THE PUBLISHING HOUSE
OF THE ROMANIAN ACADEMY

PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A
Volume 23, Number 4/2022, pp. 329–336

ON THE PARITY SLOPE OF WORDS OF LOW COMPLEXITY

Cristian COBELI[1], Alexandru ZAHARESCU[1,2]

[1]"Simion Stoilow" Institute of Mathematics of the Romanian Academy, 21 Calea Grivitei Street,
P. O. Box 1-764, Bucharest 014700, Romania

[2]University of Illinois, Department of Mathematics, 1409 West Green Street, Urbana, IL 61801, USA

Corresponding author: Cristian COBELI, E-mail: cristian.cobeli@imar.ro

Abstract. We calculate the parity divisor functions associated to the ranks of the infinite monoletter word and to the Sturmian words and prove that there is a tilt of size proportional to $\log 2 - 1/2$ towards the number of odd divisors.

***Key words**:* parity, number of divisors, Sturmian words, low complexity.

Mathematics Subject Classification (MSC2020): 68R15, 05A05.

This article is dedicated to Professor Solomon Marcus (1925 – 2016).

1. INTRODUCTION

A preliminary version of this paper was initiated in 2011 by the second author, inspired by the articles of Marcus [7], Ilie, Marcus, Petre [6] and Marcus and Monteil [8] and a fruitful discussion with Professor Solomon Marcus, in 2010, at the Faculty of Mathematics of University of Bucharest.

With just two letters a and b one can construct infinite words of a super-large variety. They are classified by their complexity. The simplest of them are the periodic words, and next to them are the words that display some patterns. At the other end are the pseudo-random words that are generated by a certain rule or are in the 'grey area' lacking any sort of regularity in display, generation or description.

The *complexity* $c_{\mathfrak{w}}(n)$ of an infinite binary word $\mathfrak{w}$ is defined to be the number of distinct subwords of length n in $\mathfrak{w}$. The simplest are the *ultimately periodic words*, for which $c_{\mathfrak{w}}(n)$ is bounded. At the next upper level are binary words of complexity $c_{\mathfrak{w}}(n) = n+1$ for $n \geq 0$ which are also called *Sturmian words* (see the monograph [1, Ch. 10]). From the long and diverse list of recent works dedicated to the subject we mention [2,5,9–11].

Our object is to show that there is a nice common property of the number of divisors of the 'signed ranks' of these words.

For any infinite binary word $\mathfrak{w} = \mathfrak{w}_0\mathfrak{w}_1\mathfrak{w}_2\dots$ with leters $\mathfrak{w}_j \in \{a,b\}$ for $j \geq 0$, define the *parity divisor functions*:

$$e_{\mathfrak{w}}(n) := \left|\{ j \in \mathbb{N} : j \text{ divides } n, \mathfrak{w}_j = b, n/j \text{ even } \}\right|,$$
$$o_{\mathfrak{w}}(n) := \left|\{ j \in \mathbb{N} : j \text{ divides } n, \mathfrak{w}_j = b, n/j \text{ odd } \}\right|.$$

To measure the distance between them we consider the *difference function* and its *average*:

$$D_{\mathfrak{w}}(n) = o_{\mathfrak{w}}(n) - e_{\mathfrak{w}}(n), \qquad A_{\mathfrak{w}}(x) = \sum_{n=1}^{x} D_{\mathfrak{w}}(n).$$

The distributions of $e_{\mathfrak{u}}(n)$ and $o_{\mathfrak{u}}(n)$ are very irregular as can be seen in Figure 2 for $\mathfrak{u} = bbb\ldots$, the monoletter infinite word whose letters are all equal to b. The same happens for more complex binary words, as observed in [3]. Except that it takes negative values also, there is no much difference in the distribution of $D_{\mathfrak{u}}(n)$ (see the graphs on the left of Figures 3, 4 and those in [3]). However, the average $A_{\mathfrak{w}}(x)$ is very different, its estimate having a linear main term. This is proved in the following theorem for the monoletter infinite word $\mathfrak{u}$.

THEOREM 1. *We have*

$$\sum_{1\le n\le x} D_{\mathfrak{u}}(n) = (\log 2)\cdot x + O\big(\sqrt{x}\big)\,.$$

From the definition of $e_{\mathfrak{w}}$ and $o_{\mathfrak{w}}$ one sees that for more complex words $\mathfrak{w}$, the frequency of letters a, b in $\mathfrak{w}$ should influence the main term of an analogue result. This is quantified by

$$\beta_{\mathfrak{w}} := \lim_{n\to\infty} \frac{1}{n}\cdot \big|\{1\le j\le n : \mathfrak{w}(j) = b\}\big|\,.$$

Note that $\beta_{\mathfrak{u}} = 1$ and $\beta_{\mathfrak{f}} = (3-\sqrt{5})/2$, where $\mathfrak{f} = a\,ba\,aba\,ababa\,ababaaba,\ldots$ is the Fibonacci Sturmian word, which is constructed recursively, by concatenation, with the Fibonacci rule (see [1, Chapter 9]). The next result gives the estimate for the average of the difference of the divisor functions for Sturmian words.

