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SEMILINEAR PROBLEMS WITH POLY-LAPLACE TYPE OPERATORS

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Abstract. The paper deals with semilinear operator equations involving iterates of a strongly monotone symmetric linear operator. In particular there are consider semilinear polyharmonic equations subject to the Navier boundary conditions. A careful analysis is made on the energetic spaces associated to such problems and a number of existence results are obtained by using a fixed point approach.

***Key words**:* polyharmonic equation, iterates of symmetric linear operators, energetic space.
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1. INTRODUCTION

There is known the bi-Laplace equation $\Delta^2 u = 0$ whose solutions are called *biharmonic* functions. The equation arises as a model for the elastic equilibrium in the theory of elasticity. Also there are known its generalizations, the poly-Laplace equations $\Delta^p u = 0,\ p > 2$, whose solutions are said to be *polyharmonic of order p* (see, [9] and [10]). The operator $\Delta^2 = \Delta\Delta$ is referred as the *bi-Laplacian* and $\Delta^p = \Delta\left(\Delta^{p-1}\right)$ is said to be the *Laplacian of order p*. The non-homogeneous versions of these equations are

$$\Delta^p u = h,$$

and when considered in a domain Ω, in case $p = 2$, there has been added the boundary condition

$$u = \frac{\mathrm{d}u}{\mathrm{d}\nu} = 0 \ \text{ on } \partial\Omega, \tag{1}$$

where ν is the unit normal vector to the boundary, or the boundary condition

$$u = \Delta u = 0 \ \text{ on } \partial\Omega \tag{2}$$

(see [8]). For $p > 2$, condition (1) can be generalized following Lauricella [7] as follows

$$u = \frac{\mathrm{d}u}{\mathrm{d}\nu} = \ \dots \ = \frac{\mathrm{d}^{p-1}u}{\mathrm{d}\nu^{p-1}} = 0 \ \text{ on } \partial\Omega,$$

and (2), as suggested by Riquier [13], by

$$u = \Delta u = \Delta^2 u = \ \dots \ = \Delta^{p-1} u = 0 \ \text{ on } \partial\Omega.$$

For the classical theory of polyharmonic functions we refer the reader to the volume [9] which brings together the entire contribution of Miron Nicolescu to this field and which allows obtaining information on contributions originating from old, less accessible publications.

Modern theory has introduced the concept of weak solution and Sobolev spaces as natural framework for the study of these operators and of the associated semilinear problems. Thus the problem

$$\begin{cases} \Delta^2 u = f(x, u, \nabla u, \Delta u) & \text{in } \Omega \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \tag{3}$$

involving the natural boundary condition $\frac{\partial u}{\partial \nu} = 0$ can be naturally addressed in $H_0^2(\Omega)$ endowed with the equivalent norm $|\Delta u|_{L^2(\Omega)}$. Other studies (see, e.g., [1], [2], [4] and [11]) have aimed to treat problems of type (3) under the boundary conditions $u = \Delta u = 0$ on $\partial\Omega$ (called Navier boundary conditions [5]) by looking for solutions in the space $H^2(\Omega) \cap H_0^1(\Omega)$ with norm $|\Delta u|_{L^2(\Omega)}$. The problem is that the condition "$\Delta u = 0$ on $\partial\Omega$" not being a natural boundary condition it does not follow from the variational formulation of the problem. This is the reason to restrict the study to a subspace of functions in order to give a meaning to the equality $\Delta u = 0$ on the boundary. This will be one of our main goals in this work. Roughly speaking we suggest that the iterative nature of the differential operator to reflect on its energetic space and consequently on some basic inequalities. We lead this discussion more generally considering instead of Laplacian a strongly monotone symmetric linear operator A. Thus our results will concern semilinear operator equations of the form

$$A^p u = h + F\left(u, Au, \ \ldots, \ A^{p-1}u\right),$$

where A^p is the p-th iterate of A, defined recursively by $A^p = AA^{p-1}$. The whole approach is based on the theory of the energetic space X_A associated to A. There are thus obtained existence results for the problem

$$\begin{cases} A^p u = h + F\left(u, Au, \ \ldots, \ A^{p-1}u\right) \\ u, \ Au, \ \ldots, \ A^{p-1}u \in X_A \end{cases} \tag{4}$$

where $h \in X_A'$ is given and F is on the position of a perturbation of h. In particular, we obtain results for semilinear poly-Laplace equations.

