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# NOTES ON MELVYN KNIGHT'S PROBLEM

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**Abstract**. Melvyn Knight's problem asks for positive integers $n$ that can be represented as $n=(x+y+z)(\frac{1}{x}+\frac{1}{y}+\frac{1}{z})$ with integers $x,y,z$. In this paper, we investigate integers $n$ that can be represented as

$$n=\frac{x+y+z}{a^2b^2c^2}(\frac{a^2}{x}+\frac{b^2}{y}+\frac{c^2}{z}) \tag{1}$$

with integers $x$, $y$, $z$, $a$, $b$, $c$. For integers $n,a,b,c$ satisfying $4|n$ or $8|n-5$, $a+b+c=-1$, and $abc$ is a square number, we show that the representation (1) is essentially unique if $na^2b^2c^2=(|a|+|b|+|c|)^2$ and is impossible if $na^2b^2c^2\neq(|a|+|b|+|c|)^2$.

***Key words**:* elliptic curves, Melvyn Knight's problem, $p$-adic numbers.
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## 1. INTRODUCTION

In [2], Bremner, Guy and Nowakowski investigated Melvyn Knight's problem which asks for integers $n$ representable as

$$n=(x+y+z)(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}) \tag{2}$$

with integers $x,y,z$. They also briefly discussed the representation (2) in the set of positive integers. Integer solutions to equation (2) depend on the rank of the elliptic curve defined by (2). When asking for positive integer solutions to (2), the situation becomes more subtle. It can happen that equation (2) has integer solutions without having positive integer solutions. The following example is taken from Bremner, Guy and Nowakowski [2].

Let $n=564$. Then equation (2) has an integer solution $(x,y,z)$, where

$$\begin{aligned} x &= 1224420050100028778116351171179952136135134918 67,\\ y &= -346069586842550486564589226218875208971306542446 0,\\ z &= 748071910153025278379458360171464649482059055280 60, \end{aligned}$$

but does not have positive integer solutions. The later follows from Tho [5, Theorem 1.1] which says that the representation (2) is impossible in positive integers if $4|n$. In this paper, we will investigate a more general form of (2). We will prove the following theorem.

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THEOREM 1. *Let $n, a, b, c$ be nonzero integers such that $4|n$ or $8|n-5$, $a+b+c=-1$, and $abc$ is a square. Consider the representation*

$$n = \frac{x+y+z}{a^2b^2c^2}\left(\frac{a^2}{x}+\frac{b^2}{y}+\frac{c^2}{z}\right) \tag{3}$$

*with positive integers $x, y, z$.*

(i) *If $na^2b^2c^2 = (|a|+|b|+|c|)^2$, then the representation* (3) *is essentially unique with*

$$x : y : z = |a| : |b| : |c|.$$

(ii) *If $na^2b^2c^2 \neq (|a|+|b|+|c|)^2$, then the representation* (3) *is impossible.*

When $4|n$ and $|a| = |b| = |c| = 1$, we recover Theorem 1.1 in Tho [5]. Note that there are infinitely many integers $a, b, c$ satisfying the condition in Theorem 1. For example, $(a,b,c) = (m, -m, -1)$ with $m \in \mathbb{Z}^+$.

## 2. SOME PRELIMINARIES

Let $p$ be a prime number. Let $\mathbb{Q}_p$ be the $p$-adic completion of $\mathbb{Q}$ with respect to $p$-adic topology. Let $\mathbb{Z}_p$ be the ring of $p$-adic integers in $\mathbb{Q}_p$. Let $\mathbb{Q}_p^3 = \{(x,y,z) \colon x,y,z \in \mathbb{Q}_p\}$, $\mathbb{Q}_p^2 = \{x^2 \colon x \in \mathbb{Q}_p\}$, and $\mathbb{Z}_p^2 = \{x^2 \colon x \in \mathbb{Z}_p\}$. For $a \in \mathbb{Q}_p^*$, denote $v_p(a)$ the highest power of $p$ dividing $a$. For $a$ and $b$ in $\mathbb{Q}_p$, the Hilbert symbol $(a,b)_p$ is defined by

$$(a,b)_p = \begin{cases} 1 \text{ if } ax^2+by^2=z^2 \text{ has a solution } (x,\, y,\, z) \neq (0,\, 0,\, 0) \text{ in } \mathbb{Q}_p^3, \\ -1 \text{ otherwise.} \end{cases}$$

