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BLOW-UP OF SOLUTIONS AND EXISTENCE OF LOCAL SOLUTIONS FOR VISCOELASTIC PARABOLIC EQUATIONS

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Abstract. In this paper, we study the initial boundary value problem for a class of viscoelastic parabolic equations with logarithmic terms. By using contraction mapping principle, the existence of local solutions is proved. Under the appropriate assumptions of memory function and initial energy, taking concavity analysis method, we obtained the finite time blow-up of the solution. Moreover, the lifespan estimates of solutions are given.

Key words: parabolic equation, local solution, blow-up, initial boundary value problem, logarithmic term.
Mathematics Subject Classification (MSC2020): 35K55, 35K58, 35K61, 35K70.

1. INTRODUCTION

Consider the following initial boundary value problems for a class of viscoelastic parabolic equations

$$u_t - \Delta u - \alpha \Delta u_t + \int_0^t g(t-\tau)\Delta u(\tau)\mathrm{d}\tau = f(u), \quad x \in \Omega,\ t > 0, \tag{1}$$

where $g(\cdot)$ is a nonnegative continuous function, $f(u)$ is a given nonlinear function, Ω is a bounded region of $R^n (n \geq 1)$ with a smooth boundary $\partial\Omega$. Equation (1) can be used to describe some phenomena in population dynamics, phase transition thermodynamics and nuclear reactor dynamics [1–4].

When $\alpha = 0$, $g(\cdot) = 0$, equation (1) is transformed into a semilinear parabolic equation

$$u_t - \Delta u = f(u), \quad x \in \Omega,\ t > 0. \tag{2}$$

By using potential well method [5], Xu and Chen [6] considered the initial boundary value problem of equation (2), when $f(u) = |u|^{p-2}u$, they proved the existence of globle solution and blow-up properties of the solution. Chen and Luo [9, 17] taking logarithmic sobolev inequality discussed the existence and nonexistence of the solution of the equation (2), and obtained the attenuation estimation of the solution, where $f(u) = u\log u$. In the case of $f(u) = |u|^{p-2}u\log|u|$, Peng and Zhou [19] use the energy method and the properties of logarithm studyed the global existence and the finite time blow up of the solution of equation (2), and give the upper bound of the blow up time.

When $\alpha = 1$ and $g(\cdot) = 0$, equation (1) is reduced to a semilinear pseudo parabolic equation

$$u_t - \Delta u - \Delta u_t = f(u), \quad x \in \Omega,\ t > 0. \tag{3}$$

Later, Liu [7] and Xu [10] modified the research results of the equation (3) in [8] and obtained the blow up of the solution and the attenuation estimation of the global solution when the initial value is $J(u_0) \leq d$. Literature [16] improved the research results of [7, 10], established the exponential decay of the solution and energy functional when the global solution exists, and give the specific decay rate. In addition, a new blasting condition is given by using the characteristic function method, and the upper bound of blasting time is discussed.

When $\alpha = 1$, Di and Shang [13, 15] studied the initial boundary value problem of (1) with $f(u) = |u|^{p-1}u$, the existence of the global solution is obtained by Galerkin method and potential well theory. Moreover, the finite time blow-up results under negative initial energy and nonnegative initial energy are acquired by the concavity method, then, the life interval estimation of the solution is given by establishing differential inequalities. By defining an energy functional different from [13], Sun and Liu [14] proved the global existence and finite time blow-up for the problem (1) at low energy level by using Galerkin method, concave analysis method and the improved potential well method, and gave the upper bound of the blow up time. When $\alpha = 0$, under appropriate assumptions about $g(\cdot)$ and p, Messaoudi [11, 12] discussed the initial boundary value problem of equation (1), the blow up results under positive initial energy and negative initial energy are obtained.

Inspired by the above research, this paper considers the initial boundary value problem of a class of viscoelastic quasi parabolic equations with logarithmic terms

$$\begin{cases} u_t - \Delta u - \Delta u_t + \int_0^t g(t-\tau)\Delta u(\tau)\mathrm{d}\tau = |u|^{p-2}u\ln|u|, \quad x\in\Omega,\ t>0, \\ u(x,0) = u_0, \quad x\in\Omega, \\ u(x,t)|_{\partial\Omega} = 0, \quad x\in\partial\Omega,\ t>0, \end{cases} \tag{4}$$

where $u_0 \in H_0^1(\Omega)$, $p > 2$. According to the literature, there are many studies on the global solution, explosion solution and solution attenuation of this kind of viscoelastic quasi parabolic equation, but there are few studies on its local solution. In this paper, firstly, the existence of local solution of problem (4) is studied by using the principle of contractive mapping. Secondly, under the assumption of appropriate $g(\cdot)$ and initial energy, the blow up property of the solution of the problem is proved, and the life interval estimation of the solution is given.

The structure of the article is as follows: the second part gives some preliminary knowledge. In the third part, we prove the existence of local solutions of problem (4). The fourth part studies the blow up properties in finite time and gives the life interval estimation of the solution.

