THE PUBLISHING HOUSE
OF THE ROMANIAN ACADEMY

PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A
Volume xx, Number yy/2023, pp. –

NEW LIMIT THEOREMS RELATED TO FERMI CONVOLUTION

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Abstract. This paper is a continuation of study of Fermi convolution from the perspective of Cauchy-Stieltjes Kernel (CSK) families. By the use of variance function machinery, we prove some new limit theorems related to Fermi convolution. We give an approximation of elements of the CSK family generated by the Fermi Gaussian distribution. We also provide a new limit theorem related to Fermi convolution and involving free multiplicative convolution.

Key words: variance function, Cauchy kernel, Fermi convolution, Fermi-Gaussian law, free multiplicative convolution.
Mathematics Subject Classification (MSC2020): 60E10, 46L54.

1. INTRODUCTION

Limit theorems in non-commutative probability are one of the main topics of interests. Limit theorems for addition of free and boolean independent random variables are quite parallel to those in probability theory as observed by Bercovici and Pata with an appendix by Philippe Biane [1] and other authors such that Chistyakov and Gotze [8] and Wang [18]. Such a parallelism is remarkable, while the effect of non-commutativity is not very visible in the results.

It is well known that classical, free and boolean convolution is related to all partitions, non-crossing partitions and interval partitions, respectively. A new convolution is investigated in [15], which is called "Fermi convolution". It is related to the set of those non-crossing partitions in which all the inner blocks are singletons. The definition of this operation is simple that at the first sight it might suggest that it is not a new type of convolution at all. Namely, the Fermi convolution of two probability measures μ and ν with mean λ_1 and λ_2 respectively, is the shift of the boolean convolution μ^0 and ν^0 by $\lambda_1+\lambda_2$ where μ^0 and ν^0 are the zero-mean shifts of μ and ν. As for probability measures with zero mean, their Fermi and boolean convolutions coincide, many properties of the boolean convolution remain valid in the Fermi case. However, for measures with nonzero mean there are some important differences between the Fermi and the boolean cases. In [17], the notion of Fermi convolution is studied from a point of view related to the theory of Cauchy-Stieltjes Kernel (CSK) families. A formula is given for variance function under Fermi convolution power. Then, basing on the convergence of a sequence of variance functions, an approximation is provided for the elements of the CSK family generated by the Fermi Poisson distribution.

This paper is a continuation of study of Fermi convolution from the perspective of CSK families. By the use of variance function machinery, we shall prove some new limit theorems related to Fermi convolution. Section 2 will describe some facts regarding CSK families and Fermi convolution as a background for the reader. In section 3, we give an approximation of elements of the CSK family generated by the Fermi Gaussian distribution. In section 4, we provide a new limit theorem related to Fermi convolution and involving free multiplicative convolution.

2. CSK families and Fermi convolution

In the setting of noncommutative probability theory and in analogy with the theory of natural exponential families (NEFs), a theory of CSK families has been recently introduced, it is based on the Cauchy-Stieltjes kernel $1/(1-\theta x)$. Bryc [4] studied CSK families for compactly supported probability measures ν. Bryc and Hassairi [5] continued the study of CSK families by extending the results in [4] to allow probability measures ν with support bounded from one side. Other properties and characterizations of CSK families are also given in [6], [7], [9], [12], [13], [14] and [16]. In the following, we recall some facts regarding CSK families.

Let ν be a non-degenerate probability measure with support bounded from above. Then

$$M_\nu(\theta) = \int \frac{1}{1-\theta x}\nu(dx) \tag{1}$$

is defined for all $\theta \in [0,\theta_+)$ with $1/\theta_+ = \max\{0, \sup \mathrm{supp}(\nu)\}$. For $\theta \in [0,\theta_+)$, we set