For $a, b \in \mathbb{R}$, the symbol $(a,b)_\infty$ is $+1$ if $a > 0$ or $b > 0$, and $-1$ otherwise. The following properties of the Hilbert symbol are true, see Serre [9, pp. 19-26]:

(i) For all $a,\, b,\, c \in \mathbb{Q}_p$, then

$$(a,\, b^2)_p = 1,$$
$$(a, bc)_p = (a,b)_p(a,c)_p.$$

(ii) For all $a,\, b \in \mathbb{Q}$, then

$$(a,b)_\infty \prod_{p \text{ prime}} (a,b)_p = 1.$$

(iii) Let $a,\, b \in \mathbb{Q}$. Write $a = p^\alpha u$, $b = p^\beta v$, where $\alpha = v_p(a)$ and $\beta = v_p(b)$. Then

$$(a,b)_p = (-1)^{\alpha\beta(p-1)/2} \left(\frac{u}{p}\right)^\beta \left(\frac{v}{p}\right)^\alpha \text{ if } p \neq 2,$$
$$(a,b)_p = (-1)^{(u-1)(v-1)/4+\alpha(v^2-1)/8+\beta(u^2-1)/8} \text{ if } p = 2,$$

where $\left(\frac{u}{p}\right)$ denotes the Legendre symbol.

## 3. THE PROVING IDEA

The following is the main idea of our method which has been applied to many different problems, see [3–8].

Assume we want to show that a rational number $u$ is positive. The trick is to find a rational number $D < 0$ such that $(D,u)_p = 1$ for all prime numbers $p$, where $(D,u)_p$ denotes the Hilbert symbol. Then the product formula for the Hilbert symbol (Serre [9, Theorem 3, p. 23]) forces $(D,u)_\infty = 1$. Since $D<0$, we must have $u>0$. Our experience shows that when $u$ is the $x$-coordinate of a rational point on an elliptic curve

$$y^2 = f(x),$$

where $f$ is a cubic polynomial with rational coefficients, quantity $D$ is usually a factor of the discriminant of $f(x)$.

## 4. THE PROOF OF THEOREM 1

Assume that there exist positive integers $x,y,z$ satisfying (3). Using the Cauchy-Schwarz inequality, we have

$$n = \frac{(x+y+z)}{a^2b^2c^2}\left(\frac{a^2}{x}+\frac{b^2}{y}+\frac{c^2}{z}\right) \geq \frac{(|a|+|b|+|c|)^2}{a^2b^2c^2}. \tag{4}$$

The equality holds if and only if $\frac{x}{|a|} = \frac{y}{|b|} = \frac{z}{|c|}$.

(i) When $na^2b^2c^2 = (|a|+|b|+|c|)^2$, the equality in (4) holds. Therefore, $x:y:z = |a|:|b|:|c|$.

(ii) From (4) we have

$$na^2b^2c^2 \geq (|a|+|b|+|c|)^2 > a^2+b^2+c^2.$$

Therefore,

$$na^2b^2c^2(x+y+z) > (a^2+b^2+c^2)(x+y+z).$$

Hence at least one of the following inequalities must hold:

$$\begin{aligned} na^2b^2c^2x &> a^2(x+y+z), \\ na^2b^2c^2y &> b^2(x+y+z), \\ na^2b^2c^2z &> c^2(x+y+z). \end{aligned}$$

Without loss of generality, we assume $na^2b^2c^2z > c^2(x+y+z)$. Therefore,

$$(na^2b^2-1)z > x+y. \tag{5}$$

Since $x,y,z$ satisfy (3), $(x:y:z)$ is a point on the projective curve

$$\mathscr{C}: (X+Y+Z)(a^2YZ+b^2ZX+c^2XY) - na^2b^2c^2XYZ = 0.$$

Since $\mathscr{C}$ has a rational point $(1:-1:0)$, it is bi-rationally equivalent to the curve

$$\mathscr{E}: V^2 = U(U^2+AU+B),$$

where

$$\begin{aligned} A &= n^2a^4b^4c^4 - 2na^2b^2c^2(a^2+b^2+c^2) + a^4+b^4+c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2, \\ B &= 16na^4b^4c^4, \end{aligned}$$

via the map $\phi : \mathscr{C} \to \mathscr{E}$ defined by

$$\phi(X,Y,Z) = \left( \frac{-4na^2b^2c^2(b^2X+a^2Y)}{(na^2b^2-1)Z-X-Y}, \frac{4na^2b^2c^2(b^2X+a^2Y)(c^2(na^2b^2-1)(X-Y)-(a^2-b^2)(X+Y+2Z))}{(X+Y)((na^2b^2-1)Z-X-Y)} \right). \tag{6}$$