THEOREM 2. *For Sturmian word $\mathfrak{w}$ and any $\delta > 0$, we have*

$$\sum_{n=1}^{x} D_{\mathfrak{w}}(n) = (\beta_{\mathfrak{w}}\log 2)\cdot x + O_\delta\big(x^{2/3+\delta}\big). \tag{1}$$

Theorems 1 and 2 show that the bias average towards the odd divisors increases linearly with the slope $\beta_{\mathfrak{w}}\log 2$. Except the frequency $\beta_{\mathfrak{w}}$, which is a particularity of the word $\mathfrak{w}$, the size of the odds-tilt is $\log 2 \approx 0.69314$, which is larger than $1/2$.

It is likely that divisor parity slope exists for other classes of words $\mathfrak{w}$ for which $\beta_{\mathfrak{w}}$ exists. Numerical experiments, of which one we show in Fig. 1, suggests that this is true for random binary words.

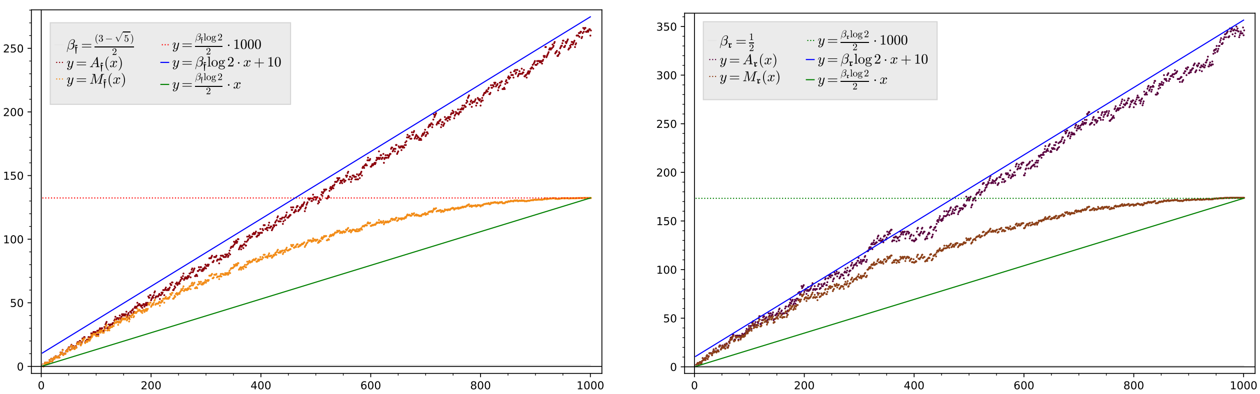

Fig. 1 – The slopes of the average $A_{\mathfrak{w}}(x) = \sum_{n=1}^{x} D_{\mathfrak{w}}(n)$ and the weighted average $M_{\mathfrak{w}}(x) = \sum_{n=1}^{x} D_{\mathfrak{w}}(n)\left(1-\frac{n}{x}\right)$ for $x = 1000$ (see the estimate in relation (8) below). Here it is compared the case where $\mathfrak{w}$ is the Fibonacci word $\mathfrak{f}$ and $\beta_{\mathfrak{f}} = (3-\sqrt{5})/2$ (left) with the case of a binary world $\mathfrak{r}$ and $\beta_{\mathfrak{r}} = 1/2$ whose letters are uniformly generated at random, and which is not Sturmian (right).

2. PARITY TILT IN THE DIVISORS OF NATURAL NUMBERS–PROOF OF THEOREM 1

Let $\mathfrak{u} = bbb\ldots$ be the monoletter infinite word whose letters are all equal to b. The values of the parity functions $e_{\mathfrak{u}}(n)$ and $o_{\mathfrak{u}}(n)$, shown for small n in Fig. 2, are very irregular and the involvement of the primes is part of the motive. We decompose n in prime factors to express $e_{\mathfrak{u}}(n)$ and $o_{\mathfrak{u}}(n)$ as follows.

