2}} + \rho^{-qt}\left\||y|^t\mathscr{F}_{W,\alpha}(\varphi)\right\|_{\alpha,q}^q.$$

We obtain the second inequality, by choosing

$$\rho = \left(\frac{2qt}{\beta C_3}\right)^{\frac{2}{\beta+2qt}} \left(\frac{\||y|^t\mathscr{F}_{W,\alpha}(\varphi)\|_{\alpha,q}^q}{\|\varphi\|_{\alpha,p}^{\frac{1}{2}}\left\||x|^{\frac{\beta}{q}}\varphi\right\|_{\alpha,p}^{\frac{1}{2}}}\right)^{\frac{2q}{\beta+2qt}}.$$

□

4. L^p-DONOHO-STRAK UNCERTAINTY PRINCIPLES

This section is devoted to establish two continuous-time uncertainty inequalities of concentration type.

Definition 1 [11]. Let Ω and Σ be a measurable subsets of $\mathbb{R}_+^{d+1}$. The timelimiting operator P_Ω, is defined by

$$P_\Omega\varphi := \varphi\chi_\Omega$$

and the Weinstein integral operator Q_Σ is given by

$$\mathscr{F}_{W,\alpha}(Q_\Sigma\varphi) = \mathscr{F}_{W,\alpha}(\varphi)\chi_\Sigma.$$

Under assumptions (A2) and (A3) then we have the integral representation [11, Proposition 4.2] of Q_Σ

$$Q_\Sigma\varphi(x) = \mathscr{F}_{W,\alpha}^{-1}(\mathscr{F}_{W,\alpha}(\varphi)\chi_\Sigma) = \int_\Sigma \Lambda_\alpha^d(x,\lambda)\mathscr{F}_{W,\alpha}(\varphi)(\lambda)d\mu_\alpha(\lambda). \tag{14}$$

Using the L^p-local uncertainty inequality introduced in Theorem 1, we get the below inequalities for $Q_\Sigma\varphi$.

LEMMA 1. *Let $s>0$. under assumptions (A1) and (A3), we have*

$$\|Q_\Sigma\varphi\|_{\alpha,q} \le \begin{cases} C_1(s)(\mu_\alpha(\Sigma))^{\frac{s}{\beta}+\frac{2}{p}-1}\||x|^s\varphi\|_{\alpha,p}, & sq<\beta; \\ C_2(s)(\mu_\alpha(\Sigma))^{\frac{1}{p}}\|\varphi\|_{\alpha,p}^{1-\frac{\beta}{sq}}\||x|^s\varphi\|_{\alpha,p}^{\frac{\beta}{sq}}, & sq>\beta; \\ 2C_1(\frac{s}{2})(\mu_\alpha(\Sigma))^{\frac{3}{2p}-\frac{1}{2}}\|\varphi\|_{\alpha,p}^{\frac{1}{2}}\||x|^s\varphi\|_{\alpha,p}^{\frac{1}{2}}, & sq=\beta, \end{cases}$$

where $C_1(s)$ and $C_2(s)$ are determined in Theorem 1.

Proof. According to (6) and Hölder's inequality, we get

$$\|Q_\Sigma\varphi\|_{\alpha,q} \leq \|\chi_\Sigma \mathscr{F}_{W,\alpha}(\varphi)\|_{\alpha,p} \leq (\mu_\alpha(\Sigma))^{\frac{2}{p}-1} \|\chi_\Sigma \mathscr{F}_{W,\alpha}(\varphi)\|_{\alpha,q}.$$

Then, the results follows from Theorem 1. □

Definition 2 [11]. Under the assumptions $(A_1)-(A_3)$ we give the following definitions.

