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# DEGREE SUM AND RESTRICTED $\{P_2,P_5\}$-FACTOR IN GRAPHS

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**Abstract**. For a graph $G$, a spanning subgraph $F$ of $G$ is called a $\{P_2,P_5\}$-factor if every component of $F$ is isomorphic to $P_2$ or $P_5$, where $P_i$ denotes the path of order $i$. A graph $G$ is called a $(\{P_2,P_5\},k)$-factor critical graph if $G-V'$ contains a $\{P_2,P_5\}$-factor for any $V'\subseteq V(G)$ with $|V'|=k$. A graph $G$ is called a $(\{P_2,P_5\},m)$-factor deleted graph if $G-E'$ has a $\{P_2,P_5\}$-factor for any $E'\subseteq E(G)$ with $|E'|=m$. The degree sum of $G$ is defined by

$$\sigma_{r+1}(G)=\min_{X\subseteq V(G)}\Big\{\sum_{x\in X} d_G(x) : X \text{ is an independent set of } r+1 \text{ vertices}\Big\}.$$

In this paper, using degree sum conditions, we demonstrate that
(i) $G$ is a $(\{P_2,P_5\},k)$-factor critical graph if $\sigma_{r+1}(G)>\frac{(3n+4k-2)(r+1)}{7}$ and $\kappa(G)\geq k+r$;
(ii) $G$ is a $(\{P_2,P_5\},m)$-factor deleted graph if $\sigma_{r+1}(G)>\frac{(3n+2m-2)(r+1)}{7}$ and $\kappa(G)\geq \frac{5m}{4}+r$.

***Key words****:* graph, path-factor, $\{P_2,P_5\}$-factor, degree sum, path factor critical graph, path factor deleted graph.
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## 1. INTRODUCTION

In this paper, we consider only finite and undirected graph without loops or multiple edges. Throughout this paper, we consider only simple connected graphs. Let $G=(V(G),E(G))$ be a graph. We denote by $V(G)$ and $E(G)$ the vertex set and the edge set of $G$, respectively. For $v\in V(G)$, we use $d_G(v)$ and $N_G(v)$ to denote the degree of $v$ and the set of vertices adjacent to $v$ in $G$, respectively. If $d_G(v)=0$ for some vertex $v\in V(G)$, then $v$ is said to be an isolated vertex in $G$. The number of isolated vertices of a graph $G$ is denoted by $i(G)$. For any subset $S\subseteq V(G)$, let $G[S]$ denote the subgraph of $G$ induced by $S$, and $G-S:=G[V(G)\setminus S]$ is the resulting graph after deleting the vertices of $S$ from $G$. The number of connected components of a graph $G$ is denoted by $\omega(G)$. We write $\kappa(G)$ for the vertex connectivity of $G$.

A spanning subgraph of $G$ is a subgraph $H$ of $G$ such that $V(H)=V(G)$ and $E(H)\subseteq E(G)$. For a family of connected graphs $\mathscr{F}$, a spanning subgraph $H$ of a graph $G$ is called an $\mathscr{F}$-factor of $G$ if its each component is isomorphic to an element of $\mathscr{F}$. In particular, $H$ is called a $\{P_2,P_5\}$-factor of $G$ if its each component is isomorphic to $P_2$ or $P_5$, where $P_i$ denotes the path of order $i$. A graph $G$ is called a $(\{P_2,P_5\},k)$-factor critical graph if $G-V'$ contains a $\{P_2,P_5\}$-factor for any $V'\subseteq V(G)$ with $|V'|=k$. A graph $G$ is called a $(\{P_2,P_5\},m)$-factor deleted graph if $G-E'$ has a $\{P_2,P_5\}$-factor for any $E'\subseteq E(G)$ with $|E'|=m$.

Since Tutte proposed the well-known Tutte 1-factor theorem [20], path-factors of graphs [2,5,6,8–11,16] and path-factor covered graphs [7,12,22–24] have been extensively studied. More results on graph factors are referred to the survey papers and books [3,21].

As early as 1985, Akiyama et al. [1] provided a good characterization for a graph admitting a $\{P_2,P_3\}$-factor, which is stated as follows.

THEOREM 1 *(Akiyama, Avis and Era [1]). A graph G has a* $\{P_2,P_3\}$*-factor if and only if* $i(G-S) \leq 2|S|$ *for all* $S \subseteq V(G)$.