$$P_{(\theta,\nu)}(dx) = \frac{1}{M_\nu(\theta)(1-\theta x)}\nu(dx).$$

The set

$$\mathscr{K}_+(\nu) = \{P_{(\theta,\nu)}(dx); \theta \in (0,\theta_+)\}$$

is called the one-sided CSK family generated by ν.
Let $k_\nu(\theta) = \int x P_{(\theta,\nu)}(dx)$ denote the mean of $P_{(\theta,\nu)}$. Then, according to [5, (page 579-580)], the map $\theta \mapsto k_\nu(\theta)$ is strictly increasing on $(0,\theta_+)$, it is given by the formula

$$k_\nu(\theta) = \frac{M_\nu(\theta)-1}{\theta M_\nu(\theta)}. \tag{2}$$

The image of $(0,\theta_+)$ by k_ν is called the (one sided) domain of the means of the family $\mathscr{K}_+(\nu)$, it is denoted $(m_0(\nu), m_+(\nu))$. This leads to a parametrization of the family $\mathscr{K}_+(\nu)$ by the mean. In fact, denoting by ψ_ν the reciprocal of k_ν, and writing for $m \in (m_0(\nu), m_+(\nu))$, $Q_{(m,\nu)}(dx) = P_{(\psi_\nu(m),\nu)}(dx)$, we have that

$$\mathscr{K}_+(\nu) = \{Q_{(m,\nu)}(dx); m \in (m_0(\nu), m_+(\nu))\}.$$

Now let

$$B = B(\nu) = \max\{0, \sup \mathrm{supp}(\nu)\} = 1/\theta_+ \in [0,\infty). \tag{3}$$

Then it is shown in [5] that the bounds $m_0(\nu)$ and $m_+(\nu)$ of the one-sided domain of means $(m_0(\nu), m_+(\nu))$ are given by

$$m_0(\nu) = \lim_{\theta \to 0^+} k_\nu(\theta) \qquad \text{and} \qquad m_+(\nu) = B - \lim_{z \to B^+} \frac{1}{G_\nu(z)}, \tag{4}$$

with $B = B(\nu)$ and $G_\nu(z)$ is the Cauchy transform of ν given by

$$G_\nu(z) = \int \frac{1}{z-x}\nu(dx). \tag{5}$$

It is worth mentioning here that one may define the one-sided CSK family for a measure ν with support bounded from below. This family is usually denoted $\mathscr{K}_-(\nu)$ and parameterized by θ such that $\theta_- < \theta < 0$, where θ_- is either $1/A(\nu)$ or $-\infty$ with $A = A(\nu) = \min\{0, \inf supp(\nu)\}$. The domain of means for $\mathscr{K}_-(\nu)$ is the interval $(m_-(\nu), m_0(\nu))$ with $m_-(\nu) = A - 1/G_\nu(A)$.
If ν has compact support, the natural domain for the parameter θ of the two-sided CSK family $\mathscr{K}(\nu) = \mathscr{K}_+(\nu) \cup \mathscr{K}_-(\nu) \cup \{\nu\}$ is $\theta_- < \theta < \theta_+$.

We come now to the notions of variance and pseudo-variance functions. The variance function

$$m \mapsto V_\nu(m) = \int (x-m)^2 Q_{(m,\nu)}(dx) \tag{6}$$

is a fundamental concept in the theory of CSK families as presented in [4]. Unfortunately, if ν hasn't a first moment which is for example the case for free 1/2-stable law, all the distributions in the CSK family generated by ν have infinite variance. This fact has led the authors in [5] to introduce a notion of pseudo-variance function $\mathbb{V}_\nu(.)$ defined by

$$\mathbb{V}_\nu(m) = m\left(\frac{1}{\psi_\nu(m)} - m\right), \tag{7}$$

If $m_0(\nu) = \int x d\nu$ is finite, then (see [5]) the pseudo-variance function is related to the variance function by

$$\mathbb{V}_\nu(m) = \frac{m}{m-m_0} V_\nu(m). \tag{8}$$

In particular, $\mathbb{V}_\nu = V_\nu$ when $m_0(\nu) = 0$.

