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CONCRETIZATION OF THE MATHEMATICAL MODELS OF LINEAR ELASTICITY THEORY BY REPRESENTATION OF GENERAL SOLUTION IN TERMS OF HARMONIC POTENTIALS IN PAPKOVICH-NEUBER FORM

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Abstract. The aim of the work is to optimize the representation of the fundamental solution of the equilibrium equations of the linear spatial theory of elasticity (Lamé equations) in the form of Papkovich-Neuber through scalar and vector harmonic functions. For the first time, a method of variation of the Lagrange functional is applied to obtain the corresponding connections for the above-mentioned harmonic functions on the lateral surface of a body. The result is the concretization of the connections on harmonic potentials. This, in turn, makes it possible to find and extend a set of exact analytical solutions of boundary value problems of the spatial theory of elasticity, which describe the distribution of stresses and corresponding external loads on the lateral surface of a given isotropic elastic body. The application of the described technique is illustrated by the example of a prismatic elastic body and the structure of the external load is analyzed.

Key words*:* mathematical modeling, three-dimensional theory of elasticity, Lamé equation, boundary value problem; theory of elasticity, stress tensor, displacement vector, external load vector, functional, harmonic function.

1. INTRODUCTION. THE ANALYSIS OF PREVIOUS RESEARCHES AND PROBLEM STATEMENT

Spatial problems of the theory of elasticity are one of the defining directions and the basis of scientific research of boundary value problems in the mechanics of a deformable solid. Therefore, the topic of research of spatial problems of the theory of elasticity is relevant for both basic and applied research. The integration of the equilibrium equation of the linear theory of elasticity as well as finding the displacement vector in an elastic body for the general case of loading are some of the most important tasks.

One of the effective approaches to solving boundary value problems in the mechanics of a deformable solid, which would allow constructing solutions of a number of problems significant from both theoretical and applied points of view, is the method of potential functions. In particular, G. Airy first proposed an approach for solving a two-dimensional problem of plane theory of elasticity by reducing it to a biharmonic equation for stress functions; E. Beltrami derived the corresponding equation of continuity in terms of stresses. In the case of a plane static problem of the theory of elasticity of the isotropic body, Kolosoff and Muskhelishvili gave the general integral of the equilibrium equations through a biharmonic function. This, in turn, made it possible to obtain expressions for the components of the stress tensor and the displacement vector through two analytical functions of a complex variable and to apply the apparatus of p- and (p,q)-analytical functions to solve the above-mentioned boundary value problems.

The application of this approach in three-dimensional problems of the elasticity theory requires the reduction of the original system of equations to a larger number of key equations (harmonic or biharmonic), for which it would be convenient to construct sets of general solutions in certain coordinate systems. In addition, the application of such general solutions to the solution of boundary value problems for solids of revolution requires that they have a sufficient number of degrees of freedom that would fully satisfy the boundary conditions on the entire lateral surface of the studied body.

However, the application of this approach in the space of three variables proved to be much more complicated in comparison with the two-dimensional case.

From the classical fundamental works [1, 2], some images of the general solution of equilibrium equations that use from two to four independent harmonic functions are known. However, it has not been proved yet that a general fundamental solution has been found, and there is a large class of problems that cannot be solved with the help of known displacement vector images.

One of the most common images of the general solution of Lamé’s equations is the representation of the displacement vector $\vec{u}$ in the form of Papkovich-Neuber [3, 4] through the spatial vector $\vec{\phi}$ and scalar ϕ_0 harmonic functions

$$\vec{u} = \vec{\nabla}\left(\phi_0 + \vec{r}\cdot\vec{\phi}\right) - 4(1-\nu)\vec{\phi}\,, \tag{1}$$

where ν is the Poisson’s ratio, $\vec{r}$ is radius vector of an arbitrarily selected point of the body.

In particular, the authors showed that one of the four functions can be equated to zero with certain restrictions on the geometry of the body and the coefficients that describe the elastic characteristics of the body.

Analysis of the completeness of fundamental solutions depending on the connectivity of the region was carried out.In [5], the author constructed a general image of the solution of the Lamé equation of the three-dimensional theory of elasticity in vector form in a curvilinear coordinate system, which is expressed only through three harmonic functions, and proposed one of the options for optimizing the representation of the fundamental solution in Papkovich-Neuber form. The expression of the vector of elastic displacements in cylindrical and elliptical coordinate systems is given and the stress-strain state is found both for an arbitrarily loaded finite elastic cylinder and for bodies of rotation [6-9].

The general representation of the fundamental solution of the Lamé equation through harmonic and biharmonic potentials made it possible to develop and apply the theory of holomorphic functions of many complex variables to construct analytical solutions of boundary value problems of the spatial theory of elasticity [10].

Important technical applications of the methods of two-dimensional theory of elasticity and the apparatus of holomorphic functions to describe the stress-strain state of elements of mine equipment are given in [11].

In this paper, using the method of variation of the Lagrange functional of an elastic body, we propose to obtain the bonds between scalar and vector harmonic functions in the representation of the fundamental solution of the Lame equilibrium equations in the form of Papkovich-Neuber. This technique will substantiate the correctness of the main boundary value problems, as well as build and expand the set of exact analytical solutions of individual classes of static boundary value problems of spatial elasticity theory, which describe the distribution of stresses and corresponding external loads on the side surface of a given single-connected elastic body.