2. PRELIMINARIES

In this section we recall the notion of energetic space (see [14]) and some related results.

2.1. The energetic space

Let X be a real Hilbert space with the inner product $(\cdot,\cdot)_X$ and the norm $|\cdot|_X$. Let Y be a linear subspace of X and $A : Y \to X$ be a strongly monotone symmetric linear operator, that is, a linear operator satisfying

$$(Au, v)_X = (u, Av)_X \quad \text{for all } u, v \in Y,$$

$$(Au, u)_X \geq c^2 |u|_X^2 \quad \text{for all } u \in Y \tag{5}$$

and some constant $c > 0$. Then, endowed with the *energetic inner product*

$$(u, v)_A := (Au, v)_X \quad (u, v \in Y)$$

and the *energetic norm*

$$|u|_A = (Au, u)_X^{1/2} \quad (u \in Y),$$

Y becomes a pre-Hilbert space. Its completion (see, e.g., [6, Section I 4.3]) denoted by X_A is called the *energetic space* of A. In virtue of (5), any Cauchy sequence in the energetic norm is also a Cauchy sequence in the norm $|\cdot|_X$. This allows us to see X_A as a subset of the original complete space X, and the elements of X_A as limits in X of Cauchy sequences from $(Y, \ |\cdot|_A)$. Furthemore, the energetic inner product and norm can be extended from Y to X_A by

$$(u, v)_{X_A} := \lim_{k\to\infty} (u_k, v_k)_A, \quad |u|_{X_A} := \lim |u_k|_A,$$

where (u_k) and (v_k) are Cauchy sequences in $(Y, \ |\cdot|_A)$ that converge in X to u and v, respectively.

2.2. Abstract Poincaré's inequality

Inequality (5) can be extended by density from Y to X_A showing that

$$|u|_{X_A} \geq c\,|u|_X \quad \text{for all } u \in X_A. \tag{6}$$

Thus c is an embedding constant for the continuous inclusion $X_A \subset X$. We call this inequality *Poincaré's inequality*.

If the embedding $X_A \subset X$ is compact, then there is a largest embedding constant c and the inequality is reached. Indeed, if we denote

$$\lambda := \inf\left\{|u|_{X_A}^2 : \ u \in X_A, \ \ |u|_X = 1\right\},$$

then $\lambda \geq c^2$ and if we take any minimizing sequence (u_k), that is

$$u_k \in X_A, \quad |u_k|_X = 1, \quad |u_k|_{X_A}^2 \to \lambda,$$

then using the compactness of the embedding $X_A \subset X$ and passing eventually to a subsequence we can assume that $u_k \to u$ in X, for some $u \in X$. Furthermore, from the identity

$$|u_k - u_m|_{X_A}^2 + |u_k + u_m|_{X_A}^2 = 2\left(|u_k|_{X_A}^2 + |u_m|_{X_A}^2\right),$$

since $|u_k + u_m|_{X_A}^2 \geq \lambda\,|u_k + u_m|_X^2$, we deduce

$$|u_k - u_m|_{X_A}^2 \leq 2\left(|u_k|_{X_A}^2 + |u_m|_{X_A}^2\right) - \lambda\,|u_k + u_m|_X^2 \to 0$$

as $k, m \to \infty$. Hence (u_k) is a Cauchy sequence in X_A. Let $v \in X_A$ be such that $u_k \to v$ in X_A. Then $u_k \to v$ in X too, and the uniqueness of the limit implies that $v = u$. Consequently, $|u_k|_{X_A} \to |u|_{X_A}$, that is $\lambda = |u|_{X_A}^2$. Thus the infimum λ is reached and $\sqrt{\lambda}$ is the best constant c in (6). Thus, in case that the embedding $X_A \subset X$ is compact, Poincaré's inequality reads as follows:

$$|u|_X \leq \frac{1}{\sqrt{\lambda}}\,|u|_{X_A} \quad \text{for all } u \in X_A.$$