2. PRELIMINARIES

Firstly, assume that p and $g(\cdot)$ meet the following conditions:

(A1) $2 \le p < \infty$ if $n = 1,2$; $2 < p \le \dfrac{2n}{n-2}$ if $n \ge 3$.

(A2) $g \in C^1(R^+, R^+)$ satisfying $g(\tau) \ge 0$, $g'(\tau) \le 0$, $\beta = 1 - \int_0^\infty g(\tau)\mathrm{d}\tau > 0$.

Secondly, the following functional is defined:

$$J(u) = \frac{1}{2}(1 - \int_0^t g(\tau)\mathrm{d}\tau)\|\nabla u\|_2^2 - \frac{1}{p}\int_\Omega |u|^p \ln|u|\mathrm{d}x + \frac{1}{p^2}\int_\Omega |u|^p\mathrm{d}x, \tag{5}$$

$$I(u) = (1 - \int_0^t g(\tau)\mathrm{d}\tau)\|\nabla u\|_2^2 - \int_\Omega |u|^p \ln|u|\mathrm{d}x, \tag{6}$$

$$E(t) = \frac{1}{2}(g\circ\nabla u)(t) + \int_0^t \|u_t\|_{H^1}^2\mathrm{d}\tau + \frac{1}{2}(1 - \int_0^t g(\tau)\mathrm{d}\tau)\|\nabla u\|_2^2 - \frac{1}{p}\int_\Omega |u|^p\ln|u|\mathrm{d}x + \frac{1}{p^2}\int_\Omega |u|^p\mathrm{d}x, \tag{7}$$

where $(g\circ\nabla u)(t) = \int_0^t g(t-\tau)\parallel \nabla u(\tau) - \nabla u(t)\parallel_2^2\mathrm{d}\tau$, $\|\cdot\|_{H^1} = \sqrt{\|\cdot\|_2^2 + \|\nabla\cdot\|_2^2}$.

From (5) and (6), we obtained

$$J(u) = \frac{1}{p}I(u) + \frac{(p-2)}{2p}(1 - \int_0^t g(\tau)\mathrm{d}\tau)\|\nabla u\|_2^2 + \frac{1}{p^2}\int_\Omega |u|^p\mathrm{d}x. \tag{8}$$

Let

$$N = \left\{u \in H_0^1(\Omega) : I(u) = 0,\ u \neq 0\right\}, \tag{9}$$

then

$$d = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \max_{\lambda \in R^+} J(\lambda u) = \inf_{u \in N} J(u). \tag{10}$$

Finally, define the set

$$V = \left\{u \in H_0^1(\Omega) \mid I(u) < 0,\ J(u) < d\right\}.$$

Definition 2.1. If $u \in L^\infty\left([0,T);H_0^1(\Omega)\right)$ with $u_t \in L^2\left([0,T);H_0^1(\Omega)\right)$ satisfying the following conditions
(i) For any $v \in L^\infty\left([0,T);H_0^1(\Omega)\right)$, such that

$$(u_t, v) + (\nabla u, \nabla v) + (\nabla u_t, \nabla v) - \left(\int_0^t g(t-\tau)\Delta u(\tau)\mathrm{d}\tau, \nabla v\right) = \left(|u|^{p-2}u\ln|u|, v\right).$$

ii) $u(x,0) = u_0(x)$ in $H_0^1(\Omega)$.
We say that u is a weak solution of the problem (4) on the interval $\Omega \times [0,T)$.

LEMMA 2.1. *Suppose that (A1) holds, then there is a normal number s dependent on Ω, n and p, so that*

$$||u||_p \leq s||\nabla u||_2.$$

LEMMA 2.2. *Assume that (A1) and (A2) hold, $u \in H_0^1(\Omega)\setminus\{0\}$. Then, for any $u \in H_0^1(\Omega)$ and $t \in [0,T)$, where T is the maximum existence time of the solution of the problem (4), we have*
(i) $\lim_{\lambda\to 0} J(\lambda u) = 0,\ \lim_{\lambda\to\infty} J(\lambda u) = -\infty$.
(ii) *In the interval* $[0,\infty]$*, there exists a unique* $\lambda^* = \lambda^*(u)$ *such that* $\frac{d}{d\lambda}J(\lambda u)\big|_{\lambda=\lambda^*} = 0$*, and* $J(\lambda u)$ *increases on interval* $[0,\lambda^*]$*, decreases on interval* $[\lambda^*, +\infty]$.
(iii) $I(\lambda u) > 0$ *for* $0 < \lambda < \lambda^*$, $I(\lambda u) < 0$ *for* $\lambda^* < \lambda < +\infty$ *and* $I(\lambda^* u) = 0$.

LEMMA 2.3. *Assume that (A1) and (A2) hold, then the constant d defined in* (10) *satisfies* $d > 0$.