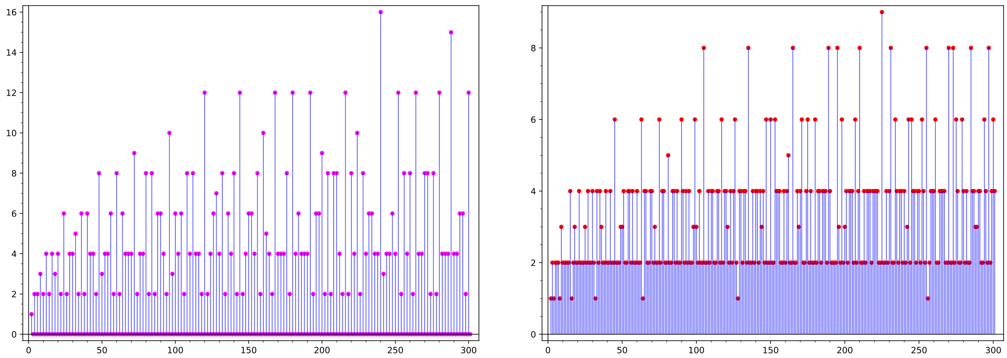

Fig. 2 – The values of $e_{\mathrm{u}}(n)$ (left) and $o_{\mathrm{u}}(n)$ (right) for $n \in [2, 300]$.

Let $n = 2^{\alpha} r$, with $\alpha > 0$, r odd and $r = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$. The divisors of n are the terms of the sum obtained after all the multiplications of the following formal product are made:

$$(1 + 2 + \cdots + 2^{\alpha})(1 + p_1 + \cdots + p_1^{\alpha_1}) \cdots (1 + p_k + \cdots + p_k^{\alpha_k}).$$

In particular, we see that the total number of divisors of n is

$$(\alpha + 1)(\alpha_1 + 1) \cdots (\alpha_k + 1) = (\alpha + 1) d(r),$$

where $d(r)$ is the number of divisors of r. If $\alpha = 0$, then n is odd, so it has no even divisors. Thus $e_{\mathrm{u}}(n) = 0$ and $o_{\mathrm{u}}(n) = o_{\mathrm{u}}(r) = (\alpha_1 + 1) \cdots (\alpha_r + 1) = d(r)$. If $\alpha \geq 1$, the number of odd divisors is also equal to $d(r)$. Further, to each odd divisor of n corresponds α even divisors, those that are obtained by multiplication with $2, 2^2, \ldots, 2^{\alpha}$. In summary:

$$o_{\mathrm{u}}(n) = d(r) \;\text{ and }\; e_{\mathrm{u}}(n) = \alpha \cdot d(r) \quad \text{for } n = 2^{\alpha} r, \text{ with } \alpha \geq 0 \text{ and } r \text{ odd}. \tag{2}$$

Two special cases, in which $e_{\mathrm{u}}(n)$ attains its minimum and its maximum, occur. Since any odd n has no even divisors, the minimum of $e_{\mathrm{u}}(n)$ is attained half of the time. Thus $e_{\mathrm{u}}(n) = 0$ and $o_{\mathrm{u}}(n) = d(n)$ for n odd. In this case $o_{\mathrm{u}}(n)$ has a local maximum, $o_{\mathrm{u}}(n) = 2^k$ for $n = 1 \cdot 3 \cdot 5 \cdots (2k+1)$. At the other end, if n is a power of 2, then 1 is the only odd divisor of n, so $e_{\mathrm{u}}(n) = \alpha$ and $o_{\mathrm{u}}(n) = 1$ for $n = 2^{\alpha}$.

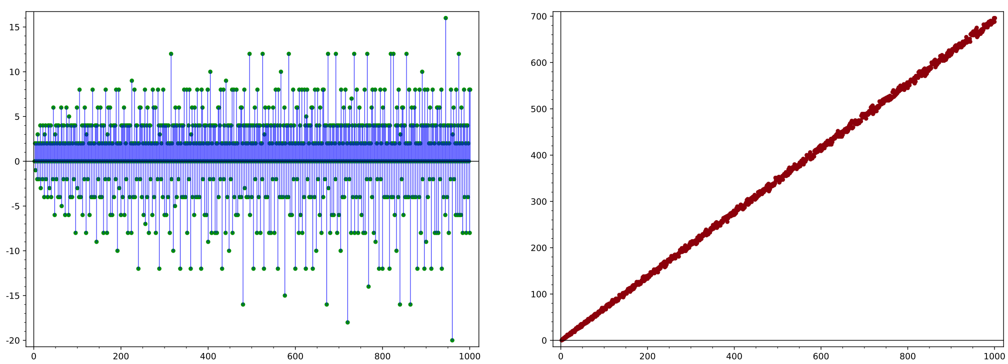

Fig. 3 – The values of $D_{\mathrm{u}}(n)$ and its partial average $\sum_{1 \leq n \leq x} D_{\mathrm{u}}(n)$ for $x \in [2, 1000]$.

Experimental results show that the partial average of $D_{\mathrm{u}}(n)$, that is, the sum $\sum_{n=1}^{x} D_{\mathrm{u}}(n)$, has a tendency to grow linearly with x (see Fig. 3) and the same behavior is apparent for the averages calculated on shorter intervals. Although, in rare situations, the growth may be ‘bumpy’ in shorter intervals that contain ‘special’ n’s.