2.3. The dual of the energetic space

Having $X_A \subset X$, for the dual spaces we have $X' \subset X_A'$ and if, based on Riesz' theorem, we assume the identification $X' = X$, then one has

$$X_A \subset X \subset X_A'.$$

In addition, from (6) we also have

$$|u|_X \geq c\,|u|_{X_A'} \quad \text{for all } u \in X. \tag{7}$$

Indeed, if $u \in X$, then for any $v \in X$, one has $\langle u, v\rangle = (u, v)_X$, where by $\langle \cdot, \cdot \rangle$ we mean the value of a linear functional at a given element. Then

$$|u|_{X_A'} = \sup_{v \in X_A \setminus \{0\}} \frac{|\langle u, v\rangle|}{|v|_{X_A}} = \sup_{v \in X_A \setminus \{0\}} \frac{|(u, v)_X|}{|v|_{X_A}} \leq \sup_{v \in X_A \setminus \{0\}} \frac{|u|_X\,|v|_X}{|v|_{X_A}} \leq \frac{1}{c}\,|u|_X\,.$$

Notice in case that the embedding $X_A \subset X$ is compact, so is the embedding $X \subset X_A'$ and in (7) we may take as in (6) the best constant $c = \sqrt{\lambda}$.

2.4. Extension of operator A

Clearly we can define the linear operator $\widetilde{A} : X_A \to X_A'$ by

$$\left\langle \widetilde{A}u, \ v \right\rangle = (u, v)_{X_A} \quad \text{for all } u, v \in X_A.$$

In particular, if $u,v \in Y$, then since $(u,v)_{X_A} = (Au,v)_X$, one has $\left\langle \widetilde{A}u,\ v \right\rangle = (Au,v)_X$, which by the density of Y into X_A can be extended to all $v \in X_A$. Thus the functionals $\widetilde{A}u$ and $(Au,\cdot)_X$ act identically in X_A. The last one is a continuous linear functional on X which in virtue of Riesz's representation theorem is identified with Au. In this sense, as continuous linear functionals on X_A, one has $\widetilde{A}u = Au$, and therefore $\widetilde{A}$ can be seen as an extension of A from Y to X_A. It is common to use the same symbol A for the extension $\widetilde{A}$. Thus $A : X_A \to X_A'$.

2.5. The inverse of operator A

In the previous subsection we have that for every $u \in X_A$ there is a unique element denoted $Au \in X_A'$ with

$$\langle Au,v\rangle = (u,v)_{X_A} \quad \text{for all } v \in X_A. \tag{8}$$

Conversely, for every $h \in X_A'$ by Riesz's theorem, there is a unique element $u \in X_A$ with

$$\langle h,v\rangle = (u,v)_{X_A} \quad \text{for all } v \in X_A.$$

Clearly $Au = h$ and thus $u = A^{-1}h$. Hence the inverse of A is the operator $A^{-1} : X_A' \to X_A$ defined by

$$\left(A^{-1}h,v\right)_{X_A} = \langle h,v\rangle \quad \text{for all } v \in X_A.$$

The two linear operators A and A^{-1} are isometries between X_A and X_A'. Indeed, letting $v = u$ in (8) gives

$$|u|_{X_A}^2 = \langle Au,u\rangle \le |Au|_{X_A'}\,|u|_{X_A}\,,$$

whence $|u|_{X_A} \le |Au|_{X_A'}$. The converse inequality comes from

$$|Au|_{X_A'} = \sup_{v\in X_A\setminus\{0\}} \frac{|\langle Au,v\rangle|}{|v|_{X_A}} = \sup_{v\in X_A\setminus\{0\}} \frac{\left|(u,v)_{X_A}\right|}{|v|_{X_A}} \le |u|_{X_A}\,.$$

Hence

$$|Au|_{X_A'} = |u|_{X_A} \quad (u \in X_A)\,, \quad \left|A^{-1}h\right|_{X_A} = |h|_{X_A'} \quad \left(h \in X_A'\right).$$

2.6. Weak solutions to linear operator equations

Consider the operator equation associated to A,

$$Au = h.$$

By a (strong) solution we mean an element $u \in Y$ such that $Au = h$. Obviously this is possible if $h \in X$. By a *weak solution* we mean an element $u \in X_A$ satisfying the identity

$$(u,v)_{X_A} = \langle h,v\rangle \quad \text{for all } v \in X_A.$$

When speaking about weak solutions we may assume more generally that $h \in X_A'$. In view of the previous subsection, for each $h \in X_A'$, the equation has a unique weak solution, namely $u = A^{-1}h$.

Note that looking for weak solutions to a semilinear equation

$$Au = \Phi(u)\,,$$

where $\Phi : X_A \to X_A'$ is any mapping, reduces to solving the fixed point equation

$$u = A^{-1}\Phi(u)\,, \quad u \in X_A.$$

3. SEMILINEAR EQUATIONS INVOLVING ITERATES OF A SYMMETRIC LINEAR OPERATOR

We now come back to problem (4), where A is a linear operator as in Introduction.