The generating measure ν is uniquely determined by the pseudo-variance function $\mathbb{V}_\nu$. In fact, if we set

$$z = z(m) = m + \frac{\mathbb{V}_\nu(m)}{m}, \tag{9}$$

then the Cauchy transform satisfies

$$G_\nu(z) = \frac{m}{\mathbb{V}_\nu(m)}. \tag{10}$$

Also the distribution $Q_{(m,\nu)}(dx)$ may be written as $Q_{(m,\nu)}(dx) = f_\nu(x,m)\nu(dx)$ with

$$f_\nu(x,m) := \begin{cases} \frac{\mathbb{V}_\nu(m)}{\mathbb{V}_\nu(m)+m(m-x)}, & m \neq 0 \quad ; \\ 1, & m = 0, \ \mathbb{V}_\nu(0) \neq 0 \quad ; \\ \frac{\mathbb{V}'_\nu(0)}{\mathbb{V}'_\nu(0)-x}, & m = 0, \ \mathbb{V}_\nu(0) = 0 \quad . \end{cases} \tag{11}$$

We now recall the effect on a CSK family of applying an affine transformation to the generating measure. Consider the affine transformation $\varphi : x \longmapsto (x-\sigma)/\beta$, where $\beta \neq 0$ and $\sigma \in \mathbb{R}$ and let $\varphi(\nu)$ be the image of ν by φ. In other words, if X is a random variable with law ν, then $\varphi(\nu)$ is the law of $(X-\sigma)/\beta$, or $\varphi(\nu) = D_{1/\beta}(\nu \boxplus \delta_{-\sigma})$, where $D_r(\mu)$ denotes the dilation of measure μ by a number $r \neq 0$, that is $D_r(\mu)(U) = \mu(U/r)$. The point m_0 is transformed to $(m_0-\sigma)/\beta$. In particular, if $\beta < 0$ the support of the measure $\varphi(\nu)$ is bounded from below so that it generates the left-sided family $\mathcal{K}_-(\varphi(\nu))$. For m close enough to $(m_0-\sigma)/\beta$, the pseudo-variance function is

$$\mathbb{V}_{\varphi(\nu)}(m) = \frac{m}{\beta(m\beta+\sigma)} \mathbb{V}_\nu(\beta m + \sigma). \tag{12}$$

In particular, if the variance function exists, then $V_{\varphi(\nu)}(m) = \frac{1}{\beta^2} V_\nu(\beta m + \sigma)$.

Note that using the special case where φ is the reflection $\varphi(x) = -x$, one can transform a right-sided CSK family to a left-sided family. If ν has support bounded from above and its right-sided CSK family $\mathcal{K}_+(\nu)$ has domain of means (m_0, m_+) and pseudo-variance function $\mathbb{V}_\nu(m)$, then $\varphi(\nu)$ generates the left-sided CSK family $\mathcal{K}_-(\varphi(\nu))$ with domain of means $(-m_+, -m_0)$ and pseudo-variance function $\mathbb{V}_{\varphi(\nu)}(m) = \mathbb{V}_\nu(-m)$.

Next, we state the following result from [5] which is crucial in the proofs of our limit theorems that will be given in Section 3 and Section 4.

PROPOSITION 1. *Let V_{ν_n} be a family of analytic functions which are variance functions of a sequence of CSK families $(\mathcal{K}(\nu_n))_{n\geq 1}$.*
If $V_{\nu_n} \xrightarrow{n\to+\infty} V$ uniformly in a (complex) neighborhood of $m_0 \in \mathbb{R}$ and if $V(m_0) > 0$, then there is $\varepsilon > 0$ such

that V is the variance function of a CSK family $\mathcal{K}(\nu)$, generated by a probability measure ν parameterized by the mean $m \in (m_0-\varepsilon, m_0+\varepsilon)$.
Moreover, if a sequence of measures $\mu_n \in \mathcal{K}(\nu_n)$ such that $m_1 = \int x\mu_n(dx) \in (m_0-\varepsilon, m_0+\varepsilon)$ does not depends on n, then $\mu_n \xrightarrow{n\to+\infty} \mu$ in distribution, where $\mu \in \mathcal{K}(\nu)$ has the same mean $\int x\mu(dx) = m_1$.