2. BASIC RELATIONS AND FORMULATION OF BOUNDARY VALUE PROBLEMS OF LINEAR SPATIAL THEORY OF ELASTICITY IN HARMONIC POTENTIALS

A single-connected isotropic elastic body is considered, which occupies a domain X of Euclidean space bounded by a surface ∂X . The body is under the action of a stationary force load,which is applied to the lateral surface ∂X .

The linear problem of the theory of elasticity in a static formulation is reduced to the construction of the solution of equilibrium equations (Lame equations):

$$\mu\Delta\vec{u} + (\lambda+\mu)\vec{\nabla}\otimes\left(\vec{\nabla}\cdot\vec{u}\right) = 0\,. \tag{2}$$

The stress tensor $\hat{\sigma}$ is presented through the displacement vector $\vec{u}$ by the ratio

$$\hat{\sigma} = \lambda\left(\vec{\nabla}\cdot\vec{u}\right)\hat{I} + \mu\left(\vec{\nabla}\otimes\vec{u} + \vec{u}\otimes\vec{\nabla}\right), \tag{3}$$

and satisfies the boundary condition on the side surface ∂X

$$\vec{\sigma}_n \equiv \left(\vec{n}\cdot\hat{\sigma}\right)\big|_{\partial X} \equiv \vec{n}\cdot\left[\lambda\left(\vec{\nabla}\cdot\vec{u}\right)\hat{I} + \mu\left(\vec{\nabla}\otimes\vec{u} + \vec{u}\otimes\vec{\nabla}\right)\right]\Big|_{\partial X} = \vec{\sigma}_n^+, \tag{4}$$

where $\hat{I}$ is unit tensor; $\vec{n}$ is vector of the external normal; $\vec{u} = u_i\left(x_1, x_2, x_3\right)\vec{\ni}_i$ is vector of displacements $\left(i = \overline{1,3}\right)$; $\{x_i\}$ are coordinates of an arbitrarily selected material point $x \in X$; $\vec{\ni}_i$ are basic orts of an arbitrary orthogonal coordinate system; $\vec{\nabla} \equiv \dfrac{\partial}{\partial\vec{r}} = \vec{\ni}_i\dfrac{\partial}{\partial x_i}$ is the Hamilton differential operator $\left(i = \overline{1,3}\right)$; $\Delta \equiv \vec{\nabla}\cdot\vec{\nabla} = \dfrac{\partial^2}{\partial x_i^2}$ is the Laplace operator; $\vec{r}\left(x_1, x_2, x_3\right) = x_i\,\vec{\ni}_i$ is radius vector of an arbitrarily selected point of the body; $\otimes$ is the operation of the dyadic product; λ, μ are the elastic Lame constants; $\vec{\sigma}_n\left(x_1, x_2, x_3\right)$ is the stress vector; $\vec{\sigma}_n^+\left(x_1, x_2, x_3\right)$ is a given vector of surface forces that satisfies the integral conditions of self-equilibrium of the external load on the lateral surface of the body ∂X :

$$\int_{\partial X}\vec{\sigma}_n^+ d\Sigma = 0\,, \qquad \int_{\partial X}\left(\vec{r}\times\vec{\sigma}_n^+\right)d\Sigma = 0\,. \tag{5}$$

In the basis of constructing the solution of the boundary value problem of the theory of elasticity we use the representation of the displacement vector in the form of Papkovich-Neuber (1) through scalar and vector harmonic functions.

Then taking into account the relationship (1), the stress tensor (4) takes the form:

$$\hat{\sigma}(x_1,x_2,x_3) = 2\mu\Big[\vec{\nabla}\otimes\vec{\nabla}\phi_0(x_1,x_2,x_3) + \left(\vec{\nabla}\otimes\vec{\nabla}\otimes\vec{\phi}(x_1,x_2,x_3)\right)\cdot\vec{r}(x_1,x_2,x_3) -$$
$$-\left(1-2\nu\right)\left(\vec{\nabla}\otimes\vec{\phi}(x_1,x_2,x_3) + \vec{\phi}(x_1,x_2,x_3)\otimes\vec{\nabla}\right) - 2\nu\left(\vec{\nabla}\cdot\vec{\phi}(x_1,x_2,x_3)\right)\hat{I}\Big].$$

Thus, the boundary value problem (2)–(5) is reduced to finding the harmonic functions $\phi_0(x_1,x_2,x_3)$, $\vec{\phi}(x_1,x_2,x_3)$ satisfying the corresponding boundary conditions:

$$\vec{\sigma}_n \equiv \left(\vec{n}\cdot\hat{\sigma}\right)\big|_{\partial X} \equiv 2\mu\left\{\vec{n}\cdot\left[\vec{\nabla}\otimes\vec{\nabla}\phi_0 + \left(\vec{\nabla}\otimes\vec{\nabla}\otimes\vec{\phi}\right)\cdot\vec{r} - (1-2\nu)\left(\vec{\nabla}\otimes\vec{\phi} + \vec{\phi}\otimes\vec{\nabla}\right) - 2\nu\left(\vec{\nabla}\cdot\vec{\phi}\right)\hat{I}\right]\Big|_{\partial x}\right\} = \vec{\sigma}_n^+\,.$$

Note that the representation of the desired solution of the initial boundary value problem through the displacement vector $\vec{u}$ in the form of Papkovich-Neuber (1) ensures the fulfillment of the first of the integral conditions (5), namely the zero equality of the main vector of external load.