3.1. Functional framework

Looking at the required conditions on the elements $u,\ Au,\ \ldots,\ A^{p-1}u$ to belong to the energetic space X_A of the operator A, we may seek solutions in the space

$$H := A^{-(p-1)}\left(X_A\right).$$

Here $A^{-k} = A^{-1}\left(A^{-(k-1)}\right)$ for $k = 2,\ \ldots,\ p-1$. Since $A^{-1} : X_A' \to X_A$ and $X_A \subset X_A'$ one has

$$H = A^{-(p-1)}\left(X_A\right) \subset A^{-(p-2)}\left(X_A\right) \subset\ \ldots\ \subset A^{-1}\left(X_A\right) \subset X_A. \tag{9}$$

We endow H with the inner product and norm

$$(u,v)_H := \left(A^{p-1}u, A^{p-1}v\right)_{X_A}, \quad |u|_H := \left|A^{p-1}u\right|_{X_A}.$$

Note that the functional $|\cdot|_H$ is indeed a norm on H since if for some $u \in H$, one has $|u|_H = 0$, then $A^{p-1}u = 0$, whence $A^{p-2}u = 0$ and so on until we obtain $u = 0$.

LEMMA 1. *The space H endowed with the inner product $(\cdot,\cdot)_H$ is a Hilbert space which continuously embeds in X_A.*

Proof. Let (u_k) be any Cauchy sequence in H. Then $\left(A^{p-1}u_k\right)$ is Cauchy in X_A, so convergent in X_A to some $v \in X_A$. Since the embedding $X_A \subset X_A'$ is continuous, we then have $A^{p-1}u_k \to v$ in X_A'. Next, the continuity of A^{-1} from X_A' to X_A implies $A^{p-2}u_k \to A^{-1}v$ in X_A. Repeating the above reasoning we arrive to the conclusion that $u_k \to A^{-(p-1)}v$ in X_A, that is $u_k \to u := A^{-(p-1)}v$ in H. This proves that $(H, |\cdot|_H)$ is complete. □

Knowing the operator A^{-1} from X_A' to X_A and the inclusions (9) we immediately can see that for every $h \in X_A'$ there is a unique $u \in H$, namely $u = A^{-p}h$, which solves the non-homogeneous equation

$$A^p u = h.$$

Consequently, solving a semi-linear equation of the form

$$A^p u = \Phi(u),$$

where $\Phi : H \to X_A'$ is any mapping, is equivalent to the fixed point equation

$$u = A^{-p}\Phi(u), \quad u \in H$$

for the operator $A^{-p}\Phi : H \to H$.

3.2. Existence and uniqueness under a Lipschitz condition

Using Banach contraction principle we obtain the following result on problem (4).

THEOREM 1. *Let $F : X^p \to X$ satisfy*

$$|F(u) - F(v)|_X \le \sum_{i=1}^{p} a_i |u_i - v_i|_X \tag{10}$$

for all $u = (u_1,\ \ldots,\ u_p),\ v = (v_1,\ \ldots,\ v_p) \in X^p$ and some nonnegative constants $a_i,\ i = 1,\ \ldots,\ p$. If

$$\theta := \sum_{i=1}^{p} \frac{a_i}{c^{2(p+1-i)}} < 1, \tag{11}$$

then problem (4) has a unique solution $u \in H$.

Proof. Problem (4) is equivalent to the fixed point equation

$$u = A^{-p}\left(h + F\left(u, Au, \ ..., \ A^{p-1}u\right)\right), \ \ u \in H.$$

Using (10) and Poincaré's inequality (6), for any $u, v \in H$, we have

$$\begin{aligned} & \left|A^{-p}F\left(u,Au, \ ..., \ A^{p-1}u\right) - A^{-p}F\left(v,Av, \ ..., \ A^{p-1}v\right)\right|_H \\ = \ & \left|A^{-1}\left(F\left(u,Au, \ ..., \ A^{p-1}u\right) - F\left(v,Av, \ ..., \ A^{p-1}v\right)\right)\right|_{X_A} \\ = \ & \left|F\left(u,Au, \ ..., \ A^{p-1}u\right) - F\left(v,Av, \ ..., \ A^{p-1}v\right)\right|_{X_A'} \\ \leq \ & \frac{1}{c}\left|F\left(u,Au, \ ..., \ A^{p-1}u\right) - F\left(v,Av, \ ..., \ A^{p-1}v\right)\right|_X \\ \leq \ & \frac{1}{c}\sum_{i=1}^{p} a_i \left|A^{i-1}(u-v)\right|_X . \end{aligned} \tag{12}$$