In order to make clear the results of the paper, we need to recall some basic concepts about Fermi convolution introduced in [15]. We will use the following notations. By $\mathscr{P}$ we denote the set of probability measures on $\mathbb{R}$, $\mathscr{P}^2$ is the subset of probability measures with finite mean and variance.

The $\widetilde{B}$-transform is introduced in [15] for $\nu \in \mathscr{P}^2$, by

$$\widetilde{B}_\nu(z) = \lambda z + 1 - \frac{z}{G_{\nu^0}\left(\frac{1}{z}\right)}, \tag{13}$$

where λ is the mean of ν and ν^0 is the zero mean shift of ν. Since a measure $\nu \in \mathscr{P}^2$ is uniquely determined by its Cauchy transform G_ν, the same is true for $\widetilde{B}_\nu$.

Let $\nu_1,\ \nu_2 \in \mathscr{P}^2$. Let $\nu = \nu_1 \bullet \nu_2$ be the Fermi convolution of ν_1 and ν_2. From [15, Theorem 3.1] we have,

$$\widetilde{B}_\nu(z) = \widetilde{B}_{\nu_1}(z) + \widetilde{B}_{\nu_2}(z). \tag{14}$$

Furthermore, $\nu \in \mathscr{P}^2$ and the mean of ν is the sum of the means of ν_1 and ν_2.

We say that the probability measure $\nu \in \mathscr{P}^2$ is infinitely divisible with respect to Fermi convolution if for each $n \in \mathbb{N}$, there exists $\nu_n \in \mathscr{P}^2$ such that

$$\nu = \underbrace{\nu_n \bullet \ldots.. \bullet \nu_n}_{n \text{ times}}.$$

According to [15, Remark 3.2], all probability measures $\nu \in \mathscr{P}^2$ are infinitely divisible in the Fermi sense.

The importance of the $\widetilde{B}$-transform stems from its linear property to Fremi convolution power, that is for all $\alpha > 0$, $\widetilde{B}_{\nu^{\bullet\alpha}}(z) = \alpha \widetilde{B}_\nu(z)$.

To support the the proofs of the limit theorems in next sections, we need to recall the following result from [17].

THEOREM 1. *Suppose $\mathbb{V}_\nu$ is the pseudo-variance function of the CSK family $\mathcal{K}_+(\nu)$ generated by a non degenerate probability measure $\nu \in \mathscr{P}^2$ with $b = \sup \operatorname{supp}(\nu) < +\infty$. For $\alpha > 0$, we have that:*

(i) The support of $\nu^{\bullet\alpha}$ is bounded from above.

(ii) For m close enough to $m_0(\nu^{\bullet\alpha}) = \alpha m_0(\nu)$,

$$\mathbb{V}_{\nu^{\bullet\alpha}}(m) = \alpha \mathbb{V}_\nu(m/\alpha) + m^2(1/\alpha - 1) + m_0(\nu)(\alpha - 1)m. \tag{15}$$

The variance functions of the CSK families generated by ν and $\nu^{\bullet\alpha}$ exists and

$$V_{\nu^{\bullet\alpha}}(m) = \alpha V_\nu(m/\alpha) + m(m - \alpha m_0(\nu))(1/\alpha - 1) + m_0(\nu)(\alpha - 1)(m - \alpha m_0(\nu)). \tag{16}$$

3. Approximation of Fermi-Gaussian CSK family

According to [15], the Fermi Gaussian distribution μ_{0,σ^2} with mean 0 and variance σ^2 is given by

$$\mu_{0,\sigma^2} = \frac{1}{2}(\delta_{-\sigma} + \delta_\sigma).$$

The corresponding Cauchy transform is

$$G_{\mu_{0,\sigma^2}}(z) = \frac{z}{z^2-\sigma^2}.$$

We have, for all $\theta \in (-1/\sigma, 1/\sigma)$

$$M_{\mu_{0,\sigma^2}}(\theta) = \frac{1}{1-\theta^2\sigma^2}$$

and

$$k_{\mu_{0,\sigma^2}}(\theta) = \theta\sigma^2.$$

The inverse of the function $k_{\mu_{0,\sigma^2}}(.)$ is

$$\psi_{\mu_{0,\sigma^2}}(m) = m/\sigma^2$$

for all $m \in (-\sigma, \sigma) = k_{\mu_{0,\sigma^2}}((-1/\sigma, 1/\sigma))$.