3. FINDING CONNECTIONS ON HARMONIC POTENTIALS IN THE FORM OF PAPKOVICH-NEUBER

This paper investigates the establishment of additional natural connections on the above-mentioned harmonic functions (1), which do not contain additional restrictions on the input characteristics of the boundary value problem.

Consider the functional

$$\Phi\left[\vec{u},\phi_0,\vec{\phi}\right] = \Phi_0\left[\vec{u}\right] + \alpha_1\left(\Phi_1\left[\phi_0\right] + \Phi_2\left[\vec{\phi}\right]\right) + \alpha_2\Phi_3\left[\vec{u},\phi_0,\vec{\phi}\right]. \tag{6}$$

Here

$$\Phi_0[\vec{u}] = \int_{\partial X} \vec{\sigma}_n^+ \cdot \vec{u}\, \mathrm{d}\Sigma - \int_X U\, \mathrm{d}V, \quad \Phi_1[\phi_0] = -\frac{1}{2}\int_X \left(\vec{\nabla}\phi_0\right)\cdot\left(\vec{\nabla}\phi_0\right) \mathrm{d}V,$$

$$\Phi_2\left[\vec{\phi}\right] = -\frac{1}{2}\int_X \left(\vec{\nabla}\otimes\vec{\phi}\right)\cdot\cdot\left(\vec{\nabla}\otimes\vec{\phi}\right)^T \mathrm{d}V, \quad \Phi_3\left[\vec{u},\phi_0,\vec{\phi}\right] = \int_X \vec{\lambda}\cdot\left(\vec{u} - \vec{\nabla}\left(\phi_0 + \vec{r}\cdot\vec{\phi}\right) + C\vec{\phi}\right)\mathrm{d}V,$$

where $U = \frac{1}{2}(\hat{\sigma}\cdot\cdot\hat{\varepsilon})$ is the internal energy of a deformable solid; $\hat{\varepsilon} = \frac{1}{2}\left(\vec{\nabla}\otimes\vec{u} + \left(\vec{\nabla}\otimes\vec{u}\right)^T\right)$ is the deformation tensor; $\vec{\lambda}$ is the Lagrange multiplier; $C = 4(1-\nu)$, $\alpha_1, \alpha_2 = \mathrm{const}$, "$\cdot\cdot$" is the operation of double convolution.

Write the condition of functional (6) stationarity:

$$\delta\Phi\left[\vec{u},\phi_0,\vec{\phi}\right] = 0 .$$

Consider a variation of the functional $\Phi_0[\vec{u}]$. This functional coincides with the Lagrange functional of the original boundary value problem:

$$\delta\Phi_0[\vec{u}] = \int_{\partial X} \delta\left(\vec{\sigma}_n^+ \cdot \vec{u}\right)\mathrm{d}\Sigma - \int_X \delta U\, \mathrm{d}V .$$

Write the variation of the energy functional

$$\delta U = \frac{1}{2}\delta\left(\hat{\sigma}\cdot\cdot\hat{\varepsilon}\right) = \frac{1}{2}\left(\delta\hat{\sigma}\cdot\cdot\hat{\varepsilon} + \hat{\sigma}\cdot\cdot\delta\hat{\varepsilon}\right). \tag{7}$$

Let us convert the expression

$$\delta\hat{\sigma}\cdot\cdot\hat{\varepsilon} = \hat{\varepsilon}\cdot\cdot\delta\hat{\sigma} = \hat{\varepsilon}\cdot\cdot\delta\left[\lambda\hat{I}e + 2\mu\hat{\varepsilon}\right] = \hat{\varepsilon}\cdot\cdot\lambda\delta\left[\hat{I}e\right] + 2\mu\hat{\varepsilon}\cdot\cdot\delta\hat{\varepsilon}, \tag{8}$$

where $e = e_{ii}$ is the main invariant of the deformation tensor $\left(i = \overline{1,3}\right)$ is the summation index.

Since $\hat{\varepsilon}\cdot\cdot\hat{I} = e$, then

$$\hat{\varepsilon}\cdot\cdot\delta\left[\hat{I}e\right] = \hat{\varepsilon}\cdot\cdot\hat{I}\delta e = e\delta e = e\delta\left[\hat{\varepsilon}\cdot\cdot\hat{I}\right] = e\hat{I}\cdot\cdot\delta\hat{\varepsilon} .$$

Therefore, the relation (8) takes the form

$$\delta\hat{\sigma}\cdot\cdot\hat{\varepsilon} = \hat{\varepsilon}\cdot\cdot\lambda\delta\left[\hat{I}e\right] + 2\mu\hat{\varepsilon}\cdot\cdot\delta\hat{\varepsilon} = \lambda e\hat{I}\cdot\cdot\delta\hat{\varepsilon} + 2\mu\hat{\varepsilon}\cdot\cdot\delta\hat{\varepsilon} = \left(\lambda e\hat{I} + 2\mu\hat{\varepsilon}\right)\cdot\cdot\delta\hat{\varepsilon} = \hat{\sigma}\cdot\cdot\delta\hat{\varepsilon} .$$