Furthermore, for ant $w \in H$, one has

$$\begin{aligned} \left|A^{p-1}w\right|_X & \leq \frac{1}{c}\left|A^{p-1}w\right|_{X_A} = \frac{1}{c}|w|_H , \\ \left|A^{p-2}w\right|_X & \leq \frac{1}{c}\left|A^{p-2}w\right|_{X_A} = \frac{1}{c}\left|A^{p-1}w\right|_{X_A'} \leq \frac{1}{c^2}\left|A^{p-1}w\right|_X \leq \frac{1}{c^3}|w|_H , \\ \left|A^{p-3}w\right|_X & \leq \frac{1}{c}\left|A^{p-3}w\right|_{X_A} = \frac{1}{c}\left|A^{p-2}w\right|_{X_A'} \leq \frac{1}{c^2}\left|A^{p-2}w\right|_X \leq \frac{1}{c^5}|w|_H . \end{aligned}$$

Repeating the above estimations for $p-4, \ ..., \ 0$, we obtain

$$\left|A^{i-1}w\right|_X \leq \frac{1}{c^{2(p-i)+1}}|w|_H , \quad i = 1, \ ..., \ p. \tag{13}$$

Then

$$\begin{aligned} \left|A^{-p}F\left(u,Au, \ ..., \ A^{p-1}u\right) - A^{-p}F\left(v,Av, \ ..., \ A^{p-1}v\right)\right|_H & \leq \left(\sum_{i=1}^{p}\frac{a_i}{c^{2(p+1-i)}}\right)|u-v|_H \\ & = \theta\,|u-v|_H , \end{aligned} \tag{14}$$

which in view of (11) shows that the operator

$$N := A^{-p}\left(h + F\left(u, Au, \ ..., \ A^{p-1}u\right)\right)$$

is a contraction on H. The conclusion now follows from Banach contraction principle. □

3.3. Existence under a linear growth condition

If instead of the Lipschitz condition (10) we only have a linear growth condition on F and we assume that the embedding $X_A \subset X$ is compact, then we can still prove the existence of at least one solution by using Schauder's fixed point theorem.

THEOREM 2. *Assume that the embedding $X_A \subset X$ is compact and that $F : X^p \to X$ is continuous and satisfies*

$$|F(u)|_X \leq C + \sum_{i=1}^{p} a_i |u_i|_X$$

for all $u = (u_1, \ ..., \ u_p) \in X^p$ and some $C > 0$ and nonnegative constants $a_i, \ i = 1, \ ..., \ p$. If condition (11) holds, then problem (4) has at least one solution $u \in H$ with

$$|u|_H \leq \frac{Cc^{-1} + |h|_{X_A'}}{1-\theta}.$$

Proof. As above we now have

$$\left|A^{-p}F\left(u,Au,\ ...,\ A^{p-1}u\right)\right|_H \leq \frac{1}{c}\left|F\left(u,Au,\ ...,\ A^{p-1}u\right)\right|_X \leq Cc^{-1}+\theta\left|u\right|_H.$$

Since $\theta<1$, $R=\left(Cc^{-1}+|h|_{X_A'}\right)/(1-\theta)>0$ and N is a self mapping of the closed ball B_R of H centered at the origin and of radius R. On the other hand $N_0(u):=A^{-p}F\left(u,Au,\ ...,\ A^{p-1}u\right)$ can be decomposed as

$$N=A^{-(p-1)}A^{-1}JFJ_0P, \quad \text{where}$$

$$\begin{array}{rcl} P & : & H\to X_A^p,\ \ Pu=\left(u,Au,\ ...,\ A^{p-1}u\right);\ \ J_0: X_A^p\to X^p,\ \ Ju=u; \\ F & : & X^p\to X;\ \ J:X\to X_A',\ \ Ju=u;\ \ \ A^{-1}:X_A'\to X_A;\ \ \ A^{-(p-1)}:X_A\to H. \end{array}$$

All these operators are continuous and bounded (send bounded sets to bounded sets) and J_0 is compact. As a result their composition N_0 is completely continuous. Now the conclusion follows from Schauder's fixed point theorem applied to N in the ball B_R. □

3.4. Existence via a priori bounds

We may replace the growth condition on F by a sign type condition as shows the following theorem.