The variance function of the CSK family generated by μ_{0,σ^2} is

$$V_{\mu_{0,\sigma^2}}(m) = \mathbb{V}_{\mu_{0,\sigma^2}}(m) = \sigma^2 - m^2.$$

The two sided CSK family generated by μ_{0,σ^2} is given by

$$\mathcal{K}(\mu_{0,\sigma^2}) = \left\{ Q_{(m,\mu_{0,\sigma^2})}(dx) = \mu_{m,\sigma^2}(dx) = \frac{1}{2\sigma}\left[(\sigma-m)\delta_{-\sigma} + (\sigma+m)\delta_{\sigma}\right] : \ m \in (-\sigma,\sigma) \right\}.$$

The family $\mathcal{K}(\mu_{0,\sigma^2})$ consists of Fermi Gaussian distributions with mean $m \in (-\sigma, \sigma)$. Basing on the notion of Fermi convolution, we provide an approximation of elements of the CSK family $\mathcal{K}(\mu_{0,\sigma^2})$.

THEOREM 2. *Suppose the variance function V_ν of a CSK family $\mathcal{K}(\nu)$ is analytic and strictly positive in a neighborhood of $m_0(\nu)$. Then there is $\delta > 0$ such that if, for $\alpha > 0$, $\mathcal{L}(Y_\alpha) \in \mathcal{K}(\nu_\alpha)$, with $\nu_\alpha = D_{1/\alpha}(\nu^{\bullet\alpha})$, has mean $\mathbb{E}(Y_\alpha) = m_0 + m/\sqrt{\alpha}$ with $|m| < \delta$, then*

$$\sqrt{\alpha}(Y_\alpha - m_0(\nu)) \xrightarrow{\alpha\to+\infty} Q_{(m,\mu_{0,\sigma^2})} \qquad \textit{in distribution},$$

where $\sigma^2 = V_\nu(m_0)$.

Proof. Since $\mathcal{L}(Y_\alpha)$ is in the CSK family $\mathcal{K}(\nu_\alpha)$ having variance function of the form

$$V_{\nu_\alpha}(m) = V_\nu(m)/\alpha + (1/\alpha - 1)m(m - m_0(\nu)) + m_0(\nu)(1 - 1/\alpha)(m - m_0(\nu)),$$

then $\mathcal{L}(\sqrt{\alpha}(Y_\alpha - m_0(\nu)))$ is in the CSK family having variance function of the form

$$\begin{aligned} V_\alpha(m) &= \alpha V_{\nu_\alpha}(m/\sqrt{\alpha} + m_0) \\ &= \alpha\left[V_\nu(m/\sqrt{\alpha}+m_0)/\alpha + \left(\frac{1}{\alpha}-1\right)\frac{m}{\sqrt{\alpha}}\left(\frac{m}{\sqrt{\alpha}}+m_0\right) + m_0\left(1-\frac{1}{\alpha}\right)\frac{m}{\sqrt{\alpha}}\right] \\ &= V_\nu(m/\sqrt{\alpha}+m_0) + m^2(1/\alpha - 1). \end{aligned}$$

