



TWO-DISTANCE VERTEX-DISTINGUISHING TOTAL COLORING OF SUBCUBIC GRAPHS

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Abstract. A 2-distance vertex-distinguishing total coloring of graph G is a proper total coloring of G such that any pair of vertices at distance of two have distinct sets of colors. The 2-distance vertex-distinguishing total chromatic number $\chi''_{d2}(G)$ of G is the minimum number of colors needed for a 2-distance vertex-distinguishing total coloring of G . In this paper, it's proved that if G is a subcubic graph, then $\chi''_{d2}(G) \leq 7$.

Key words: 2-distance vertex distinguishing total coloring, total coloring, subcubic graph.

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1. INTRODUCTION

All graphs considered in this paper are finite, simple and undirected. We use $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$ to denote the vertex set, edge set, minimum degree and maximum degree of graph G , respectively. The *distance* between two vertices u and v , denoted by $d(u, v)$, is the length of the shortest path connecting them. The graph G is denoted as *Cubic* if it is a 3-regular graph, and *subcubic* if $\Delta(G) \leq 3$. Let C_n be a cycle whose length is n .

A *total- k -coloring* of graph G is a mapping $\phi: V(G) \cup E(G) \rightarrow \{1, 2, \dots, k\}$ so that $\phi(x) \neq \phi(y)$ for any pair of adjacent or incident elements $x, y \in V(G) \cup E(G)$. The *total chromatic index* $\chi''(G)$ of graph G is defined as the smallest integer k to make sure a proper total- k -coloring exist in G . The total coloring of graph G was introduced by Behzad [1] and independently by Vizing [2], and each raised the following conjecture:

CONJECTURE 1. *Every simple graph G has $\chi''(G) \leq \Delta(G) + 2$.*

So far Conjecture 1 remains open. A well-known upper bound of $\chi''(G)$ for simple graph G may be $\Delta(G) + 10^{26}$, by Molloy and Reed [3].

For a total- k -coloring ϕ of G , we use $C_\phi(v) = \{\phi(v)\} \cup \{\phi(xv) | xv \in E(G)\}$ to denote the set of colors assigned to a vertex v and those edges incident to v . The *neighbor-distinguishing total chromatic index* $\chi''_a(G)$ of G is defined as the smallest integer k for which G can be totally- k -colored by using k colors so that $C_\phi(u) \neq C_\phi(v)$ for any pair of adjacent vertices u and v .

Zhang et al. [4] studied neighbor-distinguishing total coloring of cycles, wheels, trees, complete graphs and complete bipartite graphs, and proposed the following conjecture:

CONJECTURE 2. *Every graph G with $|V(G)| \geq 2$ has $\chi''_a(G) \leq \Delta(G) + 3$.*

Wang [5] and Chen [6], independently, proved this Conjecture holds for graphs with $\Delta(G) \leq 3$. Lu et al. [7] proved this Conjecture holds for graphs with $\Delta(G) = 4$, and Papaioannou et al. [8] verified this Conjecture for 4-regular graphs. Applying a probabilistic analysis, Coker et al. [9] verified that $\chi''_a(G) \leq \Delta(G) + C$, where C

is a constant. Huang et al. [10] proved that $\chi''_a(G) \leq 2\Delta(G)$ for any graph G with $\Delta(G) \geq 3$. The conjecture is still open for planar graphs. Furthermore, Chang et al. [11] proved that $\chi''_a(G) \leq \Delta(G) + 3$ for every planar graph G with $\Delta(G) \geq 8$.

The *2-distance vertex-distinguishing total coloring* of graph G is a proper total coloring of G such that $C_\phi(u) \neq C_\phi(v)$ for any pair of vertices u and v with $d(u, v) = 2$. The *2-distance vertex-distinguishing total chromatic index* $\chi''_{d2}(G)$ of graph G is the smallest integer k such that G has a 2-distance vertex-distinguishing total coloring using k colors.

Hu et al. [12] studied 2-distance vertex-distinguishing total coloring of paths, cycles, wheels, trees, unicycle graphs, $P_m \times P_n$ and $C_m \times P_n$. Then they proposed the following conjecture:

CONJECTURE 3. *Every simple graph G has $\chi''_{d2}(G) \leq \Delta(G) + 3$.*

In this paper, we will prove $\chi''_{d2}(G) \leq 7$ for any subcubic graphs.

2. MAIN RESULTS

Before showing our main result, we introduce a few of concepts and notation. $N_G(v)$ denotes the set of neighbors of the vertex v and $d_G(v)$ denotes the degree of the vertex v in G . A vertex of degree k is called k -vertex. Similarly, a vertex of degree at least k (at most k) is called k^+ -vertex (k^- -vertex). A 3-vertex v is called a 3_i -vertex if v is adjacent to exactly i 2-vertices for $0 \leq i \leq 3$. Let $\chi(G)$ denote the chromatic index of G , which is the least integer k for which G has a vertex coloring using k colors such that any two adjacent vertices get distinct colors.

In what follows, a 2-distance vertex-distinguishing total k -coloring of G is shortly written as a 2DVDT- k -coloring. Two vertices $u, v \in V(G)$ with $d(u, v) = 2$ are called a conflict with respect to the coloring ϕ if $C_\phi(u) = C_\phi(v)$. Otherwise, they are called compatible. For a subgraph G' of G and a 2DVDT-coloring ϕ of G' , we say that ϕ is a *legal coloring* of G' for short.