LEMMA 2.4. *Suppose that (A1) and (A2) hold, let $u(x,t)$ be a solution of (4), then, $E(t)$ is non-increasing function, that is*

$$E'(t) \leq 0.$$

Proof. Multiply the equation of problem (4) by u_t and integral on Ω, which can be obtained by partial integral

$$\|u_t\|_2^2 + \|\nabla u_t\|_2^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u(t)\|_2^2 - \int_0^t g(t-\tau)\int_\Omega \nabla u(\tau)\nabla u_t(t)\mathrm{d}x\mathrm{d}\tau = \int_\Omega |u|^{p-1}\ln|u|u_t\mathrm{d}x. \tag{11}$$

Through calculation, it can be seen that

$$\begin{aligned}
&\int_0^t g(t-\tau)\int_\Omega \nabla u(\tau)\nabla u_t(t)\mathrm{d}x\mathrm{d}\tau \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{1}{2}(g\circ\nabla u)(t) + \frac{1}{2}\int_0^t g(\tau)\mathrm{d}\tau\|\nabla u(t)\|_2^2\right) + \left(\frac{1}{2}\left(g'\circ\nabla u\right)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|_2^2\right),
\end{aligned} \tag{12}$$

$$\int_\Omega |u|^{p-1}\ln|u|u_t\mathrm{d}x = \frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u|^p\ln|u|\mathrm{d}x - \frac{1}{p^2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u|^p\mathrm{d}x. \tag{13}$$

Inserting (12) and (13) into (11), we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\left(1 - \int_0^t g(\tau)\mathrm{d}\tau\right)\|\nabla u(t)\|_2^2 + \frac{1}{2}(g\circ\nabla u)(t) - \frac{1}{p}\int_\Omega |u|^p\ln|u|\mathrm{d}x + \frac{1}{p^2}\int_\Omega |u|^p\mathrm{d}x + \int_0^t \|u_t\|_{H^1}^2\mathrm{d}\tau\right) \\
&= \frac{1}{2}\left(g'\circ\nabla u\right)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|_2^2 \leq 0.
\end{aligned}$$

The proof of the Lemma 2.4 is completed.

3. EXISTENCE OF LOCAL SOLUTIONS

The existence of local solutions of problem (4) is discussed below. For a given function v, consider linear problems

$$\begin{cases} u_t-\Delta u-\Delta u_t+\int_0^t g(t-\tau)\Delta u(\tau)\mathrm{d}\tau=|v|^{p-2}v\ln|v|, & (x,t)\in\Omega\times(0,T),\\ u(x,0)=u_0, \quad x\in\Omega,\\ u(x,t)=0, \quad (x,t)\in\partial\Omega\times(0,T),\end{cases} \tag{14}$$

where $T>0$.

LEMMA 3.1. *Assume that (A1) and (A2) hold, if $u_0\in H_0^1(\Omega)$, then, the problem* (14) *admits a global weak solution u satisfying $u\in L^\infty\left([0,T];H_0^1(\Omega)\right)$ with $u_t\in L^2\left([0,T];H_0^1(\Omega)\right)$.*

The detailed proof of this lemma see references [20–22].

THEOREM 3.1. *Suppose that* (A1) *and* (A2) *hold, if $u_0\in H_0^1(\Omega)$, then there is $T>0$, so that the problem* (4) *has a unique local solution $u(t)$ satisfying $u\in L^\infty([0,T];H_0^1(\Omega))$ with $u_t\in L^2([0,T];H_0^1(\Omega))$.*

Proof. For $T>0$, define a class of function $X_{R_0,T}$, this kind of function includes all the functions in Z that meet the initial condition of problem (4), i.e.

$$X_{R_0,T}=\{u\in Z: ||u(t)||_z\le R_0^2,\ t\in[0,T]\},$$

where $Z=\left\{u:u\in L^\infty([0,T],H_0^1(\Omega)),\ u_t\in L^2([0,T],L^2(\Omega))\right\}$ endowed with the norm $||u(t)||_z=\sup_{0\le t\le T}\left(1-\right.$ $\left.-\int_0^t g(\tau)\mathrm{d}\tau\right)||\nabla u||_2^2$, then $X_{R_0,t}$ is a complete metric space with the distance $d(u_1,u_2)=||u_1-u_2||_z$.

By Lemma 3.1, we define a nonlinear mapping $\Psi: v\to u, u=\Psi v$ in the following way, for any $v\in X_{R_0,T}$, $u=\Psi v$ is the unique solution of problem (14). Then we claim that Ψ is a contraction mapping from $X_{R_0,T}$ into itself for $T>0$ and $R_0>0$.