For an integer $d \geq 2$, a $\{P_i : i \geq d\}$-factor is briefly denoted by $P_{\geq d}$-factor. Note that a graph has $P_{\geq 2}$-factors if and only if it has $\{P_2,P_3\}$-factors. Kaneko [16] gave a necessary and sufficient condition for the existence of $P_{\geq 3}$-factors. For $d \geq 4$, it is not known that whether the existence problem of $P_{\geq d}$-factors is polynomially solvable or not, though some results about such factors on special classes of graphs have been obtained (see, for example, Kano et al. [17], Ando et al. [4], and Kawarabayashi et al. [18]).

A graph $F$ is *hypomatchable* if $F-x$ has a perfect matching for every $x \in V(F)$. A graph is a *propeller* if it is obtained from a hypomatchable graph $F$ by adding new vertices $u,v$ and edge $uv$, and joining $u$ to some vertices of $F$. Loebal and Poljak [19] proved the following theorem.

THEOREM 2 *(Loebal and Poljak [19]). Let F be a connected nontrivial graph. If F has a perfect matching, F is hypomatchable, or F is a propeller, then the existence problem of a* $\{P_2,F\}$*-factor is polynomially solvable. The problem is NP-complete for all other graphs F.*

In particular, the existence problem of a $\{P_2,P_{2d+1}\}$-factor is *NP*-complete for $d \geq 2$. As $\{P_2,P_{2d+1}\}$-factor is a useful tool for finding large matchings, Egawa, Furuya and Ozeki [15] investigated the existence of $\{P_2,P_{2d+1}\}$-factors and obtained the following theorem.

For $S \subseteq V(G)$, let $\mathscr{C}_i(G-S)$ be the set of components of order $i$ in $G-S$, where integer $i \geq 1$. Write $c_i(G-S) = |\mathscr{C}_i(G-S)|$. For $0 \leq i \leq d-1$, we use $c^o_{<2d}(G-S)$ to denote the number of odd components of $G-S$ with order less than $2d$, that is, $c^o_{<2d}(G-S) = \sum_{1\leq i\leq d} c_{2i-1}(G-S)$.

THEOREM 3 *(Egawa, Furuya and Ozeki [15]). Let* $d \geq 3$ *be an integer, and let G be a graph. If* $c^o_{<2d}(G-S) \leq \frac{5}{6d^2}|S|$ *for all* $S \subseteq V(G)$*, then G has a* $\{P_2,P_{2d+1}\}$*-factor.*

Recently, Egawa and Furuya [13, 14] obtained stronger sufficient conditions for $\{P_2,P_{2d+1}\}$-factors with $d = 2,3,4$. In particular, they proved the following theorem.

THEOREM 4 *(Egawa & Furuya [13]). A graph G has a* $\{P_2,P_5\}$*-factor if* $3c_1(G-S)+2c_3(G-S) \leq 4|S|+1$ *for all* $S \subseteq V(G)$.

Now, we introduce the parameter called *degree sum*. If a graph $G$ has $r$ independent vertices, define

$$\sigma_{r+1}(G) = \min_{X\subseteq V(G)} \Big\{ \sum_{x\in X} d_G(x) : X \text{ is an independent set of } r+1 \text{ vertices}\Big\}.$$

In this paper, we obtain two degree sum conditions for graphs to be $(\{P_2,P_5\},k)$-factor critical graphs and $(\{P_2,P_5\},m)$-factor deleted graphs, respectively.

## 2. $(\{P_2,P_5\},k)$-FACTOR CRITICAL GRAPH

THEOREM 5. *Let G be a graph of order* $n \geq 2r+k+8$*, where* $r \geq 1, k \geq 0$ *are integers. If* $\kappa(G) \geq k+r$ *and* $\sigma_{r+1}(G) > \frac{(3n+4k-2)(r+1)}{7}$*, then G is a* $(\{P_2,P_5\},k)$*-factor critical graph.*

*Proof.* Let $G' = G-V'$ for $V' \subseteq V(G)$ with $|V'| = k$. In order to verify Theorem 5, it suffices to prove that $G'$ has a $\{P_2,P_5\}$-factor. On the contrary, suppose that $G'$ admits no $\{P_2,P_5\}$-factor. Then by Theorem 4, there exists $S \subseteq V(G')$ such that $3c_1(G'-S)+2c_3(G'-S) \geq 4|S|+2$. It follows that

$$c_1(G'-S)+c_3(G'-S) \geq c_1(G'-S)+\frac{2}{3}c_3(G'-S) \geq \frac{4|S|+2}{3} \tag{1}$$

for some $S \subseteq V(G')$.