The proof of main result is based on the following facts:

LEMMA 1. ([12]) *Let G be a simple graph with $\Delta(G) \leq 2$, then $\chi''_{d2}(G) \leq 5$.*

LEMMA 2. ([13]) *If G is a connected graph and is neither an odd cycle nor a complete graph, then $\chi(G) \leq \Delta$.*

LEMMA 3. ([14]) *Every connected cubic graph G without cut edges can be edge-partitioned into a perfect matching and a class of cycles.*

THEOREM 1. *If G be a subcubic graph, then $\chi''_{d2}(G) \leq 7$.*

Proof. The proof is by contradiction. Let G be a minimum counterexample in the Theorem 1 to make its edges $E(G)$ as small as possible. Obviously, G is a connected graph and $\chi''_{d2}(G) > 7$. However, if $|E(H)| < |E(G)|$ for any graph H , then $\chi''_{d2}(H) \leq 7$. Assume that $C = \{1, 2, \dots, 7\}$ is a color set and ϕ is a 2DVDT-7-coloring. For the sake of simplicity in the following proof, we write $C_\phi(v)$ as $C(v)$ for a vertex $v \in V(G')$.

If $\Delta(G) \leq 2$, then graph G is a cycle or a path. From Lemma 1 it is follows that $\chi''_{d2}(G) \leq 5$. So, assume that $\Delta(G) = 3$. To complete the proof, we need to establish a series of auxiliary claims.

CLAIM 1. $\delta(G) \geq 2$.

Proof. Assume to the contrary that G contains a 1-vertex v . Let u be the neighbor of v . Consider the subgraph $G' = G - \{v\}$. Then G' has a 2DVDT-7-coloring ϕ using the color set C by the minimality of G . We have two possibilities:

Case 1. $d_G(u) = 2$.

Let u_1 be the neighbor of u other than v . It suffices to color uv with color $a \in C \setminus \{\phi(u), \phi(uu_1)\}$ so that u is compatible with the neighbors of u_1 , then color v with color in $C \setminus \{\phi(u), \phi(uu_1), a\}$.

Case 2. $d_G(u) = 3$.

Let u_1, u_2 be the neighbors of u other than v . If at least one of u_1 and u_2 is a 2^- -vertex, then it suffices to color uv with color $b \in C \setminus \{\phi(u), \phi(uu_1), \phi(uu_2)\}$ so that u is compatible with the neighbors of u_1 and u_2 , and color v with color in $C \setminus \{\phi(u), \phi(uu_1), \phi(uu_2), b\}$. Otherwise, $d_G(u_1) = d_G(u_2) = 3$. Let $N_G(u_1) = \{u, w_1, w_2\}$ and $N_G(u_2) = \{u, w_3, w_4\}$. Furthermore, assume that $\phi(uu_1) = 1, \phi(uu_2) = 2$ and $\phi(u) = 3$. If uv cannot be legally colored, suppose without loss of generality that $C(w_i) = \{1, 2, 3, i+3\}$ for $i = 1, 2, 3, 4$. It suffices to recolor u with color $a \in \{4, 5, 6, 7\} \setminus \{\phi(u_1), \phi(u_2)\}$, color uv with color in $\{4, 5, 6, 7\} \setminus \{a\}$, and color v with 1. Thus, G has a 2DVDT-7-coloring, a contradiction. \square

CLAIM 2. G does not contain a 3-cycle $v_1v_2v_3v_1$ satisfying one of the following:

- (1) $d_G(v_2) = d_G(v_3) = 2$ and $d_G(v_1) = 3$.
- (2) $d_G(v_2) = d_G(v_3) = 3$ and $d_G(v_1) = 2$.

Proof. (1) Assume to the contrary that G contains a 3-cycle $v_1v_2v_3v_1$ satisfying $d_G(v_2) = d_G(v_3) = 2$ and $d_G(v_1) = 3$. Let u_1 be the neighbor of v_1 other than v_1 and v_3 . Consider the subgraph $G' = G - \{v_2v_3\}$. Then G' has a 2DVDT-7-coloring ϕ using the color set C by the minimality of G . It suffices to color v_2v_3 with color in $C \setminus (C(v_2), C(v_3))$ such that v_2 and v_3 are compatible with u_1 . Thus, G has a 2DVDT-7-coloring, a contradiction.

(2) Assume to the contrary that G contains a 3-cycle $v_1v_2v_3v_1$ satisfying $d_G(v_2) = d_G(v_3) = 3$ and $d_G(v_1) = 2$. Let u_i be the neighbor of v_i other than v_1 . Consider the subgraph $G' = G - \{v_1v_3\}$. Then G' has a 2DVDT-7-coloring ϕ using the color set C by the minimality of G . Assume without loss of generality that $\phi(v_1u_1) = 1, \phi(v_2u_2) = 2, \phi(v_3u_3) = 3$ and $\phi(v_2) = 4$. It is easy to notice that $\phi(v_3) \notin \{2, 4\}$. Then we use color 2 firstly to recolor v_1 . We have to handle the following two situations:

Case 1. $d_G(u_2) = 2$.

Firstly, assume that $\phi(v_3) \in \{1, 3\}$. It suffices to color v_1v_3 with color in $\{4, 5, 6, 7\} \setminus \{\phi(v_3u_3)\}$ such that G can be legally colored. Next, assume that $\phi(v_3) \in \{5, 6, 7\}$. Say $\phi(v_3) = 5$ by symmetry. If $\phi(v_3u_3) \in \{1, 3\}$, it suffices to color v_1v_3 with color in $\{4, 6, 7\}$. Otherwise, $\phi(v_3u_3) \in \{4, 6, 7\}$. Say $\phi(v_3u_3) = 4$ by symmetry. If v_1v_3 cannot be legally colored, assume without loss of generality that $C(u'_3) = \{2, 4, 5, 6\}$ and $C(u''_3) = \{2, 4, 5, 7\}$ with $N_G(u_3) = \{v_3, u'_3, u''_3\}$. It suffices to color v_1v_3 with 3 and recolor v_1 with color 6 or 7.

Case 2. $d_G(u_2) = 3$.