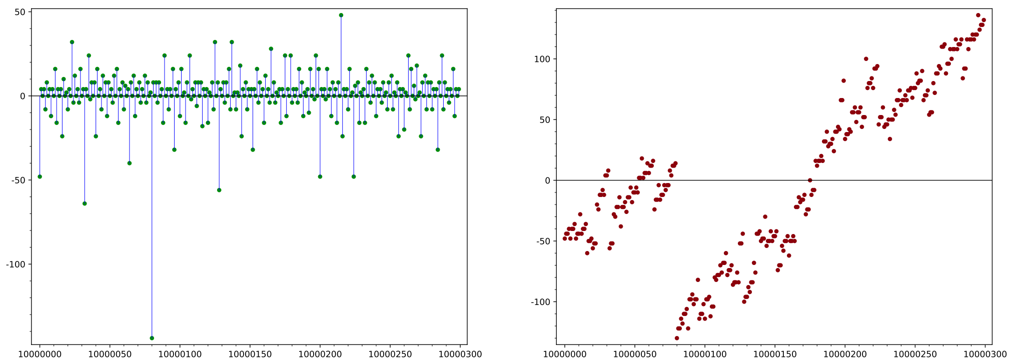

Fig. 4 – The difference function $D_{\mathrm{u}}(n) = o_{\mathrm{u}}(n) - e_{\mathrm{u}}(n)$ (left) and its partial average $\sum_{n\in\mathscr{I},n\le x} D_{\mathrm{u}}(n)$ (right) for $n,x \in \mathscr{I}$ and $\mathscr{I} = [10^7, 10^7+299]$. Here, the largest value of $|D_{\mathrm{u}}(n)|$ is attained for $n = 10\,000\,080 = 2^4\cdot 3^2\cdot 5\cdot 17\cdot 19\cdot 43$, in which case $D_{\mathrm{u}}(n) = -144$.

For example, the largest jump in Figure 4 is caused by the wealth of divisors of $n = 10000080 = 2^4\cdot 3^2\cdot 5\cdot 17\cdot 19\cdot 43$, for which $o_{\mathrm{u}}(n) = 48$, $e_{\mathrm{u}}(n) = 192$ and $D_{\mathrm{u}}(n) = 48 - 192 = -144$.

Next, by (2) the formula for the difference function is

$$D_{\mathrm{u}}(n) = o_{\mathrm{u}}(n) - e_{\mathrm{u}}(n) = (1-\alpha)d(r), \quad \text{for } n = 2^\alpha r,\ \alpha \ge 0,\ r \text{ odd}. \tag{3}$$

Let us note the particular case where n is even but not divisible by four, in which case $e_{\mathrm{u}}(n) = o_{\mathrm{u}}(n) = d(n)$, so that $D_{\mathrm{u}}(n) = 0$.

In the following, we assume x is large enough and calculate the average of $D_{\mathrm{u}}(n)$ over the positive integers $n \le x$. Using formula (3), we have:

$$\sum_{n=1}^{x} D_{\mathrm{u}}(n) = \sum_{\substack{r=1\\ r \text{ odd}}}^{x} \sum_{\substack{\alpha=0\\ 2^\alpha r\le x}}^{\left[\frac{\log x}{\log 2}\right]} (1-\alpha)d(r)\,. \tag{4}$$

Let us denote the divisor sum over the odd integers by $I(t) := \sum\limits_{\substack{r=1\\ r \text{ odd}}}^{t} d(r)$. Then, (4) becomes

$$\sum_{n=1}^{x} D_{\mathrm{u}}(n) = I(x) - I(x/4) - 2I(x/8) - \cdots - (1-\tau)I(x/2^\tau) + R(x)\,, \tag{5}$$

where $\tau = \left[\frac{\log x}{\log 2}\right]$ and $R(x) = \tau I(y)$, for some $y \le x/2^{\tau+1}$, collects the remaining terms.

The sum of odd divisors can be calculated by the well-known inclusion-exclusion Dirichlet method and this is the object of the next lemma.

LEMMA 1. *We have*

$$I(x) := \sum_{\substack{n=1\\ n \text{ odd}}}^{x} d(n) = \frac{1}{4}x\log x + x\left(\frac{\log 2}{2} + \frac{\gamma}{2} - \frac{1}{4}\right) + O\left(\sqrt{x}\right).$$

Proof. We need an estimate for the odd harmonic sum, which is $H_o(x) := \sum\limits_{\substack{1\le n\le x\\ n \text{ odd}}}^{x} \frac{1}{n} = H(x) - H(x/2)/2$, where $H(x) = \sum_{1\le n\le x} 1/n = \log x + \gamma + O(1/x)$ and γ is Euler's constant. Then

$$H_o(x) = \log x + \gamma - \frac{1}{2}\log\frac{x}{2} - \frac{\gamma}{2} + O(1/x) = \frac{1}{2}\log x + \frac{\log 2}{2} + \frac{\gamma}{2} + O(1/x). \tag{6}$$