Thus

$$\delta U = \frac{1}{2}\left(\hat{\sigma}\cdot\cdot\delta\hat{\varepsilon} + \hat{\sigma}\cdot\cdot\delta\hat{\varepsilon}\right) = \hat{\sigma}\cdot\cdot\delta\hat{\varepsilon}$$

and, accordingly, the variation of the functional (7) is presented as

$$\delta\Phi_0[\vec{u}] = \int_{\partial X}\left(\vec{\sigma}_n^+ \cdot \delta\vec{\mathrm{u}}\right)\mathrm{d}\Sigma - \int_X \left(\hat{\sigma}\cdot\cdot\delta\hat{\varepsilon}\right)\mathrm{d}V .$$

Using the Ostrogradsky-Gauss formula

$$\int_{\partial X} \vec{n}\cdot\left(\hat{\sigma}\cdot\delta\vec{u}\right)\mathrm{d}\Sigma = \int_X \vec{\nabla}\cdot\left(\hat{\sigma}\cdot\delta\vec{u}\right)\mathrm{d}V$$

and the relation

$$\vec{\nabla}\cdot\left(\hat{\sigma}\cdot\delta\vec{u}\right) = \left(\vec{\nabla}\cdot\hat{\sigma}\right)\cdot\delta\vec{u} + \hat{\sigma}\cdot\cdot\delta\hat{\varepsilon},$$

we obtain

$$\delta\Phi_0[\vec{u}] = \int_{\partial X}\left(\vec{\sigma}_n^+ - \vec{\sigma}_n\right)\cdot\delta\vec{u}\, \mathrm{d}\Sigma + \int_X \left(\vec{\nabla}\cdot\hat{\sigma}\right)\cdot\delta\vec{u}\, \mathrm{d}V, \tag{9}$$

where $\vec{\sigma}_n := \vec{n}\cdot\hat{\sigma}$ is the vector of external surface forces.

Note that the functional $\Phi_0[\vec{u}]$ is a Lagrange functional of the initial boundary value problem, and from the necessary conditions of the extremum of this functional we obtain both the equilibrium equation $\vec{\nabla}\cdot\hat{\sigma} = 0$ and the boundary conditions $\vec{\sigma}_n = \vec{\sigma}_n^+$.

Consider a variation of the functional $\Phi_1[\phi_0]$:

$$\delta\Phi_1[\phi_0] = -\frac{1}{2}\int_X \delta\left[\left(\vec{\nabla}\phi_0\right)\cdot\left(\vec{\nabla}\phi_0\right)\right]\mathrm{d}V \ .$$

Since

$$\delta\left[\left(\vec{\nabla}\phi_0\right)\cdot\left(\vec{\nabla}\phi_0\right)\right] = 2\left(\vec{\nabla}\phi_0\right)\cdot\left(\vec{\nabla}\delta\phi_0\right),$$

then using the Ostrogradsky-Gauss formula and the relation

$$\vec{\nabla}\cdot\left[\left(\vec{\nabla}\phi_0\right)\cdot\delta\phi_0\right] = \left(\vec{\nabla}\phi_0\right)\cdot\left(\vec{\nabla}\delta\phi_0\right) + \left[\vec{\nabla}\cdot\vec{\nabla}\phi_0\right]\cdot\delta\phi_0\,,$$

we get:

$$\delta\Phi_1[\phi_0] = \int_{\partial X}\left(-\frac{\partial\phi_0}{\partial n}\right)\delta\phi_0\,\mathrm{d}\Sigma + \int_X \Delta\phi_0\cdot\delta\phi_0\,\mathrm{d}V \tag{10}$$

where $\dfrac{\partial}{\partial n} = \vec{n}\cdot\vec{\nabla}$.

Consider a variation of the functional $\Phi_2\left[\vec{\phi}\right]$:

$$\delta\Phi_2\left[\vec{\phi}\right] = -\frac{1}{2}\int_X \delta\left[\left(\vec{\nabla}\otimes\vec{\phi}\right)\cdot\cdot\left(\vec{\nabla}\otimes\vec{\phi}\right)^T\right]\mathrm{d}V \ . \tag{11}$$

Since

$$\delta\left[\left(\vec{\nabla}\otimes\vec{\phi}\right)\cdot\cdot\left(\vec{\nabla}\otimes\vec{\phi}\right)^T\right] = 2\left(\vec{\nabla}\otimes\vec{\phi}\right)\cdot\cdot\left(\delta\vec{\phi}\otimes\vec{\nabla}\right),$$

then using the relation

$$\vec{\nabla}\cdot\left[\left(\vec{\nabla}\otimes\vec{\phi}\right)\cdot\delta\vec{\phi}\right] = \left(\vec{\nabla}\otimes\vec{\phi}\right)\cdot\cdot\left(\delta\vec{\phi}\otimes\vec{\nabla}\right) + \left(\Delta\vec{\phi}\right)\cdot\delta\vec{\phi}\,,$$

the expression (11) takes the form:

$$\delta\Phi_2\left[\vec{\phi}\right] = \int_{\partial X}\left(-\frac{\partial\vec{\phi}}{\partial n}\right)\cdot\delta\vec{\phi}\,\mathrm{d}\Sigma + \int_X \Delta\vec{\phi}\cdot\delta\vec{\phi}\,\mathrm{d}V. \tag{12}$$

Thus, from the necessary conditions of the extremum of functionals, we obtain the conditions of harmony of functions and the corresponding (zero) boundary conditions for these functions.