Firstly, assume that $\phi(v_3) = 1$. It suffices to color v_1v_3 with color in $\{3, 4, 5, 6, 7\} \setminus \{\phi(v_3u_3)\}$ such that G can be legally colored. Next, assume that $\phi(v_3) \in \{3, 5, 6, 7\}$. Say $\phi(v_3) = 3$ by symmetry. If $\phi(v_3u_3) = 1$, it suffices to color v_1v_3 with color in $\{4, 5, 6, 7\}$. Otherwise, $\phi(v_3u_3) \in \{4, 5, 6, 7\}$. Say $\phi(v_3u_3) = 4$ by symmetry. If v_1v_3 cannot be legally colored, assume without loss of generality that $C(u'_3) = \{2, 3, 4, 5\}, C(u''_3) = \{2, 3, 4, 6\}$ and $C(u_2) = \{2, 3, 4, 7\}$ with $N_G(u_3) = \{v_3, u'_3, u''_3\}$. It suffices to recolor v_3 with color $a \in \{1, 5, 6, 7\} \setminus \{\phi(u_3)\}$ and color v_1v_3 with color in $\{5, 6, 7\} \setminus \{a\}$. \square

CLAIM 3. G does not contain adjacent 2-vertices.

Proof. Assume to the contrary, G contains adjacent 2-vertices u, v . Let $N_G(u) = \{v, u_1\}$ and $N_G(v) = \{u, v_1\}$. If $d_G(v_1) = 3$, then let $N_G(v_1) = \{v, v'_1, v''_1\}$. By Claim 2 and $\Delta(G) = 3, vv_1 \notin E(G)$. Let $G' = G - \{u\} + \{u_1v\}$. Then G' has a 2DVDT-7-coloring ϕ using the color set C by the minimality of G . We have to handle the following two situations:

Case 1. $d_G(u_1) = 2$.

Let $N_G(u_1) = \{u, u'_1\}$. Based on ϕ , in G , color uu_1 with $\phi(vu_1)$. It suffices to color uv with color $b \in C \setminus \{\phi(v), \phi(vv_1), \phi(uu_1), \phi(u_1u'_1)\}$, and color u with color in $C \setminus (C(u_1) \cup \{b\})$.

Case 2. $d_G(u_1) = 3$.

Let $N_G(u_1) = \{u, u'_1, u''_1\}$. Assume without loss of generality that $\phi(u_1) = 4, \phi(u_1u'_1) = 1, \phi(u_1u''_1) = 2$ and $\phi(vu_1) = 3$. It follows that $\phi(v) \notin \{3, 4\}$ and $\phi(vv_1) \neq 3$. Based on ϕ , in G , color uu_1 with 3. We have to

handle two possibilities:

- Assume that $\phi(v) \in \{1, 2\}$. Say $\phi(v) = 1$ by symmetry. It suffices to color uv with color $b \in \{4, 5, 6, 7\} \setminus \{\phi(vv_1)\}$ such that v is compatible with the neighbours of v_1 , and color u with color in $\{5, 6, 7\} \setminus \{b\}$.
- Assume that $\phi(v) \in \{5, 6, 7\}$. Say $\phi(v) = 5$ by symmetry. If $\phi(vv_1) \notin \{4, 6, 7\}$, then it is similar to the former case of $\phi(v) \in \{1, 2\}$. Otherwise, $\phi(vv_1) \in \{4, 6, 7\}$. Say $\phi(vv_1) = 4$ by symmetry. If $d_G(v_1) = 2$ or $d_G(v_1) = 3$ and $C(v'_1) \neq \{4, 5, i\}$ for $i = 6, 7$. Then it suffices to color uv with $b \in \{6, 7\}$, and color u with color in $\{6, 7\} \setminus \{b\}$. If $d_G(v_1) = 3$ and $(C(v'_1), C(v''_1)) = (\{4, 5, 6\}, \{4, 5, 7\})$. It suffices to recolor v with color $a \in \{1, 2\} \setminus \{\phi(v_1)\}$, and color uv and u with 6 and 7, respectively. \square

CLAIM 4. G does not contain a 4-cycle $v_1v_2v_3v_4v_1$ such that $d_G(v_2) = d_G(v_4) = 2$.

Proof. Assume to the contrary, G contains a 4-cycle $v_1v_2v_3v_4v_1$ such that $d_G(v_2) = d_G(v_4) = 2$. By Claim 2 and Claim 3, $d_G(v_1) = d_G(v_3) = 3$ and $v_1v_3 \notin E(G)$. Then let u_1, u_3 be the neighbors of v_1, v_3 other than v_2 and v_4 , respectively. If $d_G(u_1) = d_G(u_3) = 3$, then let $N_G(u_1) = \{v_1, u'_1, u''_1\}$ and $N_G(u_3) = \{v_3, u'_3, u''_3\}$. There are three situations to be handled:

Case 1. $u_1 = u_3$.

Let $G' = G - \{v_2, v_4\}$. Then G' has a 2DVDT-7-coloring ϕ using the color set C by the minimality of G . Assume without loss of generality that $\phi(v_3u_3) = 1, \phi(v_1u_3) = 2, \phi(u_3u'_3) = 4$ and $\phi(u_3) = 4$ ($d_G(u_3) = 2$ is similar). Remove firstly the colors of v_1 and v_3 , and use 4 to color v_2, v_4 . Next, use 1, 3, 2, 3 to color $v_1, v_1v_4, v_4v_3, v_3v_2$, respectively. Finally, use $b \in \{5, 6\}$ to color v_3 to make v_3 compatible with u'_3 , and color v_1v_2 with color in $\{5, 6, 7\} \setminus \{b\}$ such that v_1 is compatible with u'_3 .

Case 2. $u_1u_3 \notin E(G)$ and $u_1 \neq u_3$.

Let $G' = G - \{v_1, v_2, v_3, v_4\} + \{u_1u_3\}$. Then G' has a 2DVDT-7-coloring ϕ using the color set C by the minimality of G . Assume without loss of generality that $\phi(u_1u_3) = 1, \phi(u_3u'_3) = 2, \phi(u_3u''_3) = 3$ and $\phi(u_3) = 4$ ($d_G(u_3) = 2$ is similar). Based on ϕ , in G , use 1 to color u_1v_1, u_3v_3 . It is easy to notice that color sets of u_1 and u_3 don't change in G , and $\phi(u_1) \neq 4$. So it suffices to color $v_2, v_2v_3, v_3, v_3v_4, v_4, v_1v_4, v_1$ with 2, 4, 5, 6, 2, 3, 4, respectively. And color v_1v_2 with color in $\{5, 6, 7\}$ such that G can be legally colored.