Next, we write $I(x)$ as a double sum that counts lattice points under a hyperbola:

$$I(x) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{x} d(n) = \sum_{\substack{1\le ab\le x \\ a,b \text{ odd}}}^{x} 1\,.$$

The contribution of the numerous smaller terms can be controlled efficiently counting them twice, in different order. Thus, we have:

$$\begin{aligned} I(x) &= \sum_{\substack{1\le a\le\sqrt{x} \\ a \text{ odd}}} \sum_{\substack{1\le b\le\frac{x}{a} \\ b \text{ odd}}} 1 + \sum_{\substack{1\le b\le\sqrt{x} \\ b \text{ odd}}} \sum_{\substack{1\le a\le\frac{x}{b} \\ a \text{ odd}}} 1 - \sum_{\substack{1\le a\le\sqrt{x} \\ a \text{ odd}}} \sum_{\substack{1\le b\le\sqrt{x} \\ b \text{ odd}}} 1 \\ &= 2\sum_{\substack{1\le a\le\sqrt{x} \\ a \text{ odd}}} \left(\frac{x}{2a} + O(1)\right) - \left(\frac{\sqrt{x}}{2} + O(1)\right)^2 \\ &= xH_o(\sqrt{x}) - \frac{x}{4} + O\left(\sqrt{x}\right). \end{aligned}$$

Then, on using the estimate (6), we find that

$$I(x) = x\left(\frac{1}{2}\log\sqrt{x} + \frac{\log 2}{2} + \frac{\gamma}{2} + O\left(\frac{1}{\sqrt{x}}\right)\right) - \frac{x}{4} + O\left(\sqrt{x}\right) = \frac{1}{2}x\log\sqrt{x} + x\left(\frac{\log 2}{2} + \frac{\gamma}{2} - \frac{1}{4}\right) + O\left(\sqrt{x}\right),$$

which concludes the proof of the lemma. □

Replacing the terms in relation (5) by their corresponding estimates from Lemma 1, we have:

$$\begin{aligned} A_{\mathrm{u}}(x) &= I(x) - I(x/4) - 2I(x/8) - \cdots - (1-\tau)I(x/2^{\tau}) + R(x) \\ &= \frac{1}{4}x\log x S_1(\tau) + \frac{\log 2}{4}xS_2(\tau) + \frac{2\log 2 + 2\gamma - 1}{4}xS_1(\tau) + O\big(\sqrt{x}S_3(\tau)\big) + R(x)\,, \end{aligned} \tag{7}$$

where we denoted

$$S_1(\tau) = 1 - \frac{1}{2^2} - \frac{2}{2^3} - \cdots - \frac{\tau-1}{2^{\tau}}, \qquad S_2(\tau) = \frac{1\cdot 2}{2^2} + \frac{2\cdot 3}{2^3} + \cdots + \frac{(\tau-1)\tau}{2^{\tau}}, \text{ and}$$
$$S_3(\tau) = 1 - \frac{1}{2^{2/2}} - \frac{2}{2^{3/2}} - \cdots - \frac{\tau-1}{2^{\tau/2}}$$

and $\tau = \left[\frac{\log x}{\log 2}\right]$. The sums $S_1(\tau), S_2(\tau), S_3(\tau)$ can be added and expressed in closed-form and then their sizes are easily evaluated. Thus we find that all terms except the second from the right hand side of (7) are no larger than $O(\sqrt{x})$. Then, since $S_2(\tau) = 4 + O\big((\log x)/x\big)$, the main term on the right hand side of (7) is $\frac{\log 2}{4}xS_2(\tau) = (\log 2)x + O(\log x)$, and the theorem follows. Note that the slope from the experiment presented in the image on the right of Figure 3 is consistent with the one given by the estimation from Theorem 1 because $\log 2 \approx 0.69314$.

3. THE DIVISOR PARITY SLOPE FOR STURMIAN WORDS – PROOF OF THEOREM 2

For any Sturmian word $\mathfrak{w}$, the following average of $D_{\mathfrak{w}}(n)$, for $1 \le n \le x$, tempered with the weight function $1 - x/n$, is estimated in [3, Theorem 1] (see the Fig. 1):

$$M_{\mathfrak{w}}(x) := \sum_{n=1}^{x} D_{\mathfrak{w}}(n)\left(1 - \frac{n}{x}\right) = \frac{\beta_{\mathfrak{w}}\log 2}{2}x + O_{\delta}\left(x^{1/3+\delta}\right) \text{ for any } \delta > 0. \tag{8}$$

To measure the influence of the weight function in $M_{\mathfrak{w}}(x)$ and compare the result when the weight function is missing, we introduce the function

$$f_y(t) := \begin{cases} y-t & \text{if } \; 0 \le t < y, \\ 0 & \text{if } \; y \le t. \end{cases}$$