(i) A function φ is ε_Ω-concentrated to Ω in $L^p_\alpha(\mathbb{R}^{d+1}_+)$-norm, if

$$\varepsilon_\Omega \|\varphi\|_{\alpha,p} \geq \|\varphi - P_\Omega\varphi\|_{\alpha,p}. \tag{15}$$

(ii) $\mathscr{F}_{W,\alpha}(\varphi)$ is ε_Σ-concentrated to Σ in $L^q_\alpha(\mathbb{R}^{d+1}_+)$-norm, with $q = p/(p-1)$, if

$$\varepsilon_\Sigma \|\mathscr{F}_{W,\alpha}(\varphi)\|_{\alpha,q} \geq \|\mathscr{F}_{W,\alpha}(\varphi) - \mathscr{F}_{W,\alpha}(Q_\Sigma\varphi)\|_{\alpha,q}. \tag{16}$$

(iii) A function ψ is bandlimited to Σ if $Q_\Sigma\psi = \psi$ and we note by $\mathscr{B}^p_\alpha(\Sigma)$ the set of all functions $\psi \in L^p(\mathbb{R}^d_+)$ that are bandlimited to Σ.

LEMMA 2. *Let $s>0$, $\mu_\alpha(\Sigma)>0$ and $\varphi \in \mathscr{B}^p_\alpha(\Sigma)$. Then, under the assumptions $(A_1)-(A_3)$, the space of bandlimited functions satisfies the following property*

$$\|P_\Omega\varphi\|_{\alpha,p} \leq \begin{cases} C_1(s)(\mu_\alpha(\Omega))^{\frac{1}{p}}(\mu_\alpha(\Sigma))^{\frac{1}{p}+\frac{s}{\beta}} \||x|^s\varphi\|_{\alpha,p}, & sq<\beta; \\ C_2(s)(\mu_\alpha(\Omega))^{\frac{1}{p}}(\mu_\alpha(\Sigma)) \|\varphi\|_{\alpha,p}^{1-\frac{\beta}{sq}} \||x|^s\varphi\|_{\alpha,p}^{\frac{\beta}{sq}}, & sq>\beta; \\ 2C_1(\frac{s}{2})(\mu_\alpha(\Omega))^{\frac{1}{p}}(\mu_\alpha(\Sigma))^{\frac{1}{2p}+\frac{1}{2}} \|\varphi\|_{\alpha,p}^{\frac{1}{2}} \||x|^s\varphi\|_{\alpha,p}^{\frac{1}{2}}, & sq=\beta, \end{cases}$$

where $C_1(s)$ and $C_2(s)$ are constants determined in Theorem 1.

Proof. The inequality is trivial if $\mu_\alpha(\Omega)=\infty$. Now, assume that $\mu_\alpha(\Omega)$ is finite. Then, under the hypothesis of the lemma, relation (14), inequality (2) and Hölder's inequality, we get

$$|\varphi(x)| \leq (\mu_\alpha(\Sigma))^{\frac{1}{p}} \|\chi_\Sigma \mathscr{F}_{W,\alpha}(\varphi)\|_{\alpha,q}.$$

Therefore,

$$\|P_\Omega\varphi\|_{\alpha,p} \leq (\mu_\alpha(\Omega))^{\frac{1}{p}}(\mu_\alpha(\Sigma))^{\frac{1}{p}} \|\chi_\Sigma \mathscr{F}_{W,\alpha}(\varphi)\|_{\alpha,q}.$$

Finally, we get the results by Theorem 1. □

THEOREM 3. *Let $s>0$, $\mu_\alpha(\Sigma)>0$ and $\varphi \in \mathscr{B}^p_\alpha(\Sigma)$ that is ε_Ω-concentrated to Ω in $L^p_\alpha(\mathbb{R}^{d+1}_+)$-norm. Then, under the assumptions $(A_1)-(A_3)$, we have*