CLAIM 1. $|S| \geq r$.

*Proof.* Suppose $|S| \leq r-1$, then by $\kappa(G) \geq k+r$ and $|V'| = k$, we have that $G'-S = G-V'-S$ is connected and thus $\omega(G'-S) = 1$. Then by (1), we get

$$\frac{4|S|+2}{3} \leq c_1(G'-S)+c_3(G'-S) \leq \omega(G'-S) = 1,$$

which implies $|S| = 0$ and $c_1(G')+c_3(G') = \omega(G') = 1$. It follows that $|G'| \leq 3$, which contradicts $|G'| = n-k \geq 2r+8$. □

CLAIM 2. $c_1(G'-S) \leq r$.

*Proof.* Assume $c_1(G'-S) \geq r+1$. Then there exist at least $r+1$ isolated vertices $x_1,x_2,\dots,x_{r+1}$ in $G'-S$ such that $d_{G'-S}(x_i) = 0$ for $1 \leq i \leq r+1$. Hence, we have

$$d_G(x_i) \leq |V'|+|S| = |S|+k \tag{2}$$

for $1 \leq i \leq r+1$.

Obviously, $\{x_1,x_2,\dots,x_{r+1}\}$ is an independent set of $G$. In terms of (2) and the degree condition of Theorem 5, we obtain

$$|S|+k \geq \max\{d_G(x_i) : 1 \leq i \leq r+1\} \geq \frac{\sigma_{r+1}(G)}{r+1} > \frac{3n+4k-2}{7},$$

which implies

$$|S| > \frac{3n-3k-2}{7}. \tag{3}$$

It follows from (1) and (4) that

$$\begin{aligned} n &\geq |S|+|V'|+c_1(G'-S)+3\times c_3(G'-S) \\ &\geq |S|+k+c_1(G'-S)+c_3(G'-S) \\ &\geq |S|+k+\frac{4|S|+2}{3} \\ &= \frac{7|S|}{3}+k+\frac{2}{3} \\ &> \frac{7}{3}\times\frac{3n-3k-2}{7}+k+\frac{2}{3} \\ &= n, \end{aligned}$$

which is a contradiction. We complete the proof of Claim 2. □

Using (1) and Claim 1, we derive

$$c_1(G'-S)+c_3(G'-S) \geq \frac{4|S|+2}{3} \geq r+\frac{r+2}{3} \geq r+1,$$

which implies that $G'-S$ admits $r+1$ components of order one or three. Let $G_1,G_2,\dots,G_{r+1}$ be $r+1$ components of $G'-S$, and choose vertex $x_i \in V(G_i)$ such that $d_{G_i}(x_i) \leq 2$ for $1 \leq i \leq r+1$. Obviously, $\{x_1,x_2,\dots,x_{r+1}\}$ is an independent set of $G$, and $d_G(x_i) \leq k+|S|+2$ for $1 \leq i \leq r+1$. By the degree condition of Theorem 5, we have that

$$k+|S|+2 \geq \max\{d_G(x_i) : 1 \leq i \leq r+1\} \geq \frac{\sigma_{r+1}(G)}{r+1} > \frac{3n+4k-2}{7}.$$

It follows that

$$|S| > \frac{3n-3k-16}{7}. \tag{4}$$

According to (1), (4), Claim 2, $r \geq 1$ and $n \geq 2r+k+8$, we obtain

$$\begin{aligned} n &\geq |S|+|V'|+3\times\left(c_1(G'-S)+c_3(G'-S)\right)-2\times c_1(G'-S) \\ &\geq |S|+k+4|S|+2-2r \\ &> 5\times\frac{3n-3k-16}{7}+k \\ &= \frac{15n-8k-80}{7}, \end{aligned}$$

that is, $n < k+10$, which is a contradiction to that $n \geq 2r+k+8 \geq k+10$. We complete the proof of Theorem 5. □

*Remark* 1. Now, we show that the degree sum condition

$$\sigma_{r+1}(G) > \frac{(3n+4k-2)(r+1)}{7}$$

in Theorem 5 cannot be replaced by

$$\sigma_{r+1}(G) \geq \frac{(3n+4k-7)(r+1)}{7}.$$

Let $k \geq 0$ and $r \geq 1$ be two integers, and $t$ be a sufficiently large integer. Construct a graph $G = K_q \vee ((rt+2k+3)K_1)$, where $q = \frac{3rt+10k+2}{4}$. Then $G$ is a $q$-connected graph of order $n = \frac{7rt+18k+14}{4}$, and

$$\frac{\sigma_{r+1}(G)}{r+1} \geq q = \frac{3rt+10k+2}{4} = \frac{3n+4k-7}{7}.$$