We also have

$$A^2 - 4B = DEFG,$$

where

$$D = (a+b+c)^2 - na^2b^2c^2, \quad E = (a+b-c)^2 - na^2b^2c^2,$$
$$F = (a-b+c)^2 - na^2b^2c^2, \quad G = (-a+b+c)^2 - na^2b^2c^2.$$

All of the above computations could be checked by MAGMA [1].

LEMMA 1. *$D, E, F, G < 0$.*

*Proof.* Since $na^2b^2c^2 \neq (|a|+|b|+|c|)^2$, (4) is a strict inequality. Thus $na^2b^2c^2 > (|a|+|b|+|c|)^2$. Then it easily follows that $D, E, F, G < 0$. □

The above lemma shows that curve $\mathscr{E}$ is non-singular, hence is an elliptic curve.

LEMMA 2. *Let $(u,v) = \phi(x,y,z)$. Then $u < 0$.*

*Proof.* From (6), we have

$$u = \frac{-4na^2b^2c^2(b^2x + a^2y)}{(na^2b^2-1)z - x - y}.$$

Since $x, y, z, a^2, b^2, c^2 > 0$ and $(na^2b^2-1)z - x - y > 0$ (see (5)), we have $u < 0$. □

Note that $u$ and $v$ satisfy

$$v^2 = u(u^2 + Au + B). \tag{7}$$

LEMMA 3. *Let $p$ be an odd prime. Then $(D,u)_p = 1$.*

*Proof.* Let $u = p^r s$, where $r, s \in \mathbb{Z}$, and $p \nmid s$.

**Case 1** : $r < 0$. From (7) we have

$$v^2 = p^{3r}s(s^2 + p^{-r}As + p^{-2r}B). \tag{8}$$

Since $-r > 0$ and $p \nmid s$, from (8) we have $3r = v_p(v^2)$. Hence, $2|r$. Let $v = p^{3r/2}t$, where $p \nmid t$. From (8) we have

$$t^2 = s(s^2 + p^{-r}As + p^{-2r}B).$$

Thus $s^3 \equiv t^2 \pmod{p}$, so $s$ is a square mod $p$. Therefore, $s \in \mathbb{Z}_p^2$. Thus $u = p^r s \in \mathbb{Q}_p^2$. Hence, $(D,u)_p = 1$.

**Case 2** : $r = 0$.

**Case 2.1** : $p \nmid D$. Then $u$ and $D$ are units in $\mathbb{Z}_p$. Hence, $(D,u)_p = 1$.

**Case 2.2** : $p|D$. Since $D|A^2 - 4B$, we have $p|A^2 - 4B$. Therefore,

$$u^2 + Au + B = (u + \frac{A}{2})^2 + \frac{4B - A^2}{4} \equiv (u + \frac{A}{2})^2 \pmod{p}. \tag{9}$$

- $p \nmid u + \frac{1}{2}A$. From (9) we have $u^2 + Au + B \in \mathbb{Z}_p^2 - \{0\}$. Thus,

$$u = \frac{v^2}{u^2 + Au + B} \in \mathbb{Q}_p^2.$$

Therefore, $(D,u)_p = 1$.