We use Proposition 1 to the sequence of variance functions $V_\alpha(m)$ and we obtain

$$V_\alpha(m) \xrightarrow{\alpha\to+\infty} V_\nu(m_0) - m^2.$$

From proposition 1, we deduce that there is $\delta > 0$ such that if $|m| < \delta$ and $\mathbb{E}(Y_\alpha) = m_0 + m/\sqrt{\alpha}$, then with $\sigma^2 = V_\nu(m_0)$,

$$\mathcal{L}(\sqrt{\alpha}(Y_\alpha - m_0(\nu))) \xrightarrow{\alpha\to+\infty} \mu_{m,\sigma^2} \in \mathcal{K}(\mu_{0,\sigma^2}) \ \text{ in distribution.}$$

□

4. A new limit theorem involving free multiplicative convolution

In this section, we derive a new limit theorem related to the Fermi convolution and involving the free multiplicative convolution. According to Proposition 1, this leads to some new variance function with non usual form. We first recall, from [16], some facts concerning the effect of free multiplicative convolution on CSK families.

Denote by $\mathscr{P}_+$ the set of Borel probability measures on $\mathbb{R}_+$ and by $\mathscr{P}_+^2$ the set of Borel probability measures on $\mathbb{R}_+$ with finite mean and variance. Let $\nu \in \mathscr{P}_+$ such that $\delta = \nu(\{0\}) < 1$, and consider the function

$$\Psi_\nu(z) = \int_0^{+\infty} \frac{zx}{1-zx} \nu(dx), \quad z \in \mathbb{C} \setminus \mathbb{R}_+. \tag{17}$$

The function Ψ_ν is univalent in the left half-plane $i\mathbb{C}^+$ and its image $\Psi_\nu(i\mathbb{C}^+)$ is contained in the circle with diameter $(\nu(\{0\})-1, 0)$. Moreover $\Psi_\nu(i\mathbb{C}^+) \cap \mathbb{R} = (\nu(\{0\})-1, 0)$. Let $\chi_\nu : \Psi_\nu(i\mathbb{C}^+) \longrightarrow i\mathbb{C}^+$ be the inverse function of Ψ_ν. Then the S-transform of ν is the function

$$S_\nu(z) = \chi_\nu(z) \frac{1+z}{z}. \tag{18}$$

The product of S-transforms is an S-transform, so that the multiplicative free convolution $\nu_1 \boxtimes \nu_2$ of the measures ν_1 and ν_2 is defined by

$$S_{\nu_1 \boxtimes \nu_2}(z) = S_{\nu_1}(z) S_{\nu_2}(z).$$

We say that a probability measure $\nu \in \mathscr{P}_+$ is infinitely divisible with respect to $\boxtimes$, if for each $n \in \mathbb{N}$, there exists $\nu_n \in \mathscr{P}_+$ such that

$$\nu = \underbrace{\nu_n \boxtimes \dots\dots \boxtimes \nu_n}_{n \text{ times}}.$$

The free multiplicative convolution power $\nu^{\boxtimes \alpha}$ is defined at least for all $\alpha \geq 1$ (see [2, Theorem 2.17]) by $S_{\nu^{\boxtimes \alpha}}(z) = S_\nu(z)^\alpha$. For more details about the S-transform and the free multiplicative convolution, see [3].

The action of the free multiplicative convolution power on the pseudo-variance function is given in [16], that is: for $\nu \in \mathscr{P}_+$ and $t > 0$ such that $\nu^{\boxtimes t}$ is well defined and for $m \in (m_-(\nu^{\boxtimes t}), m_0(\nu^{\boxtimes t})) = ((m_-(\nu))^t,\ (m_0(\nu))^t)$, we have

$$\mathbb{V}_{\nu^{\boxtimes t}}(m) = m^{2-2/t} \mathbb{V}_\nu\left(m^{1/t}\right). \tag{19}$$

Furthermore, if $m_0(\nu)$ is finite, then the variance functions of the CSK families generated by ν and $\nu^{\boxtimes \alpha}$ exists and

$$V_{\nu^{\boxtimes \alpha}}(m) = \frac{m - m_0^\alpha}{m^{1/\alpha} - m_0} m^{1-1/\alpha} V_\nu\left(m^{1/\alpha}\right). \tag{20}$$

Next, we state and prove the main result of this paragraph.