Let $V' \subseteq V(K_q)$ with $|V'| = k$, and $G' = G - V'$. We choose $S = V(K_{q-k}) \subseteq V(K_q)$, then we obtain

$$c_1(G'-S)+c_3(G'-S) = rt+2k+3 > \frac{4(q-k)+1}{3} = \frac{4|S|+1}{3}.$$

By Theorem 4, $G'$ is has no $\{P_2,P_5\}$-factor, that is, $G$ is not a $(\{P_2,P_5\},k)$-factor critical graph.

## 3. $(\{P_2,P_5\},m)$-FACTOR DELETED GRAPH

THEOREM 6. *Let $m$ and $r$ be two integers with $r \geq 1$ and $0 \leq m \leq r-1$, and let $G$ be a graph of order $n \geq 2r+4m+8$. If $\kappa(G) \geq \frac{5m}{4}+r$ and $\sigma_{r+1}(G) > \frac{(3n+2m-2)(r+1)}{7}$, then $G$ is a $(\{P_2,P_5\},m)$-factor deleted graph.*

*Proof.* Let $G' = G - E'$ for $E' \subseteq E(G)$ with $|E'| = m$. Then $V(G') = V(G)$ and $E(G') = E(G) \setminus E'$. To prove Theorem 6, it suffices to verify that $G'$ has a $\{P_2,P_5\}$-factor. On the contrary, suppose that $G'$ admits no $\{P_2,P_5\}$-factor. Then by Theorem 4, there exists $S \subseteq V(G')$ such that $3c_1(G'-S)+2c_3(G'-S) \geq 4|S|+2$. It follows that

$$c_1(G'-S)+c_3(G'-S) \geq c_1(G'-S)+\frac{2}{3}c_3(G'-S) \geq \frac{4|S|+2}{3} \tag{5}$$

for some $S \subseteq V(G')$.

Next, we shall consider two cases according to the value of $c_1(G-S)$ and derive a contradiction in each case.

*Case 1.* $c_1(G-S) \geq r+1$.

In this case, there exist at least $r+1$ isolated vertices $x_1,x_2,...,x_{r+1}$ in $G-S$ such that $d_{G-S}(x_i)=0$ for $1\le i\le r+1$. Hence, we have

$$d_G(x_i)\le d_{G-S}(x_i)+|S|=|S| \tag{6}$$

for $1\le i\le r+1$. Obviously, $\{x_1,x_2,...,x_{r+1}\}$ is an independent set of $G$. Then by (6) and the degree condition of Theorem 6, we have that

$$|S|\ge \max\{d_G(x_i):1\le i\le r+1\}\ge \frac{\sigma_{r+1}(G)}{r+1}>\frac{3n+2m-2}{7}. \tag{7}$$

It follows from (5) and (7) that

$$n\ge |S|+c_1(G'-S)+c_3(G'-S)\ge |S|+\frac{4|S|+2}{3}=\frac{7|S|+2}{3}>\frac{3n+2m}{3}>n,$$

which is a contradiction.

*Case 2.* $c_1(G-S)\le r$.

*Subcase 2.1.* $S$ is not a vertex cut set of $G$.

In this subcase, $\omega(G-S)=\omega(G)=1$. After deleting an edge in a graph, the number of its components increases by at most 1. Hence, if $|S|\ge\frac{3m+2}{4}$, then it follows that

$$\begin{aligned}
c_1(G'-S)+c_3(G'-S) &= c_1(G-S-E')+c_3(G-S-E')\\
&\le \omega(G-S-E')\\
&\le \omega(G-S)+m\\
&= m+1\\
&\le \frac{4|S|-2}{3}+1\\
&= \frac{4|S|+1}{3},
\end{aligned}$$

which contradicts (5).

If $1\le|S|<\frac{3m+2}{4}$, then by $m\le 2r-1$ and $\kappa(G)\ge\frac{5m}{4}+r$, we have

$$\kappa(G-S)\ge\kappa(G)-|S|>\frac{5m}{4}+r-\frac{3m+2}{4}=\frac{m-1}{2}+r\ge m.$$

By the integrity of $\kappa(G-S)$, we get

$$\kappa(G-S)\ge m+1. \tag{8}$$

It follows from (8) that $\kappa(G'-S)=\kappa(G-S-E')\ge\kappa(G-S)-|E'|\ge 1$. Hence, we derive

$$c_1(G'-S)+c_3(G'-S)\le\omega(G'-S)=1. \tag{9}$$

Using (5) and $|S|\ge 1$, we obtain

$$2\le\frac{4|S|+2}{3}\le c_1(G'-S)+c_3(G'-S),$$

which is a contradiction to (9).