- $p|u + \frac{1}{2}A$. Then

$$u \equiv -\frac{A}{2} \pmod{p}. \tag{10}$$

Since $p|D = 1 - na^2b^2c^2$, we have $na^2b^2c^2 \equiv 1 \pmod{p}$. Therefore,

$$\begin{aligned} A &= n^2a^4b^4c^4 - 2na^2b^2c^2(a^2+b^2+c^2) + a^4+b^4+c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2 \\ &= (na^2b^2c^2 - a^2 - b^2 - c^2)^2 - 4(a^2b^2 + b^2c^2 + c^2a^2) \\ &\equiv (1 - a^2 - b^2 - c^2)^2 - 4(a^2b^2+b^2c^2+c^2a^2) \pmod{p} \\ &\equiv ((a+b+c)^2 - a^2 - b^2 - c^2)^2 - 4(a^2b^2+b^2c^2+c^2a^2) \pmod{p} \quad (\text{ since } a+b+c=-1) \\ &\equiv (2(ab+bc+ca))^2 - 4(a^2b^2+b^2c^2+c^2a^2) \pmod{p} \\ &\equiv 8abc(a+b+c) \pmod{p} \\ &\equiv -8abc \pmod{p}. \end{aligned} \tag{11}$$

From (10) and (11) we have $u \equiv 4abc \pmod{p}$. Since $abc$ is a square and $p \nmid u$, we have $u \in \mathbb{Z}_p^2$. Therefore, $(D,u)_p = 1$.

**Case 3** : $r > 0$. From (7) we have

$$v^2 = p^r s(p^{2r}s^2 + p^r As + B). \tag{12}$$

**Case 3.1** : $p|B$. Then $p|nabc$. Therefore,

$$D = 1 - na^2b^2c^2 \equiv 1 \pmod{p}.$$

Hence, $D \in \mathbb{Z}_p^2$. Thus $(D,u)_p = 1$.

**Case 3.2** : $p \nmid B$. From (12) we have $r = v_2(v^2)$. Thus $2|r$.

• $p \nmid D$. Since $s$ and $D$ are units in $\mathbb{Z}_p$, we have $(D,s)_p = 1$. Therefore,

$$(D,u)_p = (D, p^r s)_p = (D,s)_p = 1.$$

• $p|D$. Then $na^2b^2c^2 \equiv 1 \pmod{p}$. Therefore,

$$B = 16na^4b^4c^4 \equiv 16a^2b^2c^2 \pmod{p}.$$

Similar to (11) we have $A \equiv -8abc \pmod{p}$. Therefore,

$$u^2 + Au + B \equiv u^2 - 8abcu + 16a^2b^2c^2 \equiv (u - 4abc)^2 \pmod{p}. \tag{13}$$

Since $p|u$ and $p \nmid B$, we have $p \nmid u^2 + Au + B$. Therefore, from (13) we have $u^2 + Au + B \in \mathbb{Z}_p^2 - \{0\}$. Thus

$$u = \frac{v^2}{u^2 + Au + B} \in \mathbb{Q}_p^2.$$

Therefore, $(D,u)_p = 1$. □

LEMMA 4. *If* $4|n$ *then* $(D,u)_2 = 1$.

*Proof.* If $8|nabc$, then $D \equiv 1 \pmod{8}$. Hence, $D \in \mathbb{Z}_2^2$, so $(D,u)_2 = 1$. We consider the case $8 \nmid nabc$. Then $8 \nmid n$ and $2 \nmid abc$. Let $n = 4k$, where $k \in \mathbb{Z}$, $2 \nmid k$. Let $u = 2^r s$, where $r, s \in \mathbb{Z}$, $2 \nmid s$. From (7) we have

$$v^2 = 2^r s(2^{2r}s^2 + 2^r As + B). \tag{14}$$

**Case 1** : $2|r$. Since $4|D-1$, we have

$$(D,u)_2 = (D, 2^r s)_2 = (D,s)_2 = (-1)^{(D-1)(s-1)/4} = 1.$$

**Case 2** : $2 \nmid r$.

**Case 2.1** : $r < 0$. From (14) we have

$$v^2 = 2^{3r} s(s^2 + 2^{-r} As + 2^{2-r} B).$$

Therefore, $3r = v_2(v^2)$, impossible since $2 \nmid r$.

**Case 2.2** : $r > 0$. Since $4|n$ and $2 \nmid abc$, we have

$$A = n^2a^4b^4c^4 - 2na^2b^2c^2(a^2+b^2+c^2) + a^4+b^4+c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2 \equiv 5 \pmod 8.$$

Thus $v_2(A) = 0$. Let $B_1 = ka^4b^4c^4$. Then $2 \nmid B_1$ and $B = 2^6 B_1$.