Consider a variation of the functional $\Phi_3\left[\vec{u},\phi_0,\vec{\phi}\right]$:

$$\delta\Phi_3\left[\vec{u},\phi_0,\vec{\phi}\right] = \int_X \vec{\lambda}\cdot\delta\vec{u}\,\mathrm{d}V - \int_X \vec{\lambda}\cdot\vec{\nabla}\left(\delta\phi_0 + \delta\left(\vec{r}\cdot\vec{\phi}\right)\right)\mathrm{d}V + \int_X \vec{\lambda}\cdot C\,\delta\vec{\phi}\,\mathrm{d}V. \tag{13}$$

Using the following relation

$$\vec{\nabla}\cdot\left(\vec{\lambda}\cdot\delta\phi_0\right) = \vec{\lambda}\cdot\left(\vec{\nabla}\cdot\delta\phi_0\right) + \left(\vec{\nabla}\cdot\vec{\lambda}\right)\delta\phi_0\,,$$

the variation of the functional $\delta\Phi_3\left[\vec{u},\phi_0,\vec{\phi}\right]$ (13) takes the form:

$$\begin{aligned}
\delta\Phi_3\left[\vec{u},\phi_0,\vec{\phi}\right] &= \int_X \vec{\lambda}\cdot\delta\vec{u}\,\mathrm{d}V - \int_X\left(\vec{\nabla}\cdot\left(\vec{\lambda}\cdot\delta\phi_0\right) - \left(\vec{\nabla}\cdot\vec{\lambda}\right)\delta\phi_0\right)\mathrm{d}V - \\
&-\int_X\left(\vec{\nabla}\cdot\left(\vec{\lambda}\left(\vec{r}\cdot\delta\vec{\phi}\right)\right) - \left(\vec{\nabla}\cdot\vec{\lambda}\right)\left(\vec{r}\cdot\delta\vec{\phi}\right)\right)\mathrm{d}V + \int_X C\vec{\lambda}\cdot\delta\vec{\phi}\,\mathrm{d}V = \\
&= \int_X\left[\vec{\lambda}\cdot\delta\vec{u} + C\vec{\lambda}\cdot\delta\vec{\phi} + \left(\vec{\nabla}\cdot\vec{\lambda}\right)\delta\phi_0 + \left(\vec{r}\cdot\left(\vec{\nabla}\cdot\vec{\lambda}\right)\cdot\delta\vec{\phi}\right)\right]\mathrm{d}V - \\
&-\int_{\partial X}\left[\vec{n}\cdot\left(\vec{\lambda}\cdot\delta\phi_0\right) + \vec{n}\cdot\left(\vec{\lambda}\left(\vec{r}\cdot\delta\vec{\phi}\right)\right)\right]\mathrm{d}\Sigma\,.
\end{aligned} \tag{14}$$

Thus, based on the relations (9), (10), (12), (14), the variation of the original functional (6) is written as follows:

$$
\begin{aligned}
&\delta\Phi\left[\vec{u},\phi_0,\vec{\phi}\right]=\delta\Phi_0\left[\vec{u}\right]+\alpha_1\left(\delta\Phi_1\left[\phi_0\right]+\delta\Phi_2\left[\vec{\phi}\right]\right)+\alpha_2\delta\Phi_3\left[\vec{u},\phi_0,\vec{\phi}\right]=\\
&=\left(\int_{\partial X}\left(\vec{\sigma}_n^+-\vec{\sigma}_n\right)\cdot\delta\vec{u}\,d\Sigma+\int_X\left(\vec{\nabla}\cdot\hat{\sigma}\right)\cdot\delta\vec{u}\,dV\right)+\alpha_1\left(\int_{\partial X}\left(-\frac{\partial\phi_0}{\partial n}\right)\delta\phi_0\,d\Sigma+\int_X\Delta\phi_0\cdot\delta\phi_0\,dV\right)+\\
&+\alpha_1\left(\int_{\partial X}\left(-\frac{\partial\vec{\phi}}{\partial n}\right)\cdot\delta\vec{\phi}\,d\Sigma+\int_X\Delta\vec{\phi}\cdot\delta\vec{\phi}\,dV\right)+\\
&+\alpha_2\int_X\left[\vec{\lambda}\cdot\delta\vec{u}+C\vec{\lambda}\cdot\delta\vec{\phi}+\left(\vec{\nabla}\cdot\vec{\lambda}\right)\delta\phi_0+\left(\vec{r}\cdot\left(\vec{\nabla}\cdot\vec{\lambda}\right)\cdot\delta\vec{\phi}\right)\right]dV-\\
&-\alpha_2\int_{\partial X}\left[\left(\vec{n}\cdot\left(\delta\phi_0\cdot\vec{\lambda}\right)\right)+\vec{n}\cdot\left(\vec{\lambda}\left(\vec{r}\cdot\delta\vec{\phi}\right)\right)\right]d\Sigma\,.
\end{aligned}
\tag{15}
$$