If $|S|=0$, then by (5), we have

$$c_1(G')+c_3(G')=c_1(G'-S)+c_3(G'-S)\ge\frac{4|S|+2}{3}=\frac{2}{3}. \tag{10}$$

Note that $\kappa(G') \geq \kappa(G) - |E'| \geq \frac{5m}{4} + r - m \geq r$, and thus $\omega(G') = 1$. This together with (10) implies $c_1(G') + c_3(G') = \omega(G') = 1$. Hence, $G'$ is a graph of order one or three, which contradicts $|G'| = n \geq 2r + 4m + 8 > 3$.

*Subcase 2.2.* $S$ is a vertex cut set of $G$.

In this subcase, we have $\omega(G-S) \geq 2$ and $|S| \geq \kappa(G) \geq \frac{5m}{4} + r$. In terms of (5) and $m \leq r-1$, we obtain

$$\begin{aligned} c_1(G-S) + c_3(G-S) &\geq c_1(G'-S) + c_3(G'-S) - 2m \\ &\geq \frac{4|S|+2}{3} - 2m \\ &\geq \frac{-m+4r+2}{3} \\ &= \frac{-m+r-1}{3} + r + 1 \\ &\geq r+1, \end{aligned}$$

which implies that there exist $r+1$ components of order at most three in $G-S$, denoted by $H_1, H_2, \ldots, H_{r+1}$. We choose $x_i \in V(H_i)$ with $d_{H_i}(x_i) \leq 2$ for $1 \leq i \leq r+1$. Obviously, $\{x_1, x_2, \ldots, x_{r+1}\}$ is an independent set of $G$. Then it follows from the degree condition of Theorem 6 that

$$|S| + 2 \geq \max\{d_G(x_i) : 1 \leq i \leq r+1\} \geq \frac{\sigma_{r+1}(G)}{r+1} > \frac{3n+2m-2}{7}.$$

It follows that

$$|S| > \frac{3n+2m-16}{7}. \tag{11}$$

In light of (5), (11), $c_1(G-S) \leq r$ and $n \geq 2r + 4m + 8$, we deduce

$$\begin{aligned} n &\geq |S| + 3 \times \big(c_1(G-S) + c_3(G-S)\big) - 2 \times c_1(G-S) \\ &\geq |S| + 3 \times \big(\frac{4|S|+2}{3} - 2m\big) - 2r \\ &= 5|S| - 6m + 2 - 2r \\ &> 5 \times \frac{3n+2m-16}{7} - 6m + 2 - 2r \\ &= \frac{15n-32m-66}{7} - 2r, \end{aligned}$$

that is, $n < 4m + \frac{33+7r}{4}$. Since $r \geq 1$, we obtain $n < 4m + \frac{33+7r}{4} \leq 4m + 8 + 2r$, which is a contradiction to that $n \geq 4m + 8 + 2r$. We complete the proof of Theorem 6. □

*Remark* 2. Now, we show that the degree sum condition

$$\sigma_{r+1}(G) > \frac{(3n+2m-2)(r+1)}{7}$$

in Theorem 6 cannot be replaced by

$$\sigma_{r+1}(G) \geq \frac{(3n-2)(r+1)}{7}.$$

Let $m \geq 0$ and $r \geq 1$ be two integers, and $t$ be a sufficiently large integer. Construct a graph $G = K_p \vee ((rt+1)K_1 \cup (mK_2))$, where $p = \frac{3rt+6m+1}{4}$. Then $G$ is a $p$-connected graph of order $n = \frac{7rt+14m+5}{4}$, and

$$\frac{\sigma_{r+1}(G)}{r+1} \geq p = \frac{3rt+6m+1}{4} = \frac{3n-2}{7}.$$

Let $E' = E(mK_2)$ and $G' = G - E'$. We choose $S = V(K_p) \subseteq V(G')$, then we obtain

$$c_1(G'-S)+c_3(G'-S) = rt+1+2m > \frac{4p+1}{3} = \frac{4|S|+1}{3}.$$

By Theorem 4, $G'$ is has no $\{P_2,P_5\}$-factor, that is, $G$ is not a $(\{P_2,P_5\},m)$-factor deleted graph.

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