Let $v\in X_{R_0,T}$, for $t\in[0,T]$, multiply the equation in problem (14) by u_t and integrate on $[0,t]$,we obtained

$$\begin{aligned} &2\int_0^t\|u_t\|_{H^1}^2\,\mathrm{d}\tau+||\nabla u||_2^2+(g\circ\nabla u)(t)-\int_0^t g(\tau)\mathrm{d}\tau||\nabla u||_2^2\\ &=||\nabla u_0||_2^2+\int_0^t((g'\circ\nabla u)(t)-g(t)||\nabla u||_2^2)\mathrm{d}\tau+2\int_0^t\int_\Omega|v|^{p-1}\ln|v|u_t\mathrm{d}x\mathrm{d}\tau\\ &\le||\nabla u_0||_2^2+2\int_0^t\int_\Omega|v|^{p-1}\ln|v|u_t\mathrm{d}x\mathrm{d}\tau\\ &\le||\nabla u_0||_2^2+\int_0^t|||v|^{p-1}\ln|v|||_2^2\mathrm{d}\tau+\int_0^t\|u_t\|_{H^1}^2\,\mathrm{d}\tau,\end{aligned} \tag{15}$$

then

$$(1-\int_0^t g(\tau)\mathrm{d}\tau)||\nabla u(t)||_2^2\le||\nabla u_0||_2^2+\int_0^t|||v|^{p-1}\ln|v|||_2^2\mathrm{d}\tau. \tag{16}$$

By Hölder inequality, Lemma 2.1 and $\ln\alpha<\alpha$ for $\alpha>1$, we get

$$\begin{aligned}|||v|^{p-1}\ln|v|||_2^2=\int_\Omega\left(|v|^{p-1}\ln|v|\right)^2\mathrm{d}x&=\int_{\{x\in\Omega,v(x)\le1\}}\left(|v|^{p-1}\ln|v|\right)^2\mathrm{d}x+\int_{\{x\in\Omega,v(x)>1\}}\left(|v|^{p-1}\ln|v|\right)^2\mathrm{d}x\\ &\le\frac{1}{[e(p-1)]^2}|\Omega|+\int_\Omega|v|^{2p}\,\mathrm{d}x\\ &\le\frac{1}{[e(p-1)]^2}|\Omega|+s^{2p}(1-\int_0^t g(\tau)\mathrm{d}\tau)^{-p}R_0^{2p}.\end{aligned} \tag{17}$$

Substitute (17) into (16), then we have

$$(1-\int_0^t g(\tau)\mathrm{d}\tau)||\nabla u(t)||_2^2 \le ||\nabla u_0||_2^2 + (\frac{1}{[e(p-1)]^2}|\Omega| + s^{2p}(1-\int_0^t g(\tau)\mathrm{d}\tau)^{-p}R_0^{2p})T.$$

Let $R_0 > ||\nabla u_0||_2$, $T < (R_0^2 - ||\nabla u_0||_2^2)/(\frac{1}{[e(p-1)]^2}|\Omega| + s^{2p}(1-\int_0^t g(\tau)\mathrm{d}\tau)^{-p}R_0^{2p})$, the above formula is less than or equal to R_0^2, i.e. $u \in X_{R_0,T}$, then Ψ is a self mapping.

The following will prove that Ψ is a compressed mapping, let $v_1, v_2 \in X_{R_0,T}$ and $u_1 = \Psi v_1$, $u_2 = \Psi v_2$ be the corresponding solution for problem (14). Setting $U = u_1 - u_2$, $V = v_1 - v_2$, then U satisfies the following system

$$\begin{cases} U_t - \Delta U - \Delta U_t + \int_0^t g(t-\tau)\Delta U(\tau)\mathrm{d}\tau = |v_1|^{p-2}v_1 \ln|v_1 - |v_2|^{p-2}v_2 \ln|v_2|, \ (x,t) \in \Omega \times (0,T), \\ U(x,0) = 0, \ x \in \Omega. \end{cases} \tag{18}$$

Multiply the equation in (18) by U_t and integrate it over $[0,t]$ to get

$$\begin{aligned} 2\int_0^t \|U_t\|_{H^1}^2 \,\mathrm{d}\tau + (1-\int_0^t g(\tau)\mathrm{d}\tau)||\nabla U(t)||_2^2 + (g \circ \nabla U)(t) - \int_0^t ((g' \circ \nabla U)(t) - g(t)||\nabla U(\tau)||_2^2)\mathrm{d}\tau \\ = 2\int_0^t \int_\Omega (|v_1|^{p-1}\ln|v_1| - |v_2|^{p-1}\ln|v_2|)U_t(\tau)\mathrm{d}x\mathrm{d}\tau. \end{aligned} \tag{19}$$