Case 3. $u_1u_3 \in E(G)$ and $u_1 \neq u_3$.

Let $G' = G - \{v_2, v_4\}$. Then G' has a 2DVDT-7-coloring ϕ using the color set C by the minimality of G . Assume without loss of generality that $\phi(u_1u_3) = 2, \phi(u_3u'_3) = 3, \phi(u_3v_3) = 1$ and $\phi(u_3) = 4$ ($d_G(u_3) = 2$ is similar). It is easy to notice that $\phi(u_1) \neq 2$ and $\phi(u_1v_1) \neq 2$. Remove the colors of v_1 and v_3 . It firstly suffices to color $v_1, v_4, v_3v_4, v_3, v_2$ with color 2, 4, 2, 3, 4, respectively. Next, we use $b \in \{5, 6, 7\}$ to color v_2v_3 such that v_3 is compatible with u_1 and u'_3 . Finally, we use $a \in \{1, 3, 5, 6, 7\} \setminus \{b, \phi(u_1v_1)\}$ to color v_2v_2 , and color v_1v_4 with color in $\{3, 5, 6, 7\} \setminus \{a, \phi(v_1u_1)\}$. Obviously, v_1 has at least 4 distinctive color sets. Since v_1 has at most three vertices of conflict, G has a 2DVDT-7-coloring. Thus, a contradiction. \square

CLAIM 5. G does not contain 3_3 -vertex.

Proof. Assume to the contrary, G contains a 3_3 -vertex v . Let $N_G(v) = \{v_1, v_2, v_3\}$ and $d_G(v_1) = d_G(v_2) = d_G(v_3) = 2$. And let u_i be the neighbor of v_i other than v for $i = 1, 2, 3$. By Claim 3, $d_G(u_1) = d_G(u_2) = d_G(u_3) = 3$. Let $N_G(u_i) = \{v_i, u'_i, u''_i\}$ for $i = 1, 2, 3$. By Claim 4, $u_1 \neq u_2$. Consider the subgraph $G' = G - \{v_1\} - \{vv_2\} + \{u_1v_2\}$. Then G' has a 2DVDT-7-coloring ϕ using the color set C by the minimality of G . Assume without loss of generality that $\phi(v_2u_1) = 1, \phi(u_1u'_1) = 2, \phi(u_1u''_1) = 3$ and $\phi(u_1) = 4$. Based on ϕ , in G , we color u_1v_1 with 1. Remove the color of v . There are two possibilities to be handled:

Case 1. $\phi(vv_3) \neq 1$.

We color vv_2 with 1. It is easy to notice that the color set of v_2 doesn't change in G . We have to handle three situations:

(1) $\phi(vv_3) \in \{2, 3\}$. Say $\phi(vv_3) = 2$ by symmetry. Firstly, we use 5 to color v_1 and $b_1 \in \{4, 6, 7\}$ to color vv_1 such that v_1 is compatible with v_2 . It follows that vv_1 can be colored with at least two colors. Next, we have two possibilities need to be handled according to the color of v_3 :

- $\phi(v_3) \in \{1, 4, 5, 6, 7\}$. It suffices to color v with color in $\{3, 4, 6, 7\} \setminus \{\phi(v_2), \phi(v_3), b_1\}$. Obviously, v

has at least one color set that can be distinguished from u_i for $i = 1, 2, 3$.

- $\phi(v_3) = 3$. We color v with $b_2 \in \{4, 6, 7\} \setminus \{\phi(v_2), b_1\}$. If $C(v_2) = \{1, 5, i\}$, $\phi(v_2) = i$, and $C(u_3) = \{1, 2, m, p\}$ for $i \in \{4, 6, 7\}$ and $m, p \in \{4, 6, 7\} \setminus \{i\}$. Then G cannot be legally colored. Thus, it suffices to recolor v_1, vv_1, v with 6, 4, 7, respectively. Otherwise, we are done.

(2) $\phi(vv_3) = 4$. We have three possibilities need to be handled according to the color of v_3 :

- $\phi(v_3) \in \{5, 6, 7\}$. Let $\phi(v_3) = a$. It suffices to color v_1 with a , vv_1 with $b_1 \in \{5, 6, 7\} \setminus \{a\}$ such that v_1 is compatible with v_2 , and color v with $b_2 \in \{2, 3, 5, 6, 7\} \setminus \{\phi(v_2), a, b_1\}$.

- $\phi(v_3) = 1$. It suffices to color v_1 with 5, vv_1 with $b_1 \in \{6, 7\}$ such that v_1 is compatible with v_2 , and color v with $b_2 \in \{2, 3, 6, 7\} \setminus \{\phi(v_2), b_1\}$.

- $\phi(v_3) \in \{2, 3\}$. Say $\phi(v_3) = 2$ by symmetry. It suffices to color v_1 with 2, vv_1 with $b_1 \in \{5, 6, 7\} \setminus \{a\}$ such that v_1 is compatible with v_2 and u'_1 , and color v with $b_2 \in \{3, 5, 6, 7\} \setminus \{\phi(v_2), b_1\}$.

Obviously, v has at least one color set that can be distinguished from u_i for $i = 1, 2, 3$.

(3) $\phi(vv_3) \in \{5, 6, 7\}$. Say $\phi(vv_3) = 5$ by symmetry. We have four possibilities need to be handled according to the color of v_3 :

- $\phi(v_3) \in \{6, 7\}$. Say $\phi(v_3) = 6$ by symmetry. It suffices to color v_1 with 5, vv_1 with $b_1 \in \{4, 7\}$ such that v_1 is compatible with v_2 , and color v with $b_2 \in \{2, 3, 4, 7\} \setminus \{\phi(v_2), b_1\}$.