For now, let L be a fixed number $0 < L < x$ whose precise value will be chosen towards the end of the proof. Define

$$S_L(x) := \sum_{n\ge 1} D_{\mathfrak{w}}(n)\big(f_{x+L}(n) - f_x(n)\big). \tag{9}$$

Notice that the series that defines $S_L(x)$ is actually finite, the multipliers of $D_{\mathfrak{w}}(n)$ vary continuously and for small n they are constant, because

$$f_{x+L}(n) - f_x(n) = \begin{cases} L & \text{if } \; 0 \le t < x, \\ x+L-n & \text{if } \; x \le t < x+L, \\ 0 & \text{if } \; x+L \le t. \end{cases}$$

Then (9) becomes

$$S_L(x) = L \sum_{1\le n\le x} D_{\mathfrak{w}}(n) - \sum_{x\le n\le x+L} D_{\mathfrak{w}}(n)(x+L-n) = LA_{\mathfrak{w}}(x) + O\Bigg(L \sum_{x\le n\le x+L} |D_{\mathfrak{w}}(n)|\Bigg). \tag{10}$$

Using Ramanujan's bound $d(n) = O_\delta(n^\delta)$, for all $\delta > 0$ [4, Section 18.1, Theorem 315], it follows that

$$|D_{\mathfrak{w}}(n)| \le |o_{\mathfrak{w}}(n)| + |e_{\mathfrak{w}}(n)| \le d(n) = O_\delta\big(n^\delta\big), \tag{11}$$

for all $\delta > 0$. On combining (11) and (9) we find that

$$A_{\mathfrak{w}}(x) = \frac{1}{L} S_L(x) + O_\delta\big(Lx^\delta\big), \;\; \text{for all } \delta > 0. \tag{12}$$

On the other hand, by the definition (9) of $S_L(x)$, we have

$$\begin{aligned} S_L(x) &= \sum_{n\ge 1} D_{\mathfrak{w}}(n) f_{x+L}(n) - \sum_{n\ge 1} D_{\mathfrak{w}}(n) f_x(n)\big) \\ &= \sum_{1\le n\le x+L} D_{\mathfrak{w}}(n)(x+L-n) - \sum_{1\le n\le x} D_{\mathfrak{w}}(n)(x-n) \\ &= (x+L) \sum_{1\le n\le x+L} D_{\mathfrak{w}}(n) \left(1 - \frac{n}{x+L}\right) - x \sum_{1\le n\le x} D_{\mathfrak{w}}(n) \left(1 - \frac{n}{x}\right) \\ &= (x+L) M_{\mathfrak{w}}(x+L) - x M_{\mathfrak{w}}(x). \end{aligned} \tag{13}$$

Then, by (8), it follows:

$$\begin{aligned} S_L(x) &= (x+L)\left(\frac{\beta_{\mathfrak{w}} \log 2}{2}(x+L) + O_\delta\big(x^{1/3+\delta}\big)\right) - x\left(\frac{\beta_{\mathfrak{w}} \log 2}{2} x + O_\delta\big(x^{1/3+\delta}\big)\right) \\ &= \frac{\beta_{\mathfrak{w}} \log 2}{2}(2Lx + L^2) + O_\delta\big(x^{4/3+\delta}\big), \end{aligned} \tag{14}$$

because we have assumed that $0 < L < x$. On inserting this estimate in (12), we obtain

$$\begin{aligned} A_{\mathfrak{w}}(x) &= \frac{\beta_{\mathfrak{w}} \log 2}{2}(2x+L) + O_\delta\big(x^{4/3+\delta}/L\big) + O_\delta\big(Lx^\delta\big) \\ &= (\beta_{\mathfrak{w}} \log 2) \cdot x + O(L) + O_\delta\big(x^{4/3+\delta}/L\big) + O_\delta\big(Lx^\delta\big). \end{aligned}$$

Here, the second error term is absorbed by the third and to balance the remaining ones, choose $L = x^{2/3}$. Therefore, $A_{\mathfrak{w}}(x) = (\beta_{\mathfrak{w}} \log 2) \cdot x + O_\delta\left(x^{2/3+\delta}\right)$, for all $\delta > 0$, which concludes the proof of Theorem 2.

4. THE FRACTAL FACES mod 3 OF A STURMIAN WORD

For readers interested to continue the investigation of the subject we end this paper by including several images which show the fractalic character gained by the Sturmian words. This is due to just one unit added to their complexity compared to that of the ultimately periodic words. We have generated the Sturmian words by a rotation process as follows. Let $\theta > 0$ be an irrational number and let $\varphi \in [0,1)$ be an offset angle. Define $x_n := \varphi + n\theta \pmod 1$ for $n \geq 0$. Then, let $\mathfrak{w}_n = a$ if $x_n \in [0,\theta)$ and $\mathfrak{w}_n = b$ otherwise. Denote by $R(\varphi,\theta,N)$ the finite word consisting of the first N letters that were generated in this way. For example, $R(0.2, 4\log 2, 10) = abaaa\,abaaa$.