By Lagrange mean value theorem, we have

$$\begin{aligned} |v_1|^{p-1}\ln|v_1| - |v_2|^{p-1}\ln|v_2| = ((p-1)\ln|\varepsilon|+1)|\varepsilon|^{p-2}(v_1 - v_2) = V|\varepsilon|^{p-2}(1+(p-1)\ln|\varepsilon|) \\ \le V(|v_1|+|v_2|)^{p-2} + V(p-1)(|v_1|+|v_2|)^{p-1}, \end{aligned} \tag{20}$$

where $|\varepsilon| = v_1 + \xi(v_2 - v_1) < v_1 + v_2$, for $0 < \xi < 1$, then applying Hölder inequality, Young inequality and Lemma 2.1, we get

$$\begin{aligned} &\int_0^t \int_\Omega (|v_1|^{p-1}\ln|v_1| - |v_2|^{p-1}\ln|v_2|)U_t(\tau)\mathrm{d}x\mathrm{d}\tau \\ &\le \int_0^t \int_\Omega V(|v_1|+|v_2|)^{p-2}U_t(\tau)\mathrm{d}x\mathrm{d}\tau + \int_0^t \int_\Omega V(p-1)(|v_1|+|v_2|)^{p-1}U_t(\tau)\mathrm{d}x\mathrm{d}\tau \\ &\le \int_0^t ||V||_p|||v_1|+|v_2|||_{2p}^{p-2}||U_t||_2\mathrm{d}\tau + \int_0^t (p-1)||V||_{2p}|||v_1|+|v_2|||_{2p}^{p-1}||U_t||_2\mathrm{d}\tau \\ &\le s^{p-1}\int_0^t ||\nabla V||_2||U_t||_2(||\nabla v_1||_2 + ||\nabla v_2||_2)^{p-2}\mathrm{d}\tau + s^p\int_0^t (p-1)||\nabla V||_2||U_t||_2(||\nabla v_1||_2 + ||\nabla v_2||_2)^{p-1}\mathrm{d}\tau \\ &\le \frac{1}{2}[s^{p-1}(2R_0(1-\int_0^t g(\tau)\mathrm{d}\tau)^{-\frac{1}{2}})^{p-2} + s^p(2R_0(1-\int_0^t g(\tau)\mathrm{d}\tau)^{-\frac{1}{2}})^{p-1}]^2\int_0^t ||\nabla V||_2^2\mathrm{d}\tau + \frac{1}{2}\int_0^t ||u_t||_2^2\mathrm{d}\tau. \end{aligned} \tag{21}$$

From (19) – (21), we obtained

$$2\int_0^t ||U_t||_{H^1}^2\mathrm{d}\tau + (1-\int_0^t g(\tau)\mathrm{d}\tau)||\nabla U(t)||_2^2 \le C\int_0^t ||\nabla V||_2^2\mathrm{d}\tau + \int_0^t ||u_t||_2^2\mathrm{d}\tau, \tag{22}$$

where $C := \left[s^{p-1}(2R_0(1-\int_0^t g(\tau)\mathrm{d}\tau)^{-\frac{1}{2}})^{p-2} + s^p(2R_0(1-\int_0^t g(\tau)\mathrm{d}\tau)^{-\frac{1}{2}})^{p-1}\right]^2$, thus

$$\left(1-\int_0^t g(\tau)\mathrm{d}\tau\right)||\nabla U(t)||_2^2 \le C\int_0^t ||\nabla V||_2^2\mathrm{d}\tau \le \left(1-\int_0^t g(\tau)\mathrm{d}\tau\right)^{-1} CT||V||_z, \tag{23}$$

namely

$$||U||_z \le (1-\int_0^t g(\tau)\mathrm{d}\tau)^{-1}CT||V||_z.$$

Let $T < (1-\int_0^t g(\tau)\mathrm{d}\tau)C^{-1}$, then Ψ is a compressed mapping. In summary, when we choose $R_0 > ||\nabla u_0||_2$ and $T < \min\left[(R_0^2 - ||\nabla u_0||_2^2)/(\frac{1}{[e(p-1)]^2}|\Omega| + s^{2p}(1-\int_0^t g(\tau)\mathrm{d}\tau)^{-p}R_0^{2p}), (1-\int_0^t g(\tau)\mathrm{d}\tau)C^{-1}\right]$, Ψ is a compressed self mapping. By Banach fixed point theorem, Theorem 3.1 proofed.

4. FINITE TIME BLOW-UP OF SOLUTIONS

LEMMA 4.1 ([18]). *Let $Q(t) \in C^2(R^+, R^+)$ and $\delta > 0$ satisfying*

$$Q''(t) - 4(\delta+1)Q'(t) + 4(\delta+1)Q(t) \ge 0,\ t \ge 0. \tag{24}$$

If $Q'(0) > r_2 b(0)$, then $Q'(t) > 0$ for $t > 0$, where $r_2 = 2(\delta+1) - 2[(\delta+1)\delta]^{\frac{1}{2}}$ is the smallest root of the quadratic equation $r^2 - 4(\delta+1)r + 4(\delta+1) = 0$.