THEOREM 3. *Assume that* $F:X_A^p\to X_A'$ *is completely continuous and satisfies*

$$\left\langle F(v),\ v_p\right\rangle \leq \alpha\left|v_p\right|_X^2 \tag{15}$$

for all $v=(v_1,\ ...,\ v_p)\in X_A^p$ *and some* $\alpha\in[0,c^2)$. *Then problem (4) has at least one solution* $u\in H$. *Moreover, any solution* $u\in H$ *of the problem satisfies*

$$|u|_H\leq|h|_{X_A'}/\left(1-\alpha c^{-2}\right). \tag{16}$$

Proof. Using a similar reasoning as in the previous proof we can show that the operator $A^{-p}F:H\to H$ is completely continuous. We now prove that the set of all possible solutions of the equations

$$u=\mu A^{-p}\left(h+F\left(u,Au,\ ...,A^{p-1}u\right)\right)$$

for $\mu\in[0,1]$ is bounded as (16) shows. Indeed, if u is such a solution, then

$$\begin{aligned} |u|_H^2 &= \mu\left(A^{-p}h,\ u\right)_H+\mu\left(A^{-p}F\left(u,Au,\ ...,A^{p-1}u\right),\ u\right)_H \\ &= \mu\left(A^{-1}h,\ A^{p-1}u\right)_{X_A}+\mu\left(A^{-1}F,\ A^{p-1}u\right)_{X_A} \\ &\leq |h|_{X_A'}|u|_H+\mu\left(A^{-1}F,\ A^{p-1}u\right)_{X_A}. \end{aligned}$$

Next since $v:=\left(u,Au,\ ...,\ A^{p-1}u\right)\in X_A^p$, based on (15), one has

$$\left(A^{-1}F,\ A^{p-1}u\right)_{X_A}=\left\langle F,\ A^{p-1}u\right\rangle \leq \alpha\left|A^{p-1}u\right|_X^2\leq\frac{\alpha}{c^2}|u|_H^2.$$

It follows that

$$|u|_H^2\leq\alpha c^{-2}|u|_H^2+|h|_{X_A'}|u|_H,$$

whence (16). The existence of a solution is guaranteed by the Leray-Schauder principle. □

4. SEMILINEAR PROBLEMS WITH POLY-LAPLACE OPERATORS

The results established in Section 3 can be easily applied to problems involving poly-Laplace operators, more exactly to the problem

$$\begin{cases} \Delta^p u = h + f\left(x, u, \Delta u, \ ..., \ \Delta^{p-1} u\right) \ \text{ in } \Omega \\ u = \Delta u = \ ... \ = \Delta^{p-1} u = 0 \ \text{ on } \partial\Omega. \end{cases} \tag{17}$$

Here $\Omega \subset \mathbb{R}^n$ is bounded open, $X = L^2(\Omega)$, $A = -\Delta$, $X_A = H_0^1(\Omega)$, $X_A' = H^{-1}(\Omega)$, $h \in H^{-1}(\Omega)$ and $f : \Omega \times \mathbb{R}^p \to \mathbb{R}$. Hence

$$H = (-\Delta)^{-(p-1)} H_0^1(\Omega), \quad (u,v)_H = \int_\Omega \nabla\Delta^{p-1} u \cdot \nabla\Delta^{p-1} v, \quad |u|_H = \left|\Delta^{p-1} u\right|_{H_0^1(\Omega)}.$$

Also the compactness of the imbedding $X_A \subset X$ holds and the imbedding constant in Poincaré's inequality is $c = \sqrt{\lambda_1}$, where λ_1 is the first eigenvalue of the Dirichlet problem for Laplacian (for the theory of elliptic problems, see, e.g., [3] or [12]).

In this case, F is the superposition operator

$$F\left(u, (-\Delta) u, \ ..., \ (-\Delta)^{p-1} u\right)(x) = f\left(x, u(x), \Delta u(x), \ ..., \ \Delta^{p-1} u(x)\right)$$

$(x \in \Omega, \ u \in H)$.

Theorem 1 yields the following result.