The rules for drawing the paths in Figures 5-8 are as follows. Let $\mathfrak{w} = \mathfrak{w}_0\mathfrak{w}_1\mathfrak{w}_2\ldots$ be a Sturmian word with letters $\mathfrak{w}_j \in \{a,b\}$, for $j \geq 0$. All steps have length equal to 1. Start from the origin looking up. For each $n \geq 0$ go forward one step. Afterwards, turn right 120° if $\mathfrak{w}_n = b$ and $n \equiv 0 \pmod 3$ or turn left 120° if $\mathfrak{w}_n = b$ and $n \equiv 2 \pmod 3$. Otherwise, keep the same direction for the next step. The end-point is marked by a dot. This rule extends the similar one modulo 2 used to draw Fibonacci words (see [9]).

We remark that quite often it happens that the path self-intersects and overlaps even several times. A classification of the particularly large variety of these paths as well as their interesting properties requires further investigation.

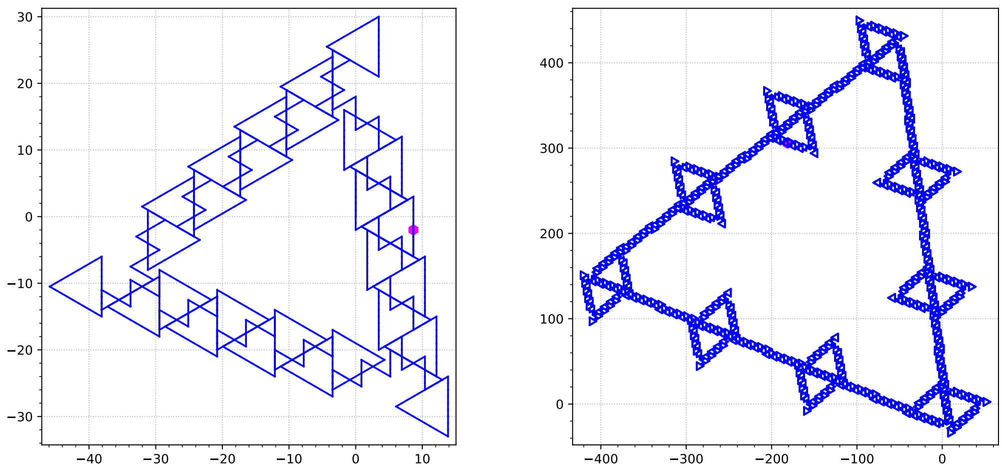

Fig. 5 – $R(\varphi,\theta,N)$ for $\varphi = 0.2$, $\theta = 4\log 2$, $N = 1\,000$ (left) and $N = 40\,000$ (right).

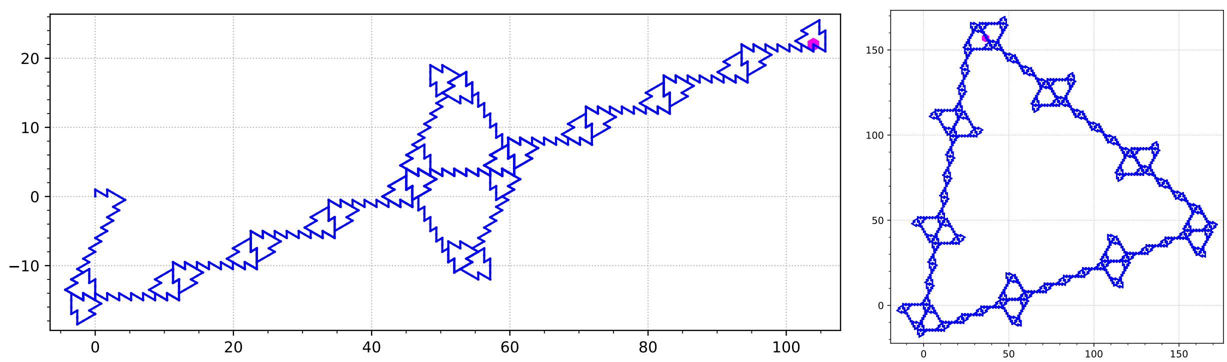

Fig. 6 – $R(\varphi,\theta,N)$ for $\varphi = 0.26$, $\theta = \pi/18$, $N = 1\,000$ (left) and $N = 10\,000$ (right).