- $\phi(v_3) = 1$. It suffices to color v_1 with 6, vv_1 with $b_1 \in \{4, 7\}$ such that v_1 is compatible with v_2 , and color v with $b_2 \in \{2, 3, 4, 7\} \setminus \{\phi(v_2), b_1\}$.

- $\phi(v_3) \in \{2, 3\}$. Say $\phi(v_3) = 2$ by symmetry. It suffices to color v_1 with 5, vv_1 with $b_1 \in \{4, 6, 7\}$ such that v_1 is compatible with v_2 , and color v with $b_2 \in \{3, 4, 6, 7\} \setminus \{\phi(v_2), b_1\}$.

- $\phi(v_3) = 4$. It suffices to color v_1 with 6, vv_1 with $b_1 \in \{4, 7\}$ such that v_1 is compatible with v_2 , and color v with $b_2 \in \{2, 3, 7\} \setminus \{\phi(v_2), b_1\}$.

Obviously, v has at least one color set that can be distinguished from u_i for $i = 1, 2, 3$.

Case 2. $\phi(vv_3) = 1$.

It is easy to notice that $\phi(v_2) \notin \{1, 4\}$ and $\phi(v_2u_2) \neq 1$. Remove the color of v_2 . We have three situations to handle according to the value of $\phi(v_3)$:

(1) $\phi(v_3) = 4$. There are two possibilities to be handled:

- $\phi(v_2u_2) \neq 4$. Firstly, assume that $\phi(u_2) \neq 4$. It firstly suffices to color v_2, v_1 with 4, 5, respectively. Then use $b_1 \in \{2, 3, 5, 6, 7\} \setminus \{\phi(v_2u_2)\}$ to color vv_2 , $b_2 \in \{6, 7\} \setminus \{b_1\}$ to color vv_1 , and $b_3 \in \{2, 3, 6, 7\} \setminus \{b_1, b_2\}$ to color v . Next, assume that $\phi(u_2) = 4$. It firstly suffices to color v_2, v_1 with 5, respectively. Then use color $b_1 \in \{2, 3, 4, 6, 7\} \setminus \{\phi(v_2u_2)\}$ to color vv_2 such that v_2 is compatible with u'_2 and u''_2 , $b_2 \in \{6, 7\} \setminus \{b_1\}$ to color vv_1 , and $b_3 \in \{2, 3, 6, 7\} \setminus \{b_1, b_2\}$ to color v .

- $\phi(v_2u_2) = 4$. Firstly, assume that $\phi(u_2) \neq 2$. It firstly suffices to color v_2, v_1 with 2, 5, respectively. Then use color $b_1 \in \{3, 5, 6, 7\}$ to color vv_2 such that v_2 is compatible with u'_2 and u''_2 , $b_2 \in \{6, 7\} \setminus \{b_1\}$ to color vv_1 , and $b_3 \in \{3, 6, 7\} \setminus \{b_1, b_2\}$ to color v . Next, assume that $\phi(u_2) = 2$. It firstly suffices to color v_2, v_1 with 5, respectively. Then use color $b_1 \in \{2, 3, 6, 7\}$ to color vv_2 such that v_2 is compatible with u'_2 and u''_2 , $b_2 \in \{6, 7\} \setminus \{b_1\}$ to color vv_1 , and $b_3 \in \{2, 3, 6, 7\} \setminus \{b_1, b_2\}$ to color v .

Obviously, v has at least one color set that can be distinguished from u_i for $i = 1, 2, 3$.

(2) $\phi(v_3) \in \{2, 3\}$. Say $\phi(v_3) = 2$ by symmetry. There are two possibilities to be handled:

- $\phi(u_2) \neq 1$. By $\phi(v_2u_2) \neq 1$, it firstly suffices to color v_2, v_1 with 1, 5, respectively. Then use color $b_1 \in \{2, 3, 4, 5, 6, 7\} \setminus \{\phi(v_2u_2)\}$ to color vv_2 such that v_2 is compatible with u'_2, u''_2 and v_3 , $b_2 \in \{4, 6, 7\} \setminus \{b_1\}$ to color vv_1 such that v_1 is compatible with v_2 , and $b_3 \in \{3, 4, 6, 7\} \setminus \{b_1, b_2\}$ to color v .

- $\phi(u_2) = 1$. Firstly, assume that $\phi(v_2u_2) \neq 4$. It firstly suffices to color v_2, v_1 with 4, 5, respectively. Then use color $b_1 \in \{2, 3, 5, 6, 7\} \setminus \{\phi(v_2u_2)\}$ to color vv_2 such that v_2 is compatible with u'_2 and u''_2 , $b_2 \in \{4, 6, 7\} \setminus \{b_1\}$ to color vv_1 , and $b_3 \in \{3, 6, 7\} \setminus \{b_1, b_2\}$ to color v . Next, assume that $\phi(v_2u_2) = 4$. It firstly suffices to color v_2, v_1 with 2, 5, respectively. Then use color $b_1 \in \{3, 5, 6, 7\}$ to color vv_2 such that v_2 is compatible with u'_2 and u''_2 , $b_2 \in \{4, 6, 7\} \setminus \{b_1\}$ to color vv_1 , and $b_3 \in \{3, 4, 6, 7\} \setminus \{b_1, b_2\}$ to color v .

Obviously, v has at least one color set that can be distinguished from u_i for $i = 1, 2, 3$.

(3) $\phi(vv_3) \in \{5, 6, 7\}$. Say $\phi(vv_3) = 5$ by symmetry. There are two possibilities to be handled:

- $\phi(u_2) \neq 1$. By $\phi(v_2u_2) \neq 1$, it firstly suffices to color v_2, v_1 with 1, 6, respectively. Then use color $b_1 \in \{2, 3, 4, 5, 6, 7\} \setminus \{\phi(v_2u_2)\}$ to color vv_2 such that v_2 is compatible with u'_2, u''_2 and $v_3, b_2 \in \{4, 7\} \setminus \{b_1\}$ to color vv_1 such that v_1 is compatible with v_2 , and $b_3 \in \{2, 3, 4, 7\} \setminus \{b_1, b_2\}$ to color v . It is easy to notice that vv_2 can be colored with at least two colors, then vv_1 can be colored with at least one color. Obviously, v has at least one color set that can be distinguished from u_i for $i = 1, 2, 3$.