COROLLARY 1. *Let f satisfy the Carathéodory conditions, $f(\cdot, 0) \in L^2(\Omega)$ and*

$$|f(x,u) - f(x,v)| \le \sum_{i=1}^p a_i |u_i - v_i| \tag{18}$$

for all $u = (u_1, \ ..., \ u_p)$, $v = (v_1, \ ..., \ v_p) \in \mathbb{R}^p$ and some nonnegative constants a_i, $i = 1, \ ..., \ p$. If $\theta < 1$ (θ being given by (11) with $c = \sqrt{\lambda_1}$), then problem (17) has a unique solution $u \in (-\Delta)^{-(p-1)} H_0^1(\Omega)$.

Proof. According to the main theorem abut Nemytskii's superposition operator, F maps $L^2(\Omega; \mathbb{R}^p)$ to $L^2(\Omega)$. In addition, for any $u, v \in L^2(\Omega; \mathbb{R}^p)$, from (18) we find

$$|F(u) - F(v)|_{L^2(\Omega)} \le \sum_{i=1}^p a_i |u_i - v_i|_{L^2(\Omega)}.$$

Thus Theorem 1 is applicable and gives the result. □

Theorem 2 yields the following result.

COROLLARY 2. *Let f satisfy the Carathéodory conditions and*

$$|f(x,u)| \le \psi(x) + \sum_{i=1}^p a_i |u_i|$$

for all $u = (u_1, \ ..., \ u_p) \in \mathbb{R}^p$, a.e. $x \in \Omega$, some nonnegative constants a_i, $i = 1, \ ..., \ p$ and a function $\psi \in L^2(\Omega)$. If $\theta < 1$ (θ being given by (11) with $c = \sqrt{\lambda_1}$), then problem (17) has at least one solution $u \in (-\Delta)^{-(p-1)} H_0^1(\Omega)$ with

$$\left|\Delta^{p-1} u\right|_{H_0^1(\Omega)} \le \frac{|\psi|_{L^2(\Omega)} / \sqrt{\lambda_1} + |h|_{H^{-1}(\Omega)}}{1 - \theta}.$$

Using Theorem 3 we obtain the following result.

COROLLARY 3. *Let f satisfy the Carathéodory conditions and*

$$|f(x,u)| \leq \psi(x) + \sum_{i=1}^{p} a_i |u_i|^q \tag{19}$$

for all $u = (u_1, \ ..., \ u_p) \in \mathbb{R}^p$, a.e. $x \in \Omega$, some nonnegative constants a_i, $i = 1, \ ..., \ p$, a number $1 \leq q < 2^* - 1 = 2^*/(2^*)' = (n+2)/(n-2) \ (n \geq 3))$ and a function $\psi \in L^{q_0}(\Omega)$, where $q_0 \in ((2^*)', 2^*/q]$. In addition assume that

$$v_p f(x,v) \leq \alpha v_p^2$$

for every $v \in \mathbb{R}^p$ and some $\alpha \in [0, \lambda_1)$. Then problem (17) has at least one solution $u \in (-\Delta)^{-(p-1)} H_0^1(\Omega)$. Moreover, any solution $u \in (-\Delta)^{-(p-1)} H_0^1(\Omega)$ satisfies

$$\left|\Delta^{p-1} u\right|_{H_0^1(\Omega)} \leq |h|_{H^{-1}(\Omega)} / \left(1 - \frac{\alpha}{\lambda_1}\right).$$

Proof. Let $q_1 = q_0 q$. Clearly $q_1 \in [1, 2^*]$. Hence the embedding $H_0^1(\Omega) \subset L^{q_1}(\Omega)$ is continuous, while since $q_0 > (2^*)'$, the embedding $L^{q_0}(\Omega) \subset H^{-1}(\Omega)$ is compact. In addition since $q = q_1/q_0$, from (19) we have that Nemytskii's superposition operator N_f is well-defined, continuous and bounded from $L^{q_1}(\Omega)^p$ to $L^{q_0}(\Omega)$. Then our operator $F(u) = f(\cdot, u(\cdot))$ can be decomposed as $F = JN_f P$, where

$$\begin{aligned} P &: \ H_0^1(\Omega)^p \to L^{q_1}(\Omega)^p, \ \ Pu = u; \\ N_f &: \ L^{q_1}(\Omega)^p \to L^{q_0}(\Omega), \ \ N_f(v)(x) = f(x, v(x)); \\ J &: \ L^{q_0}(\Omega) \to H^{-1}(\Omega), \ \ J_1 u = u. \end{aligned}$$

Since J is compact one deduces that $F : H_0^1(\Omega)^p \to H^{-1}(\Omega)$ is completely continuous.