Find the minimum of the basic variation ratio (15)

$$
\begin{aligned}
\delta\Phi\left[\vec{u},\phi_0,\vec{\phi}\right]=&\int_X\left[\left(\vec{\nabla}\cdot\hat{\sigma}+\alpha_2\vec{\lambda}\right)\delta\vec{u}+\left(\alpha_1\Delta\phi_0+\alpha_2\left(\vec{\nabla}\cdot\vec{\lambda}\right)\right)\cdot\delta\phi_0+\left(\alpha_1\Delta\vec{\phi}+\alpha_2\left(\vec{\nabla}\cdot\vec{\lambda}\right)\cdot\vec{r}+\alpha_2C\vec{\lambda}\right)\cdot\delta\vec{\phi}\right]dV+\\
&+\int_{\partial X}\left[\left(\vec{\sigma}_n^+-\vec{\sigma}_n\right)\cdot\delta\vec{u}+\left(\alpha_1\left(-\frac{\partial\vec{\phi}}{\partial n}\right)-\alpha_2\left(\vec{n}\cdot\vec{\lambda}\right)\cdot\vec{r}\right)\cdot\delta\vec{\phi}+\left(\alpha_1\left(-\frac{\partial\phi_0}{\partial n}\right)-\alpha_2\left(\vec{n}\cdot\vec{\lambda}\right)\right)\cdot\delta\phi_0\right]d\Sigma=0\,.
\end{aligned}
$$

As a result, we obtain the following boundary value problem for finding the displacement vector $\vec{u}$, harmonic functions $\phi_0=\phi_0(x_1,x_2,x_3)$, $\vec{\phi}=\vec{\phi}(x_1,x_2,x_3)$ and $\vec{\lambda}$:

- in the domain X

$$
\begin{gathered}
\vec{\nabla}\cdot\hat{\sigma}+\alpha_2\vec{\lambda}=0\,,\\
\alpha_1\Delta\phi_0+\alpha_2\left(\vec{\nabla}\cdot\vec{\lambda}\right)=0\,,\qquad \alpha_1\Delta\vec{\phi}+\alpha_2\left(\vec{\nabla}\cdot\vec{\lambda}\right)\vec{r}+\alpha_2C\vec{\lambda}=0\,;
\end{gathered}
\tag{16}
$$

- on the surface ∂X

$$
\begin{gathered}
\vec{\sigma}_n^+-\vec{\sigma}_n=0\,,\\
\alpha_1\frac{\partial\phi_0}{\partial n}+\alpha_2\vec{n}\cdot\vec{\lambda}=0\,,\qquad \alpha_1\frac{\partial\vec{\phi}}{\partial n}+\alpha_2\left(\vec{n}\cdot\vec{\lambda}\right)\vec{r}=0\,.
\end{gathered}
\tag{17}
$$

From the system of equations (16), we find the vector

$$\vec{\lambda}=-\frac{1}{\alpha_2}\vec{\nabla}\cdot\hat{\sigma}\,.$$

In this regard, the boundary value problem (16) – (17) can be presented as follows:

– in the domain X

$$\alpha_1\Delta\phi_0=\alpha_2\vec{\nabla}\cdot\left(\vec{\nabla}\cdot\hat{\sigma}\right),\qquad \alpha_1\Delta\vec{\phi}=\left(\vec{\nabla}\cdot\left(\vec{\nabla}\cdot\hat{\sigma}\right)\right)\vec{r}+C\left(\vec{\nabla}\cdot\hat{\sigma}\right);\tag{18}$$

– on the surface ∂X

$$
\begin{gathered}
\vec{\sigma}_n^+=\vec{\sigma}_n\,,\\
\alpha_1\frac{\partial\phi_0}{\partial n}=\vec{n}\cdot\left(\vec{\nabla}\cdot\hat{\sigma}\right),\qquad \alpha_1\frac{\partial\vec{\phi}}{\partial n}=\left(\vec{n}\cdot\left(\vec{\nabla}\cdot\hat{\sigma}\right)\right)\vec{r}\,.
\end{gathered}
\tag{19}
$$

Note that from the systems of equations (18), by fulfilling the conditions of harmony of functions ϕ_0 and $\vec{\phi}$

$$\Delta\phi_0=0\,,\qquad \Delta\vec{\phi}=0$$

we obtain, as a consequence, the equation of equilibrium (2), given in terms of stresses.

From the analysis of the boundary conditions (19) on the surface, we establish that

$$\vec{\sigma}_n^+ = \vec{\sigma}_n,$$
$$\frac{\partial\vec{\phi}}{\partial n} = \frac{\partial\phi_0}{\partial n}\vec{r}\,. \tag{20}$$

Thus, according to the representation of the displacement vector in the form of Papkovich-Neuber (1), the problem is reduced to finding harmonic functions and which must satisfy the conditions (20). The second condition is the desired (in a sense “natural”) condition for harmonic functions, which specifies the relationship between them.

4. ANALYSIS OF THE STRUCTURE OF THE EXTERNAL LOAD FOR A PRISMATIC ELASTIC BODY

Consider a prismatic elastic body $\{-a_i \le x_i \le a_i\}$ bounded by the surfaces $\partial X_i : x_i = \pm a_i \ \left(i = \overline{1,3}\right)$, where $\{x_i\}$ are the Cartesian coordinates of an arbitrarily chosen material point $x \in X$.