• $r > 6$. From (14) we have

$$v^2 = 2^{r+6} s(2^{2r-6}s^2 + 2^{r-6}As + B_1).$$

Therefore, $r + 6 = v_2(v^2)$, impossible since $2 \nmid r$.

• $r < 6$. From (14) we have

$$v^2 = 2^{2r} s(2^r s^2 + As + 2^{6-r} B_1). \tag{15}$$

Thus $v_2(v) = r$. Let $v = 2^r t$, where $2 \nmid t$. From (15) we have

$$t^2 = s(2^r s^2 + As + 2^{6-r} B_1). \tag{16}$$

Note that in (16) we have $A \equiv 5 \pmod 8$, $2 \nmid r$, $0 < r < 6$, $2 \nmid B_1$, $2 \nmid s$.

(i) $r = 1$. Reducing (16) mod 4 gives

$$1 \equiv s(2+s) \equiv 2s+1 \pmod 4.$$

impossible since $2 \nmid s$.

(ii) $r = 3$. Reducing (16) mod 8 gives

$$1 \equiv 5s^2 \pmod 8,$$

impossible.

(iii) $r = 5$. Reducing (16) mod 4 gives

$$1 \equiv s(s+2) \equiv 1+2s \pmod 4,$$

impossible since $2 \nmid s$.

□

LEMMA 5. *If* $8|n-5$ *then* $(D,u)_2 = 1$.

*Proof.* If $2|abc$, since $a+b+c = -1$, two of $a,b,c$ are even and the remaining number is odd. Hence, $8|a^2b^2c^2$. Thus

$$D = 1 - na^2b^2c^2 \equiv 1 \pmod 8.$$

Therefore, $D \in \mathbb{Z}_2^2$. So $(D,u)_2 = 1$. We consider the case $2 \nmid abc$. Since, $n \equiv 5 \pmod 8$, we have

$$D = 1 - na^2b^2c^2 \equiv 4 \pmod 8.$$

Let $D = 4D_1$, where $D_1 \in \mathbb{Z}$, $2 \nmid D_1$. Then

$$\begin{aligned}
A &= n^2a^4b^4c^4 - 2na^2b^2c^2(a^2+b^2+c^2) + a^4+b^4+c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2 \\
&= (na^2b^2c^2 - a^2 - b^2 - c^2)^2 - 4(a^2b^2 + b^2c^2 + c^2a^2) \\
&= ((a+b+c)^2 - D - a^2 - b^2 - c^2)^2 - 4(a^2b^2 + b^2c^2 + c^2a^2) \\
&= (2(ab+bc+ca) - D)^2 - 4(a^2b^2 + b^2c^2 + c^2a^2) \\
&= D^2 - 4D(ab+bc+ca) + 8abc(a+b+c) \\
&= 16D_1^2 - 16D_1(ab+bc+ca) - 8abc \\
&= 8A_1.
\end{aligned}$$

where

$$A_1 = 2D_1^2 - 2D_1(ab+bc+ca) - abc. \tag{17}$$

Since $abc$ is an odd square, we have $abc \equiv 1 \pmod 8$. Therefore,

$$2(ab+bc+ca) = 1 - a^2 - b^2 - c^2 \equiv -2 \pmod 8.$$

Thus in (17) we have

$$A_1 \equiv 1 + 2D_1 \pmod 8. \tag{18}$$

In particular,

$$A_1 \equiv -1 \pmod 4. \tag{19}$$

Let $u = 2^r s$, where $r, s \in \mathbb{Z}$ and $2 \nmid s$. Let $B_1 = na^4b^4c^4$. From (7) we have

$$v^2 = 2^r s(2^{2r}s^2 + 2^{r+3}A_1 s + 2^4 B_1). \tag{20}$$

**Case 1**: $r < 0$. From (20) we have

$$v^2 = 2^{3r}s(s^2 + 2^{3-r}A_1 s + 2^{4-2r}B_1). \tag{21}$$

Therefore, $v_2(v^2) = 3r$. Thus $2|r$, so $r \leq -2$. Let $v = 2^{3r/2}t$, where $2 \nmid t$. From (21) we have

$$t^2 = s(s^2 + 2^{3-r}A_1 s + 2^{4-2r}B_1). \tag{22}$$

Reducing (22) mod 8 gives $s \equiv 1 \pmod 8$. Therefore, $s \in \mathbb{Z}_2^2$. Thus $u = 2^r s \in \mathbb{Q}_2^2$. Therefore, $(D,u)_2 = 1$.