- $\phi(u_2) = 1$. Firstly, assume that $\phi(v_2u_2) \neq 4$. It firstly suffices to color v_2, v_1 with 4, 6, respectively. Then use color $b_1 \in \{2, 3, 5, 6, 7\} \setminus \{\phi(v_2u_2)\}$ to color vv_2 such that v_2 is compatible with u'_2 and $u''_2, b_2 \in \{4, 7\} \setminus \{b_1\}$ to color vv_1 , and $b_3 \in \{2, 3, 7\} \setminus \{b_1, b_2\}$ to color v . It is easy to notice that v_1, v_2 and v_3 do not conflict with each other. Obviously, v has at least one color set that can be distinguished from u_i for $i = 1, 2, 3$. Next, assume that $\phi(v_2u_2) = 4$. We firstly color v_2, v_1 with 2, 6, respectively. Then use color $b_1 \in \{3, 5, 6, 7\}$ to color vv_2 such that v_2 is compatible with u'_2 and $u''_2, b_2 \in \{4, 7\} \setminus \{b_1\}$ to color vv_1 , and $b_3 \in \{3, 4, 7\} \setminus \{b_1, b_2\}$ to color v . It is easy to notice that v_1, v_2 and v_3 do not conflict with each other. If vv_2 can only be colored by 3 and 7, and $C(u_3) = \{1, 3, 4, 7\}$. Then $C(u_2) \neq \{1, 3, 4, 7\}$ and G can not be legally colored. Thus, it suffices to recolor v_1, vv_1, v with 7, 6, 4, respectively. Otherwise, we are done. \square

CLAIM 6. G does not contain 3_2 -vertex.

Proof. Assume to the contrary, G contains a 3_2 -vertex v . Let $N_G(v) = \{v_1, v_2, v_3\}$, $d_G(v_1) = d_G(v_2) = 2$ and let u_i be the neighbor of v_i other than v for $i = 1, 2$. By Claim 3 and Claim 5, $d_G(u_1) = d_G(u_2) = d_G(v_3) = 3$. Let $N_G(u_i) = \{v_i, u'_i, u''_i\}$ for $i = 1, 2$, and $N_G(v_3) = \{v, u_3, u_4\}$. Assume without loss of generality that $2 \leq d_G(u'_1) \leq d_G(u'_2) \leq 3$ and $d_G(u''_1) = d_G(u''_2) = 3$ by Claim 5. By Claim 4, $u_1 \neq u_2$. Then consider the subgraph $G' = G - \{v_1\} - \{vv_2\} + \{u_1v_2\}$. Then G' has a 2DVDT-7-coloring ϕ using the color set C by the minimality of G . Assume without loss of generality that $\phi(v_2u_1) = 1, \phi(u_1u'_1) = 2, \phi(u_1u''_1) = 3$ and $\phi(u_1) = 4$. It easy to notice that $\phi(v_2) \notin \{1, 4\}$ and $\phi(v_2u_2) \neq 1$. Based on ϕ , in G , we color u_1v_1 with 1. Remove the color of v . There are two possibilities to be handled:

Case 1. $\phi(vv_3) \neq 1$.

We color vv_2 with 1. It is easy to notice that the color set of v_2 doesn't change in G . We have to handle three situations:

(1) $\phi(vv_3) \in \{2, 3\}$. Say $\phi(vv_3) = 2$ by symmetry. It suffice to use 3 to color $v_1, b_1 \in \{4, 5, 6, 7\}$ to color vv_1 such that v_1 is compatible with v_2 , and $b_3 \in \{4, 5, 6, 7\} \setminus \{\phi(v_2), \phi(v_3), b_1\}$ to color v . Obviously, v has at least one color set that can be distinguished from u_i for $i = 1, 2, 3, 4$.

(2) $\phi(vv_3) = 4$. The following four possibilities are discussed:

- Assume that $\phi(v_2) = 2$ and $(C(u_2), C(u_3), C(u_4)) = (\{1, 4, 5, 6\}, \{1, 4, 5, 7\}, \{1, 4, 6, 7\})$. Then it respectively suffices to color v_1, vv_1 with 5, 3, and color v with $\{5, 6, 7\} \setminus \{\phi(v_3)\}$.

- Assume that $\phi(v_2) \in \{5, 6, 7\}$ and $\phi(v_3) = 2$. Say $\phi(v_2) = 5$ by symmetry. If $\phi(v_2u_2) = 3$ and $(C(u_2), C(u_3), C(u_4)) = (\{1, 4, 3, 5\}, \{1, 4, 3, 7\}, \{1, 4, 6, 7\})$. We recolor vv_2 with 2 or 6 such that v_2 is compatible with u'_2 , and color v_1, vv_1, v with 6, 7, 3, respectively. If $\phi(v_2u_2) = 3, C(u_2) \neq \{1, 3, 4, i\}$ for $i = 5, 7$, and $\{1, 4, 6, 7\} \in (C(u_3), C(u_4))$. We respectively color v_1, v with 6, 3, and color vv_1 with 5 or 7. If $\phi(v_2u_2) = 3$ and $\{1, 4, 6, 7\} \notin (C(u_3), C(u_4))$. We color v_1, vv_1, v with 5, 6, 7. If $\phi(v_2u_2) \neq 3$ and $(C(u_2), C(u_3), C(u_4)) = (\{1, 4, 5, 6\}, \{1, 4, 5, 7\}, \{1, 4, 6, 7\})$. We color v_1, vv_1, v with 5, 3, 6. Otherwise, it suffices to color v_1 with 3, vv_1 with $b_1 \in \{5, 6, 7\}$ and v with color in $\{6, 7\} \setminus \{b_1\}$.