Write the relation (20) in the Cartesian coordinate system

$$\frac{\partial\vec{\phi}}{\partial n} = \sum_{i=1}^{3} n_i \frac{\partial}{\partial x_i}\left(\phi_j(x_1,x_2,x_3)\vec{e}_j\right) \quad \left(j = \overline{1,3}\right),$$
$$\frac{\partial\phi_0}{\partial n}\vec{r} = \left[\sum_{i=1}^{3} n_i \frac{\partial\phi_0(x_1,x_2,x_3)}{\partial x_i}\right] x_j\vec{e}_j\,,$$

where $\phi_0(x_1,x_2,x_3)$, $\vec{\phi} = \phi_i(x_1,x_2,x_3)\vec{e}_i$ are vector and scalar harmonic functions; $\vec{e}_i$ are the basic orts of the Cartesian system; $\vec{r}(x_1,x_2,x_3) = x_i\vec{e}_i$ is the radius-vector of point $x \in X$; $\frac{\partial}{\partial n} \equiv \vec{n}\cdot\vec{\nabla} = \sum_{i=1}^{3} n_i \frac{\partial}{\partial x_i}$.

We obtain a system of equations:

- for everyone $j = \left(\overline{1,3}\right)$

$$n_1\frac{\partial\phi_j(x_1,x_2,x_3)}{\partial x_1} + n_2\frac{\partial\phi_j(x_1,x_2,x_3)}{\partial x_2} + n_3\frac{\partial\phi_j(x_1,x_2,x_3)}{\partial x_3} =$$
$$= x_j\left[n_1\frac{\partial\phi_0(x_1,x_2,x_3)}{\partial x_1} + n_2\frac{\partial\phi_0(x_1,x_2,x_3)}{\partial x_2} + n_3\frac{\partial\phi_0(x_1,x_2,x_3)}{\partial x_3}\right].$$

We can get the corresponding relations on the surfaces of a given elastic body:

– on the surface ∂X_j : $x_j = \pm a_j$, $n_j = \pm 1$ for every one $j = \left(\overline{1,3}\right)$

$$\left.\left(\pm\frac{\partial\phi_1\left(x_1,x_2,x_3\right)}{\partial x_j} = x_1\left[\pm\frac{\partial\phi_0\left(x_1,x_2,x_3\right)}{\partial x_j}\right]\right)\right|_{x_j=\pm a_j},$$
$$\left.\left(\pm\frac{\partial\phi_2\left(x_1,x_2,x_3\right)}{\partial x_j} = x_2\left[\pm\frac{\partial\phi_0\left(x_1,x_2,x_3\right)}{\partial x_j}\right]\right)\right|_{x_j=\pm a_j}, \tag{21}$$
$$\left.\left(\pm\frac{\partial\phi_3\left(x_1,x_2,x_3\right)}{\partial x_j} = x_3\left[\pm\frac{\partial\phi_0\left(x_1,x_2,x_3\right)}{\partial x_j}\right]\right)\right|_{x_j=\pm a_j}.$$

From the systems of relations (21) we obtain the required connections for harmonic functions $\phi_i(x_1, x_2, x_3)$ $\left(i = \overline{1,3}\right)$:

- on the surface ∂X_i: $x_i = \pm a_i$, $n_i = \pm 1$ $\left(i = \overline{1,3}\right)$

$$\left.\left(\pm\frac{1}{x_1}\frac{\partial\phi_1(x_1,x_2,x_3)}{\partial x_i} = \pm\frac{1}{x_2}\frac{\partial\phi_2(x_1,x_2,x_3)}{\partial x_i} = \pm\frac{1}{x_3}\frac{\partial\phi_3(x_1,x_2,x_3)}{\partial x_i}\right)\right|_{x_i=\pm a_i}. \tag{22}$$

5. COMPARATIVE ANALYSIS OF THE RESULTS AND DISCUSSION

The application of the apparatus of harmonic functions proved to be effective in solving a wide class of boundary value problems in the mechanics of a deformable body. However, the application of this approach in the three-dimensional formulation of linear elasticity theory requires reducing the basic system of equations to a number of key equations for which it would be possible to build "catalogs" [8] of general solutions in certain coordinate systems. The main issue is that there are a sufficient number of degrees of freedom of the general solution. It is necessary to fully satisfy the boundary conditions on the entire lateral surface of the studied body.

For example, in [8] the authors used two representations of J. Dugall's fundamental solution of equilibrium equations through harmonic functions $\varphi_i(r,\theta,z)$, $\omega_i(r,\theta,z)$, $\psi_i(r,\theta,z)$ $\left(i=1,2\right)$ to construct a general solution of the problem of the theory of elasticity for a continuous cylinder of finite length. The solution is presented as a superposition of two components in a cylindrical coordinate system $\left(r,\theta,z\right)$.

The first of these is determined by the ratios

$$U_r = r\frac{\partial^2\phi_1}{\partial z^2} + \frac{\partial\omega_1}{\partial r} + \frac{1}{r}\frac{\partial\psi_1}{\partial\theta}, \quad U_\theta = \frac{1}{r}\frac{\partial\omega_1}{\partial\theta} - \frac{\partial\psi_1}{\partial r}, \quad U_z = -r\frac{\partial}{\partial z}\left(r\frac{\partial\phi_1}{\partial r} + 4(1-\nu)\phi_1\right) + \frac{\partial\omega_1}{\partial z}$$

and allows to satisfy boundary conditions on a cylindrical surface. The second component, which is determined by the ratios

$$U_r = 2z\frac{\partial^2\phi_2}{\partial r\partial z} + (3-4\nu)\frac{\partial\phi_2}{\partial r} + \frac{\partial\omega_2}{\partial r} + \frac{2}{r}\frac{\partial\psi_2}{\partial\theta},$$