THEOREM 3. *Let $\nu \in \mathscr{P}_+^2$ with mean $m_0(\nu) > 0$. Then denoting $\gamma = \frac{Var(\nu)}{(m_0(\nu))^2} = \frac{V_\nu(m_0)}{m_0^2}$, we have*

$$D_{1/(nm_0^n)}\left(\nu^{\boxtimes n}\right)^{\bullet n} \xrightarrow{n \to +\infty} \kappa_\gamma \qquad \textit{in distribution},$$

where κ_γ is such that $m_0(\kappa_\gamma) = 1$, $(m_-(\kappa_\gamma), m_0(\kappa_\gamma)) \subset (0,1)$ and the variance function of the CSK family generated by κ_γ is given for $m \in (m_-(\kappa_\gamma), m_0(\kappa_\gamma))$, by

$$V_{\kappa_\gamma}(m) = \frac{m(m-1)}{m_0^2 \ln(m)} V_\nu(m_0) - (1-m)^2 = \frac{\gamma m(m-1)}{\ln(m)} - (1-m)^2. \tag{21}$$

Proof. We have that

$$m_0\left(D_{1/(nm_0(\nu)^n)}\left(\nu^{\boxtimes n}\right)^{\bullet n}\right)=\frac{1}{(nm_0(\nu)^n)}m_0\left(\left(\nu^{\boxtimes n}\right)^{\bullet n}\right)=\frac{1}{(m_0(\nu))^n}m_0\left(\nu^{\boxtimes n}\right)=1.$$

Furthermore,

$$\begin{aligned}
V_{D_{\frac{1}{nm_0^n}}((\nu^{\boxtimes n})^{\bullet n})}(m) &= \frac{1}{n^2m_0^{2n}}V_{(\nu^{\boxtimes n})^{\bullet n}}(nmm_0^n)\\
&= \frac{1}{nm_0^{2n}}V_{\nu^{\boxtimes n}}(mm_0^n)+m(1-m)(1-1/n)+(1-1/n)(m-1)\\
&= \frac{(mm_0^n-m_0^n)m^{1-1/n}m_0^{n-1}V_\nu\left(m^{1/n}m_0\right)}{nm_0^{2n}[(mm_0^n)^{1/n}-m_0]}-(1-1/n)(1-m)^2.\\
&= \frac{(m-1)m^{1-1/n}}{m_0^2\frac{m^{1/n}-1}{1/n}}V_\nu\left(m^{1/n}m_0\right)-(1-1/n)(1-m)^2.\\
&\xrightarrow{n\to+\infty} \frac{m(m-1)}{m_0^2\ln(m)}V_\nu(m_0)-(1-m)^2.
\end{aligned}$$

According to Proposition 1, this implies that

$$D_{1/(nm_0^n)}\left(\nu^{\boxtimes n}\right)^{\bullet n}\xrightarrow{n\to+\infty}\kappa_\gamma \qquad \text{in distribution,}$$

where

$$V_{\kappa_\gamma}(m)=\frac{m(m-1)}{m_0^2\ln(m)}V_\nu(m_0)-(1-m)^2=\frac{\gamma m(m-1)}{\ln(m)}-(1-m)^2,$$

and $m_0(\kappa_\gamma)=m_0\left(D_{1/(nm_0(\nu)^n)}\left(\nu^{\boxtimes n}\right)^{\bullet n}\right)=1.$

On the other hand, it is well known that if a sequence of probability measures μ_n in $\mathscr{P}_+$ is such that $\mu_n\xrightarrow{n\to+\infty}\mu$ in distribution, then $\mu\in\mathscr{P}_+$. Therefore $\kappa_\gamma\in\mathscr{P}_+$ and $m=\int_0^{+\infty}xQ_{(m,\kappa_\gamma)}(dx)>0$. This with $m_0(\kappa_\gamma)=1$ implies that $(m_-(\kappa_\gamma),m_0(\kappa_\gamma))\subset(0,1)$.

□

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Received February, 25, 2023