We now check condition (15). For $v \in H_0^1(\Omega)^p$, one has

$$\langle F(v), \, v_p \rangle \ = \ \langle JN_f P(v), \, v_p \rangle \ = \int_\Omega v_p(x) f(x, v(x)) \leq \alpha \int_\Omega v_p(x)^2 = \alpha |v_p|^2_{L^2(\Omega)}.$$

Hence the assumptions of Theorem 3 are fulfilled and the conclusion follows. □

In contrast with the general case of equations involving iterates of a linear operator A, the case of the Laplace operator is a special one due to the representation of the Laplacian $\Delta = \nabla \cdot \nabla$ as a composition of two differential operators, the gradient and the divergence. This particularity allows nonlinear terms of semilinear equations also to depend on gradient. Thus, instead of problem (17) we can consider more generally the problem

$$\begin{cases} \Delta^p u = h + f\left(x, u, \Delta u, \ ..., \ \Delta^{p-1} u, \ \nabla u, \ \nabla \Delta u, \ ..., \ \nabla \Delta^{p-1} u\right) \ \text{ in } \Omega \\ u = \Delta u = \ ... \ = \Delta^{p-1} u = 0 \ \text{ on } \partial\Omega. \end{cases} \tag{20}$$

Then looking to extend to this problem the results in Corollaries 1 and 2, the expression of constant θ in (11) should be completed by terms involving odd powers of $1/c$ $(1/\sqrt{\lambda_1})$. For example, if $f : \Omega \times \mathbb{R}^p \times \mathbb{R}^{np} \to \mathbb{R}$ is such that

$$|f(x, u, \mathbf{u}) - f(x, v, \mathbf{v})| \leq \sum_{i=1}^{p} (a_i |u_i - v_i| + b_i |\mathbf{u}_i - \mathbf{v}_i|)$$

for all $u, v \in \mathbb{R}^p$ and $\mathbf{u}, \mathbf{v} \in (\mathbb{R}^n)^p$ (where applied to vectors from $\mathbb{R}^n$, notation $|\cdot|$ stands for the Euclidian norm), then trying to follow the estimation made for (12) we arrive to the final sum

$$\frac{1}{c} \sum_{i=1}^{p} \left(a_i \left|\Delta^{i-1}(u - v)\right|_{L^2(\Omega)} + b_i \left|\nabla \Delta^{i-1}(u - v)\right|_{L^2(\Omega; \mathbb{R}^n)} \right).$$

According to (13) we have

$$\left|\Delta^{i-1}w\right|_{L^2(\Omega)} \leq \frac{1}{c^{2(p-i)+1}}\left|w\right|_H, \quad i=1,\ \ldots,\ p,$$

which help in the estimation

$$\left|\nabla\Delta^{i-1}w\right|_{L^2(\Omega;\mathbb{R}^n)} = \left|\Delta^{i-1}w\right|_{H_0^1(\Omega)} = \left|\Delta^i w\right|_{H^{-1}(\Omega)} \leq \frac{1}{c}\left|\Delta^i w\right|_{L^2(\Omega)} \leq \frac{1}{c^{2(p-i)}}\left|w\right|_H.$$

Then the analogue of (14) for the new operator

$$N_0\left(u\right) := \Delta^{-p} f\left(\cdot, u, \Delta u, \ldots, \Delta^{p-1}u, \nabla u, \nabla\Delta u, \ldots, \nabla\Delta^{p-1}u\right),$$

is the estimate

$$\left|N_0\left(u\right) - N_0\left(v\right)\right|_H \leq \sum_{i=1}^{p}\left(\frac{a_i}{c^{2(p-i)+2}} + \frac{b_i}{c^{2(p-i)+1}}\right)\left|u-v\right|_H.$$

Thus the contraction condition guaranteeing the existence and uniqueness of the solution of (20) is now

$$\widetilde{\theta} := \sum_{i=1}^{p}\left(\frac{a_i}{c^{2(p-i)+2}} + \frac{b_i}{c^{2(p-i)+1}}\right) < 1.$$

An analogue result to Corollary 2 can be established under the growth condition on f,

$$\left|f\left(x,u,\mathbf{u}\right)\right| \leq C + \sum_{i=1}^{p}\left(a_i\left|u_i\right| + b_i\left|\mathbf{u}_i\right|\right)$$

and the same condition $\widetilde{\theta} < 1$ on the constants a_i and b_i.

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