**Case 2**: $r = 0$. Reducing (20) mod 8 gives $s \equiv 1 \pmod 8$. Thus $u = s \in \mathbb{Z}_2^2$. Therefore, $(D,u)_2 = 1$.

**Case 3**: $r = 1$. From (20) we have

$$v^2 = 2^3 s(s^2 + 4A_1 s + 4B_1).$$

Hence, $v_2(v^2) = 3$, impossible.

**Case 4**: $r = 2$. From (20) we have

$$v^2 = 2^6 s(s^2 + 2A_1 s + B_1) \tag{23}$$

**Case 4.1**: $A_1 s \equiv 1 \pmod 4$. Combining with (19) gives $s \equiv -1 \pmod 4$. Let $A_1 + s = 4e + 2$, where $e \in \mathbb{Z}$. Since $abc$ is an odd square, $a^2b^2c^2 \equiv 1 \pmod{16}$, there are two cases to consider:

- $n \equiv 5 \pmod{16}$. Then $na^4b^4c^4 \equiv 5 \pmod{16}$. Let $na^4b^4c^4 = 16f + 5$, where $f \in \mathbb{Z}$. We have

$$D = 1 - na^2b^2c^2 \equiv -4 \pmod{16}.$$

Therefore, $D = 4D_1$ with $D_1 \equiv -1 \pmod 4$. From (18) we have

$$A_1 \equiv -1 \pmod 8.$$

Let $A_1 = 8d - 1$, where $d \in \mathbb{Z}$. Then

$$\begin{aligned} s^2 + 2A_1 s + B_1 &= (s + A_1)^2 + na^4b^4c^4 - A_1^2 \\ &= (4e+2)^2 + 16f + 5 - (8d-1)^2 \\ &= 8(2e^2 + 2e + 2f - 8d^2 + 2d + 1). \end{aligned}$$

Hence, $v_2(s^2 + 2A_1 s + B_1) = 3$. From (23) we have $v_2(v^2) = 9$, impossible.

• $n \equiv 13 \pmod{16}$. Then

$$D = 1 - na^2b^2c^2 \equiv 4 \pmod{16}.$$

Therefore, $D = 4D_1$ with $D_1 \equiv 1 \pmod 4$. Thus,

$$(D,u)_2 = (4D_1, 4s)_2 = (D_1, s)_2 = (-1)^{(D_1-1)(s-1)/4}. = 1.$$

**Case 4.2** : $A_1 s \equiv 3 \pmod 4$. Combining with (19) gives $s \equiv 1 \pmod 4$. Therefore,

$$(D,u)_2 = (D,4s)_2 = (D,s)_2 = (-1)^{(s-1)(D-1)/4} = 1.$$

**Case 5** : $r > 2$. From (20) we have

$$v^2 = 2^{r+4} s (2^{2r-4} s^2 + 2^{r-1} A_1 s + B_1). \tag{24}$$

Hence, $v_2(v^2) = r + 4$, so $2|r$. Since $r > 2$, we have $r \geq 4$. Let $v = 2^{2+r/2} t$, where $2 \nmid t$. From (24) we have

$$t^2 = s(2^{2r-4} s^2 + 2^{r-1} A_1 s + B_1). \tag{25}$$

Since $B_1 \equiv 1 \pmod 4$ and $r \geq 4$, reducing (25) mod 4 gives $s \equiv 1 \pmod 4$. Therefore,

$$(D,u)_2 = (4D_1, 2^r s)_2 = (D_1, s)_2 = (-1)^{(D_1-1)(s-1)/4} = 1.$$

□

LEMMA 6. *We have* $(D,u)_\infty = 1$.

*Proof.* Since $(D,u)_p = 1$ for all prime numbers $p$ and

$$(D,u)_\infty \prod_{p \text{ prime}} (D,u)_p = 1,$$

we have $(D,u)_\infty = 1$. □

From Lemmas 1 and 6 we have $u > 0$, contradicting Lemma 2.

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