LEMMA 4.2. ([18]) *If $J(t)$ is a non-increasing function on $[t_0, \infty)$ and satisfies the following differential equation*

$$J'(t)^2 \ge a + bJ(t)^{2+\frac{1}{\delta}},\ t \ge t_0, \tag{25}$$

where $a > 0$ and $b \in R$, then there exists a finite positive number T^* such that $\lim_{t\to T^*} J(t) = 0$ and an upper bound of T^* can be estimated respectively in the following case:

(i) *when $b < 0$ and $J(t_0) < \min(1, (\frac{a}{-b})^{\frac{1}{2}})$, $T^* \le t_0 + \sqrt{\frac{1}{-b}} \ln \frac{\sqrt{\frac{a}{-b}}}{\sqrt{\frac{a}{-b}} - J(t_0)}$.*

(ii) *when $b = 0$, $T^* \le t_0 + \frac{J(t_0)}{\sqrt{a}}$.*

(iii) *when $b > 0$, $T^* \le t_0 + 2^{\frac{3\delta+1}{2\delta}} \frac{\delta h}{\sqrt{a}}\{1 - [1 + hJ(t_0)]^{-\frac{1}{2\delta}}\}$, where $h = (\frac{a}{b})^{2+\frac{1}{\delta}}$.*

LEMMA 4.3. *Suppose that $(A1)$ and $(A2)$ hold, and if $u_0 \in V$ and $E(0) < d$. Then we have $u(t) \in V$ for $t \in [0, T]$, and*

$$(p-2)(1-\int_0^t g(\tau)d\tau)||\nabla u||_2^2 + \frac{2}{p}\int_\Omega |u|^p dx \ge 2pd.$$

Proof. Suppose that there is a time $t_0 \in (0, T)$ such that $u(x,t) \in V$ for any $t \in [0, t_0)$, but $u(x, t_0) \notin V$, from the definition of V and the continuity of $J(u)$and $I(u)$, we have either

(i) $J(u(x,t_0)) = d$ or (ii) $I(u(x,t_0)) = 0$.

By the Lemma 2.4, we have $E(t_0) \le E(0)$, then

$$J(u(x,t_0)) \le E(t_0) \le E(0) < d.$$

So case (i) is impossible.

Assume that (ii) holds, it can be obtained from (10) and Lemma 2.3 that

$$d \le J(u(x,t_0)) \le E(u(t_0)) < E(u(0)) < d.$$

So case (ii) is also impossible.

For $u(t) \in V$, it is known from Lemma 2.2 that there is $\lambda^* \in (0,1)$, so that $I(\lambda^* u) = 0$, it can be obtained from the definition of d

$$d \le J(\lambda^* u) = \frac{1}{p}I(\lambda^* u) + \frac{p-2}{2p}(1-\int_0^t g(\tau)\mathrm{d}\tau)||\lambda^*\nabla u||_2^2 + \frac{1}{p^2}\int_\Omega |\lambda^* u|^p \mathrm{d}x.$$

Then $(p-2)(1-\int_0^t g(\tau)\mathrm{d}\tau)||\nabla u||_2^2+\frac{2}{p}\int_\Omega |u|^p\mathrm{d}x \geq 2pd.$

Let

$$F(t)=\int_0^t \|u(\tau)\|_{H^1}^2\mathrm{d}\tau+(T-t)\|u_0\|_{H^1}^2, \tag{26}$$

then

$$F'(t)=\|u(t)\|_{H^1}^2-\|u_0\|_{H^1}^2, \tag{27}$$

$$\begin{aligned} F''(t)&=2(u,u_t)+2(\nabla u,\nabla u_t)=-2(\nabla u,\nabla u)+2(\int_0^t g(t-\tau)\nabla u(\tau)\mathrm{d}\tau,\nabla u)+2(|u|^{p-2}u\ln|u|,u)\\ &=-2\|\nabla u\|_2^2+2\int_0^t g(t-\tau)(\nabla u(\tau),\nabla u)\mathrm{d}\tau+2\int_\Omega |u|^p\ln|u|\mathrm{d}x. \end{aligned} \tag{28}$$

LEMMA 4.4. *Assume that* $\int_0^\infty g(\tau)d\tau \leq \frac{p-3}{p-2}$, *then*

$$F''(t)-2p\int_0^t \|u_t\|_{H^1}^2\,d\tau \geq -2pE(0)+\alpha\left[(p-2)\left(1-\int_0^t g(\tau)d\tau\right)||\nabla u||_2^2+\frac{2}{p}\int_\Omega |u|^p dx\right], \tag{29}$$

where $\alpha=1-\frac{1}{(p-2)\beta}$.