$$\|\varphi\|_{\alpha,p} \leq \begin{cases} \frac{C_1(s)}{1-\varepsilon_\Omega}(\mu_\alpha(\Omega))^{\frac{1}{p}}(\mu_\alpha(\Sigma))^{\frac{1}{p}+\frac{s}{\beta}} \||x|^s\varphi\|_{\alpha,p}, & sq<\beta; \\ \left(\frac{C_2(s)}{1-\varepsilon_\Omega}\right)^{\frac{sq}{\beta}}(\mu_\alpha(\Omega))^{\frac{sq}{\beta p}}(\mu_\alpha(\Sigma))^{\frac{sq}{\beta}} \||x|^s\varphi\|_{\alpha,p}, & sq>\beta; \\ \left(\frac{2C_1(\frac{s}{2})}{1-\varepsilon_\Omega}\right)^2(\mu_\alpha(\Omega))^{\frac{2}{p}}(\mu_\alpha(\Sigma))^{\frac{1}{p}+1} \||x|^s\varphi\|_{\alpha,p}, & sq=\beta, \end{cases}$$

where $C_1(s)$ and $C_2(s)$ are constants determined in Theorem 1.

Proof. Let $\varphi \in \mathscr{B}^p_\alpha(\Sigma)$, $1<p\leq 2$. Since φ is ε_Ω-concentrated to Ω in $L^p_\alpha(\mathbb{R}^{d+1}_+)$-norm, then according to the inequality (15), we get

$$\|\varphi\|_{\alpha,p} \leq \varepsilon_\Omega \|\varphi\|_{\alpha,p} + \|P_\Omega\varphi\|_{\alpha,p}.$$

Therefore,

$$\|\varphi\|_{\alpha,p} \leq \frac{1}{1-\varepsilon_\Omega} \|P_\Omega \varphi\|_{\alpha,p}.$$

Finally, we obtain the results by Lemma 2. □

THEOREM 4. *Let $s>0$, $\mu_\alpha(\Sigma)>0$ and $\varphi \in L^p_\alpha(\mathbb{R}^{d+1}_+)$ such that $\mathscr{F}_{W,\alpha}(\varphi)$ is ε_Σ-concentrated to Σ in $L^q_\alpha(\mathbb{R}^{d+1}_+)$-norm. Then, under the assumptions $(A_1)-(A_3)$, we have*

$$\|\mathscr{F}_{W,\alpha}(\varphi)\|_{\alpha,q} \leq \begin{cases} \frac{C_1(s)}{1-\varepsilon_\Sigma}(\mu_\alpha(\Sigma))^{\frac{s}{\beta}} \, \||x|^s\varphi\|_{\alpha,p}, & sq<\beta; \\ \frac{C_2(s)}{1-\varepsilon_\Sigma}(\mu_\alpha(\Sigma))^{\frac{1}{q}} \, \|\varphi\|_{\alpha,p}^{1-\frac{\beta}{sq}} \, \||x|^s\varphi\|_{\alpha,p}^{\frac{\beta}{sq}}, & sq>\beta; \\ \frac{2C_1(\frac{s}{2})}{1-\varepsilon_\Sigma}(\mu_\alpha(\Sigma))^{\frac{1}{2q}} \, \|\varphi\|_{\alpha,p}^{\frac{1}{2}} \, \||x|^s\varphi\|_{\alpha,p}^{\frac{1}{2}}, & sq=\beta, \end{cases}$$

where $C_1(s)$ and $C_2(s)$ are constants determined in Theorem 1.

Proof. Let $\varphi \in L^p_\alpha(\mathbb{R}^{d+1}_+)$, $1<p\leq 2$. Since $\mathscr{F}_{W,\alpha}(\varphi)$ is ε_Σ-concentrated to Σ in $L^q_\alpha(\mathbb{R}^{d+1}_+)$-norm, then according to the inequality (16), we get

$$\|\mathscr{F}_{W,\alpha}(\varphi)\|_{\alpha,q} \leq \varepsilon_\Sigma \|\mathscr{F}_{W,\alpha}(\varphi)\|_{\alpha,q} + \|\chi_\Sigma \mathscr{F}_{W,\alpha}(\varphi)\|_{\alpha,q}.$$

Therefore,

$$\|\mathscr{F}_{W,\alpha}(\varphi)\|_{\alpha,q} \leq \frac{1}{1-\varepsilon_\Omega} \|\chi_\Sigma \mathscr{F}_{W,\alpha}(\varphi)\|_{\alpha,q}.$$

Finally, we obtain the results by Theorem 1. □

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