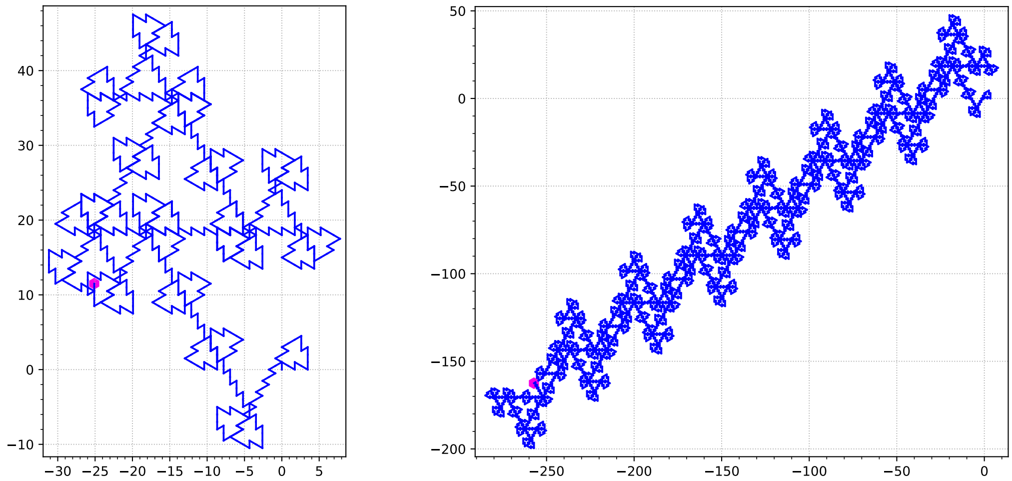

Fig. 7 – $R(\varphi,\theta,N)$ for $\varphi = 0.16, \theta = \pi/20$, $N = 1\,000$ (left) and $N = 10\,000$ (right).

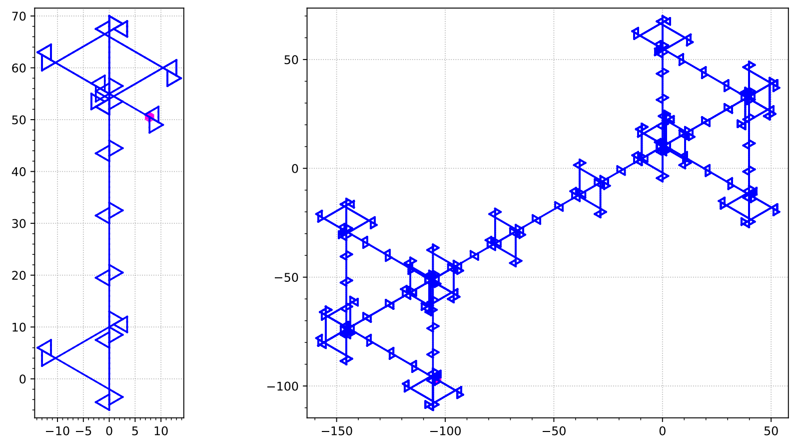

Fig. 8 – $R(\varphi,\theta,N)$ for $\varphi = 0.2, \theta = \sqrt{7}$, $N = 1\,000$ (left) and $N = 20\,000$ (right).

REFERENCES

1. J. P. ALLOUCHE, J. SHALLIT, *Automatic sequences. Theory, applications, generalizations*, Cambridge University Press, Cambridge, 2003.
2. S. BRLEK, M. CHEMILLIER, C. REUTENAUER, *Editorial. Music and combinatorics on words: a historical survey*, Journal of Mathematics and Music, **12**, *3*, pp. 125–133, 2018.
3. C. COBELI, A. ZAHARESCU, *A bias parity slope on the simplest non-periodic binary words*, Chaos, Solitons & Fractals, **169**, art. 113257, 2023.
4. G.H. HARDY, E.M. WRIGHT, *An introduction to the theory of numbers*, 6th edition, Revised by D.R. Heath-Brown and J.H. Silverman, With a foreword by Andrew Wiles, Oxford University Press, Oxford, 2008.
5. T. HOFFMAN, B. STEINHURST, *Hausdorff dimension of generalized Fibonacci word fractals*, Fractals, **26**, *1*, art. 1850012, 2018.
6. L. ILIE, S. MARCUS, I. PETRE, *Periodic and Sturmian languages*, Information Processing Letters, **98**, *6*, pp. 242–246, 2006.
7. S. MARCUS, *Quasiperiodic infinite words*, European Association for Theoretical Computer Science, **82**, pp. 170–174, 2004.
8. S. MARCUS, T. MONTEIL, *Quasiperiodic infinite words: multi-scale case and dynamical properties*, arXiv preprint, arXiv:0603354v1, 2006.
9. J.L. RAMÍREZ, G.N. RUBIANO, R. DE CASTRO, *A generalization of the Fibonacci word fractal and the Fibonacci snowflake*, Theoretical Computer Science, **528**, pp. 40–56, 2014.
10. A. VAN ENTER, H. KOIVUSALO, J. MIĘKISZ, *Sturmian ground states in classical lattice-gas models*, Journal of Statistical Physics, **178**, *3*, pp. 832–844, 2020.
11. C. ZENG, Y. XUE, Y. HUANG, *Fractal networks with Sturmian structure*, Physica A, **574**, art. 125977, 2021.

Received June 13, 2022