Proof. By applying Young inequality and Lemma 2.4, from (28), we obtained

$$\begin{aligned} &F''(t)-2p\int_0^t \|u_t\|_{H^1}^2\,\mathrm{d}\tau\\ &=-2\|\nabla u\|_2^2+2\int_0^t g(t-\tau)(\nabla u(\tau),\nabla u)\mathrm{d}\tau+2\int_\Omega |u|^p\ln|u|\mathrm{d}x+-2p\int_0^t \|u_t\|_{H^1}^2\,\mathrm{d}\tau\\ &\geq -2p\int_0^t \|u_t\|_{H^1}^2\,\mathrm{d}\tau-2\left(1-\int_0^t g(t-\tau)\mathrm{d}\tau\right)\|\nabla u\|_2^2+2\int_\Omega |u|^p\ln|u|\mathrm{d}x-2\left(\frac{p}{2}(g\circ\nabla u)(t)+\frac{1}{2p}\int_0^t g(t-\tau)\mathrm{d}\tau\|\nabla u\|_2^2\right)\\ &\geq -2pE(0)+(p-2)\left(1-\int_0^t g(t-\tau)\mathrm{d}\tau\right)\|\nabla u\|_2^2-\frac{1}{p}\int_0^t g(t-\tau)\mathrm{d}\tau)\|\nabla u\|_2^2+\frac{2}{p}\int_\Omega |u|^p\mathrm{d}x\\ &\geq -2pE(0)+\left(1-\frac{1}{(p-2)\beta}\right)\left[(p-2)(1-\int_0^t g(\tau)\mathrm{d}\tau)||\nabla u||_2^2+\frac{2}{p}\int_\Omega |u|^p\mathrm{d}x\right]. \end{aligned} \tag{30}$$

The proof of Lemma 4.4 is completed.

LEMMA 4.5. *Suppose that* $(A1)$ *and* $(A2)$ *holds, u is the solution of problem* (4), *if one of the following conditions is true:*

(1) $E(0)<0$, (2) $E(0)=0$ *and* $F'(0)\geq 0$, (3) $0<E(0)<\alpha d$ *and* $I(u_0)<0$, (4) $E(0)\geq \alpha d$.
Then $F'(t)>0$ *for* $t>t_3$, *where:*

- *in case* (1), $t_3=max\{\frac{F'(0)}{2pE(0)},0\}$.
- *in case* (2), $t_3=0$.
- *in case* (3), $t_3=\max\{\frac{-F'(0)}{2p(\alpha d-E(0))},0\}$.

Proof. (1) If $E(0)<0$, then from (29), we have

$$F'(t)\geq F'(0)-2pE(0)t,\quad t\geq 0.$$

Thus, $F'(t) > 0$ for $t > t_3$, where $t_3 = \max\{\frac{F'(0)}{2pE(0)}, 0\}$.

(2) If $E(0) = 0$, then from (29), we get $F''(t) > 0$ for $t > 0$, since $F'(0) \geq 0$, we have $F'(t) > 0$ for $t > 0$.

(3) If $0 < E(0) < \alpha d$ and $I(u_0) < 0$, then from Lemma 4.3, we get

$$F''(t) \geq 2p(\alpha d - E(0)) > 0.$$

Integrating both sides of the above equation from 0 to t gives

$$F'(t) \geq F'(0) + 2p(\alpha d - E(0))t, \quad t \geq 0.$$

Therefore, we get $F'(t) > 0$ for $t > t_3$, where $t_3 = \max\{\frac{-F'(0)}{2p(\alpha d - E(0))}, 0\}$.

(4) If $E(0) \geq \alpha d$, then

$$F''(t) - 2p\int_0^t \|u_t\|_{H^1}^2 \,\mathrm{d}\tau + 2pE(0) \geq 0.$$

By using Hölder inequality and Young inequality, we obtained

$$2\int_0^t (u,u_t)_{H^1}\mathrm{d}\tau \leq 2\int_0^t ||u||_{H^1}||u_t||_{H^1}\mathrm{d}\tau \leq \int_0^t ||u||_{H^1}^2\mathrm{d}\tau + \int_0^t ||u_t||_{H^1}^2\mathrm{d}\tau. \tag{31}$$

$$2\int_0^t (u,u_t)_{H^1}\mathrm{d}\tau = ||u_t||_{H^1} - ||u_0||_{H^1}.$$

Thus

$$\int_0^t ||u_t||_{H^1}^2\mathrm{d}\tau \geq ||u_t||_{H^1} - ||u_0||_{H^1} - \int_0^t ||u||_{H^1}^2\mathrm{d}\tau \geq F'(t) - F(t). \tag{32}$$

Then

$$F''(t) - 2pF'(t) + 2pF(t) + 2pE(0) \geq 0. \tag{33}$$

By applying Lemma 4.1 with $\delta = \frac{p-2}{2}$ and $Q(t) = F(t) + E(0)$, then we have $F'(t) > 0$ for $t > 0$.

THEOREM 4.1. *Assume that (A1) and (A2) hold and u is a solution of (4). If one of the following statements is satisfied:*