- Assume that $\phi(v_2), \phi(v_3) \in \{5, 6, 7\}$. Say $\phi(v_2) = 5, \phi(v_3) = 6$ by symmetry. It suffices to color v_1 with 3, vv_1 with $b_1 \in \{5, 6, 7\}$ such that v_1 is compatible with v_2 , and color v with color in $\{2, 7\} \setminus \{b_1\}$.

- Otherwise, we color v_1 with 3, vv_1 with $b_1 \in \{5, 6, 7\}$ such that v_1 is compatible with v_2 , and color v with color in $\{2, 5, 6, 7\} \setminus \{\phi(v_2), \phi(v_3), b_1\}$. Obviously, v has at least one color set that can be distinguished from u_i for $i = 1, 2, 3, 4$.

(3) $\phi(vv_3) \in \{5, 6, 7\}$. Say $\phi(vv_3) = 5$ by symmetry. It suffices to color v_1 with 5, vv_1 with $b_1 \in \{3, 4, 6, 7\}$ such that v_1 is compatible with v_2 , and color v with color in $\{2, 3, 4, 6, 7\} \setminus \{\phi(v_2), \phi(v_3), b_1\}$. Obviously, v has at least one color set that can be distinguished from u_i for $i = 1, 2, 3, 4$.

Case 2 $\phi(v_3) = 1$.

Remove the color of v_2 . There are three situations to be handled depending on the value of $\phi(v_3)$:

(1) $\phi(v_3) \in \{2, 3, 4\}$. Say $\phi(v_3) = 2$ by symmetry. Firstly, we color v_1 with 3. Next, there are two possibilities to be discussed:

- $\phi(u_2) \neq 1$. It suffices to color v_2 with 1, vv_2 with $b_1 \in \{2, 3, 4, 5, 6, 7\} \setminus \{\phi(v_2u_2)\}$ such that v_2 is compatible with u'_2 , vv_1 with $b_2 \in \{4, 5, 6, 7\} \setminus \{b_1\}$ such that v_1 is compatible with v_2 , and v with color in $\{4, 5, 6, 7\} \setminus \{b_1, b_2\}$.

- $\phi(u_2) = 1$. Firstly, assume that $\phi(v_2u_2) = 4$. It suffices to color v_2 with 2, vv_2 with $b_1 \in \{3, 5, 6, 7\}$ such that v_2 is compatible with u'_2 , vv_1 with $b_2 \in \{4, 5, 6, 7\} \setminus \{b_1\}$, and v with color in $\{4, 5, 6, 7\} \setminus \{b_1, b_2\}$. Next, assume that $\phi(v_2u_2) \neq 4$. It suffices to color v_2 with 4, vv_2 with $b_1 \in \{2, 3, 5, 6, 7\} \setminus \{\phi(v_2u_2)\}$ such that v_2 is compatible with u'_2 , vv_1 with $b_2 \in \{4, 5, 6, 7\} \setminus \{b_1\}$, and v with color in $\{5, 6, 7\} \setminus \{b_1, b_2\}$.

Obviously, v has at least one color set that can be distinguished from u_i for $i = 1, 2, 3, 4$.

(2) $\phi(v_3) \in \{5, 6, 7\}$. Say $\phi(v_3) = 5$ by symmetry. Firstly, we color v_1 with 5. Next, we have two possibilities to be discussed:

- $\phi(u_2) \neq 1$. It suffices to color v_2 with 1, vv_2 with $b_1 \in \{2, 3, 4, 5, 6, 7\} \setminus \{\phi(v_2u_2)\}$ such that v_2 is compatible with u'_2 , vv_1 with $b_2 \in \{3, 4, 6, 7\} \setminus \{b_1\}$ such that v_1 is compatible with v_2 , and v with color in $\{2, 3, 4, 6, 7\} \setminus \{b_1, b_2\}$.

- $\phi(u_2) = 1$. Firstly, assume that $\phi(v_2u_2) = 4$. It suffices to color v_2 with 5, vv_2 with $b_1 \in \{2, 3, 6, 7\}$ such that v_2 is compatible with u'_2 , vv_1 with $b_2 \in \{3, 4, 6, 7\} \setminus \{b_1\}$, and v with color in $\{2, 3, 4, 6, 7\} \setminus \{b_1, b_2\}$. Next, assume that $\phi(v_2u_2) \neq 4$. It suffices to color v_2 with 4, vv_2 with $b_1 \in \{2, 3, 5, 6, 7\} \setminus \{\phi(v_2u_2)\}$ such that v_2 is compatible with u'_2 , vv_1 with $b_2 \in \{3, 4, 6, 7\} \setminus \{b_1\}$, and v with color in $\{2, 3, 6, 7\} \setminus \{b_1, b_2\}$.

Obviously, v has at least one color set that can be distinguished from u_i for $i = 1, 2, 3, 4$. \square

CLAIM 7. G does not contain 3_1 -vertex.

Proof. Assume to the contrary, G contains a 3_1 -vertex v . Let $N_G(v) = \{v_1, v_2, v_3\}$, $d_G(v_1) = 2$, and let u_1 be the neighbor of v_1 other than v . By Claim 3 and Claim 6, $d_G(u_1) = d_G(v_2) = d_G(v_3) = 3$. Let $N_G(u_1) = \{v_1, u'_1, u''_1\}$, and $N_G(v_i) = \{v, v'_i, v''_i\}$ for $i = 2, 3$. By Claim 6, $d_G(u'_1) = d_G(u''_1) = 3$. By Claim 2, $u_1 \neq v_2$. Then consider the subgraph $G' = G - \{v_1\} + \{vu_1\}$. Then G' has a 2DVDT-7-coloring ϕ using the color set C by the minimality of G . Assume without loss of generality that $\phi(vu_1) = 1, \phi(vv_2) = 2, \phi(vv_3) = 3$ and $\phi(v) = 4$. Based on ϕ , in G , we color u_1v_1 with 1. It is easy notice that the color set of u_1 doesn't change in G . If $\{2, 3, 4, i\} \in (C(v'_2), C(v''_2), C(v'_3), C(v''_3))$ for $i = 5, 6, 7$. We color v_1 with color $\{2, 3\} \setminus \{\phi(u_1)\}$, recolor v with $b \in \{5, 6, 7\} \setminus \{\phi(v_2), v_3\}$, and color vv_1 with color in $\{5, 6, 7\} \setminus \{b\}$. Otherwise, it suffices to color v_1 with color $\{2, 3\} \setminus \{\phi(u_1)\}$ and vv_1 with color in $\{5, 6, 7\}$. Obviously, v has at least one color set that can be distinguished from u_1, v_i, v''_i for $i = 2, 3$. \square