(1) $E(0) < 0$, (2) $E(0) = 0$ *and* $F'(0) \geq 0$, (3) $0 < E(0) < \alpha d$ *and* $I(u_0) < 0$, (4) $E(0) \geq \alpha d$,
then the solution $u(t)$ blow-up at a finite time T^* in the sense of

$$\lim_{t \to T^{*-}} \|u(x,t)\|_{H^1}^2 = +\infty.$$

- *In case* (1), $T^* \leq t_* - \frac{A(t_*)}{A'(t_*)}$ *and if* $A(t_*) < \min(1, \sqrt{\frac{a}{-b}})$, *then* $T^* \leq t_* + \sqrt{\frac{1}{-b}} \ln \frac{\sqrt{\frac{a}{-b}}}{\sqrt{\frac{a}{-b}} - A(t_*)}$.
- *In case* (2), $T^* \leq t_* + \frac{A(t_*)}{\sqrt{a}}$.
- *In case* (3), $T^* \leq t_* - \frac{A(t_*)}{A'(t_*)}$ *and if* $A(t_*) < \min(1, \sqrt{\frac{a_1}{-b_1}})$, *then* $T^* \leq t_* + \frac{1}{\sqrt{-b_1}} \ln \frac{\sqrt{\frac{a_1}{-b_1}}}{\sqrt{\frac{a_1}{-b_1}} - A(t_*)}$.
- *In case* (4), $T^* \leq t_* + 2^{\frac{3p-4}{2(p-2)}} \frac{(p-2)h}{2\sqrt{a}}\{1 - [1 + A(t_*)]^{-\frac{1}{p-2}}\}$, *where* $h = (\frac{a}{b})^{\frac{2p-2}{p-2}}$.

Proof. Let

$$A(t) = F(t)^{-\frac{p-2}{2}}, \quad t \geq 0.$$

Then

$$A'(t) = -\frac{p-2}{2}F(t)^{-\frac{p-2}{2}-1}F'(t),$$

$$A''(t) = -\frac{p-2}{2}A(t)^{1+\frac{4}{p-2}}[F''(t)F(t) - \frac{p}{2}(F'(t))^2]. \tag{34}$$

By using Lemma 4.3 and Hölder inequality, from (26),(27) and (29), we obtained

$$\begin{aligned}
&F''(t)F(t) - \frac{p}{2}(F'(t))^2 \\
&\geq \left\{2p\int_0^t \|u_t\|_{H^1}^2 \mathrm{d}\tau - 2pE(0) + \alpha[(p-2)(1-\int_0^t g(\tau)\mathrm{d}\tau)||\nabla u||_2^2 + \frac{2}{p}\int_\Omega |u|^p \mathrm{d}x]\right\}F(t) - \frac{p}{2}(4F(t)\int_0^t ||u_t||_{H^1}^2) \\
&\geq \left\{-2pE(0) + \alpha[(p-2)(1-\int_0^t g(\tau)\mathrm{d}\tau)||\nabla u||_2^2 + \frac{2}{p}\int_\Omega |u|^p \mathrm{d}x]\right\}F(t) \\
&= \left\{-2pE(0) + \alpha[(p-2)(1-\int_0^t g(\tau)\mathrm{d}\tau)||\nabla u||_2^2 + \frac{2}{p}\int_\Omega |u|^p \mathrm{d}x]\right\}A(t)^{-\frac{2}{p-2}} \\
&\geq \Big(2p\alpha d - 2pE(0)\Big)A(t)^{-\frac{2}{p-2}}.
\end{aligned} \tag{35}$$

Substituting (35) into (34) yields

$$A''(t) \leq p(p-2)(E(0) - \alpha d)A(t)^{1+\frac{2}{p-2}}. \tag{36}$$

If the case(1) or case(2) holds, by (36) we get

$$A''(t) \leq p(p-2)E(0)A(t)^{1+\frac{2}{p-2}}. \tag{37}$$

By Lemma 4.5, multiplying (37) by $A'(t)$ and integrating on $[t_*, t]$, we have

$$A'(t) \geq a + bA(t)^{2+\frac{2}{p-2}}, \quad t \geq t_*, \tag{38}$$

where

$$a = A'(t_*)^2 - \frac{p(p-2)^2}{p-1}E(0)A(t_*)^{2+\frac{2}{p-2}}, \quad b = \frac{p(p-2)^2}{p-1}E(0). \tag{39}$$

If the case(3) holds, then we obtained

$$A''(t) \leq -p(p-2)(\alpha d - E(0))A(t)^{1+\frac{2}{p-2}}. \tag{40}$$

By using the same arguments as in (37), we see that

$$A'(t) \geq a_1 + b_1A(t)^{2+\frac{2}{p-2}}, \quad t \geq t_*, \tag{41}$$

where

$$a_1 = A'(t_*)^2 - \frac{p(p-2)^2}{p-1}(E(0) - \alpha d)A(t_*)^{2+\frac{2}{p-2}}, \quad b_1 = \frac{p(p-2)^2}{p-1}(E(0) - \alpha d). \tag{42}$$

In case(4), we can get (39) if and only if $E(0) < \frac{(p-1)A(t_*)^2}{p(p-2)^2A(t_*)^{2+\frac{2}{p-2}}}$. Therefore, when $\delta = \frac{p-2}{2}$ and $t_0 = t_*$, by Lemma 4.2, there exists a finite time T^* such that $\lim_{t\to T^{*-}} A(t) = 0$, i.e. $\lim_{t\to T^{*-}} \|u(x,t)\|_{H^1}^2 = +\infty$. This finished the proof of Theorem 4.1.

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