CLAIM 8. G does not contain a cut edge.

Proof. Assume to the contrary that G contains a cut edge uv . It follows that $G' = G - \{uv\}$ consists of two components G'_1 and G'_2 for $u \in V(G'_1)$ and $v \in V(G'_2)$. Let $G_1 = G[V(G'_1) \cup \{v\}]$ and $G_2 = G[V(G'_2) \cup \{u\}]$. It follows that G_1 and G_2 are proper subgraphs of G . By the minimalization of G , G_1 and G_2 have 2DVDT-7-coloring ϕ_1 such that $C_{\phi_1}(u) = \{1, 2, 3, 4\}$ with $\phi_1(uv) = 1$ and ϕ_2 such that $C_{\phi_2}(v) = \{1, 5, 6, 7\}$ with $\phi_2(uv) = 1$ use the color set C , respectively. (This can be accomplished by exchanging reasonably the colors of G_2 under ϕ_2 .) Noticed that combining ϕ_1 and ϕ_2 to produce a 2DVDT-7-coloring of G . This is a contradiction. \square

So far it follows that G is a 3-regular simple graph without cut edges. Since K_4 can be 2DVDT-7-colorable with color set C by [12], we assume that $G \neq K_4$. By Lemma 2, G is 3-colorable. Thus, first we use 1, 2, 3 to color all vertices of G such that adjacent vertices receive distinct colors.

Next, we color all edges of G with colors in $\{3, 4, 5, 6, 7\}$. By Lemma 3, G can be edge-partitioned into a perfect matching M and a class \mathcal{A} of cycles. Color all edges of M with the same color 7. For each cycle $A = v_1v_2v_3 \cdots v_kv_1$ in \mathcal{A} , we use colors of $\{3, 4, 5, 6\}$ to color its edges. For convenience, we define each edge v_iv_{i+1} as e_i for $v_{k+1} = v_1$. There are two situations to be handled as follows:

Case 1. Only two colors appear on the vertices of A .

Assume without loss of generality that $\phi(v_1) = 1$ and $\phi(v_2) = 2$.

If $k \equiv 0 \pmod{3}$, then $\{e_1, e_4, \dots, e_{k-5}, e_{k-2}\} \rightarrow \{4\}$, $\{e_2, e_5, \dots, e_{k-4}, e_{k-1}\} \rightarrow \{5\}$, $\{e_3, e_6, \dots, e_{k-3}, e_k\} \rightarrow \{6\}$.

If $k \equiv 1 \pmod{3}$, then $\{e_1, e_4, \dots, e_{k-6}, e_{k-3}\} \rightarrow \{4\}$, $\{e_2, e_5, \dots, e_{k-5}, e_{k-2}\} \rightarrow \{5\}$, $\{e_3, e_6, \dots, e_{k-4}, e_{k-1}\} \rightarrow \{6\}$, $\{e_k\} \rightarrow \{5\}$.

If $k \equiv 2 \pmod{3}$, then $\{e_1, e_4, \dots, e_{k-7}, e_{k-4}\} \rightarrow \{4\}$, $\{e_2, e_5, \dots, e_{k-6}, e_{k-3}\} \rightarrow \{5\}$, $\{e_3, e_6, \dots, e_{k-5}, e_{k-2}\} \rightarrow \{6\}$, $\{e_{k-1}, e_k\} \rightarrow \{4, 3\}$.

Case 2. There are three consecutive vertices of A that receive distinct colors.

Assume without loss of generality that $\phi(v_1) = 1$, $\phi(v_2) = 2$ and $\phi(v_3) = 3$.

If $k \equiv 0 \pmod{3}$, then $\{e_1, e_4, \dots, e_{k-5}, e_{k-2}\} \rightarrow \{4\}$, $\{e_2, e_5, \dots, e_{k-4}, e_{k-1}\} \rightarrow \{5\}$, $\{e_3, e_6, \dots, e_{k-3}, e_k\} \rightarrow \{6\}$.

If $k \equiv 1 \pmod{3}$, then $\{e_1, e_4, \dots, e_{k-6}, e_{k-3}\} \rightarrow \{4\}$, $\{e_2, e_5, \dots, e_{k-5}, e_{k-2}\} \rightarrow \{5\}$, $\{e_3, e_6, \dots, e_{k-4}, e_{k-1}\} \rightarrow \{6\}$, $\{e_k\} \rightarrow \{5\}$.

If $k \equiv 2 \pmod{3}$, then $\{e_1, e_4, \dots, e_{k-7}, e_{k-4}\} \rightarrow \{4\}$, $\{e_2, e_5, \dots, e_{k-6}, e_{k-3}\} \rightarrow \{5\}$, $\{e_3, e_6, \dots, e_{k-5}, e_{k-2}\} \rightarrow \{6\}$, $\{e_{k-1}\} \rightarrow \{4\}$. If $\phi(v_k) = 2$, then color e_k with 3. If $\phi(v_k) = 3$, then color e_k with 5.

This makes G 2DVDT-7-colorable. Thus, it is a contradiction. The whole proof of Theorem 1 is completed. \square

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