$$U_\theta = \frac{2z}{r}\frac{\partial^2\phi_2}{\partial\theta\partial z} + \frac{(3-4\nu)}{r}\frac{\partial\phi_2}{\partial\theta} + \frac{1}{r}\frac{\partial\omega_2}{\partial\theta} - 2\frac{\partial\psi_2}{\partial r},$$

$$U_z = 2z\frac{\partial^2\phi_2}{\partial z^2} - (3-4\nu)\frac{\partial\phi_2}{\partial z} + \frac{\partial\omega_2}{\partial z}$$

satisfies the boundary conditions at the ends of the cylinder. The solutions obtained by the authors in the form of potential harmonic functions contain sets of unknown coefficients, which are sufficient to satisfy the corresponding boundary conditions. This example indicates the importance of the number of degrees of freedom in the representations of the fundamental solution of equilibrium equations through harmonic potentials. In contrast to [8], this paper shows that the representation in form (1) should not ignore the degrees of freedom in the image of fundamental solutions due to harmonic potentials (see formulas (1), (20)).

In [6], the general solution of the Lame equilibrium equations in the Papkovich-Neuber form for a class of boundary value problems was presented through three harmonic functions R, ψ, Q in the form:

$$\vec{u} = z\,\mathrm{grad}R - (3-4\nu)R\vec{e}_3 + \mathrm{grad}\psi + \mathrm{rot}\left(Q\vec{e}_3\right).$$

In this case, the number of degrees of freedom in the general solution was reduced by one. In contrast to the results obtained there, the analogical representation of the fundamental solution (1) was optimized in this paper without losing the degrees of freedom (number of functions) to satisfy the boundary conditions of the elastic isotropic body. In particular, the vector condition of the bonds between given harmonic vector and scalar functions is given by relation (20).

Based on the fundamental solution (1) in [10], the authors presented a complex displacement vector $\vec{w}(z_1,z_2,z_3)$ and a complex stress tensor $\hat{P}(z_1,z_2,z_3)$ through holomorphic functions of two complex variables (z_1,z_2), namely

$$\vec{w}(z_1,z_2,z_3)=\vec{\nabla}^*\Phi_0(z_1,z_2)+\left(\vec{\nabla}^*\otimes\vec{\Phi}(z_1,z_2)\right)\times\vec{r}(z_1,z_2,z_3)-(3-4\nu)\vec{\Phi}(z_1,z_2);$$

$$\hat{P}(z_1,z_2,z_3)=2\mu\Big[\vec{\nabla}^*\otimes\vec{\nabla}^*\Phi_0(z_1,z_2)+\left(\vec{\nabla}^*\otimes\vec{\nabla}^*\otimes\vec{\Phi}(z_1,z_2)\right)\times\vec{r}(z_1,z_2,z_3)-$$
$$-(1-2\nu)\left(\vec{\nabla}^*\otimes\vec{\Phi}(z_1,z_2)+\vec{\Phi}(z_1,z_2)\otimes\vec{\nabla}^*\right)-2\nu\left(\vec{\nabla}^*\cdot\vec{\Phi}(z_1,z_2)\right)\hat{I}\Big],$$

and formulated the main complex-conjugate problem of the spatial theory of elasticity for the corresponding holomorphic functions. Based on the structure of the displacement vector $\vec{w}(z_1,z_2,z_3)$ and the stress tensor $\hat{P}(z_1,z_2,z_3)$, complex basic solutions $\hat{P}^{(n)}$ of order n were obtained. The starting point for them is the scalar holomorphic function $\Phi_0(z_1,z_2)$ in the form of a homogeneous polynomial $Q_0^{(n)}(z_1,z_2)$ of power $n+2$ and the holomorphic vector function $\vec{\Phi}(z_1,z_2)$ in the form of a vector homogeneous polynomial $\vec{Q}^{(n)}(z_1,z_2)$ degree $n+1$:

$$\Phi_0^{(n+2)}(z_1,z_2)\equiv Q_0^{(n+2)}(z_1,z_2)=a^{((n+2)0)}z_1^{n+2}+a^{(0(n+2))}z_2^{n+2};$$
$$\vec{\Phi}^{(n+1)}(z_1,z_2)\equiv\vec{Q}^{(n+1)}(z_1,z_2)=\vec{b}^{((n+1)0)}z_1^{n+1}+\vec{b}^{(0(n+1))}z_2^{n+1}.$$

The obtained basic solutions $\hat{P}^{(k)}$ of order k according to the proposed schematic scheme were the basis for formulating the corresponding structure of boundary conditions on the surface of the body ∂X . As a result of such a holistic approach, natural connections (20) between scalar and vector harmonic potentials were obtained with the help of the methodology developed in this article. This made it possible to constructively clarify the structure of the real and imaginary parts of the basic solutions $\hat{P}^{(k)}$ of order k and, accordingly, the structure of the real and imaginary parts of the external load vector. The approach developed in this paper made it possible to find a subset of exact analytical solutions of spatial boundary value problems, in particular, for an elastic prismatic body (22).

The effectiveness of this approach lies in the ability to obtain accurate analytical solutions of three-dimensional boundary value problems of linear elasticity theory in contrast to other approaches that involve obtaining a solution in the form of series.

6. CONCLUSIONS

Using the method of variation of the Lagrange functional of an elastic body, the natural connections between the scalar and vector harmonic potentials in the representation of the fundamental solution of the Lame equilibrium equations in the form of Papkovich-Neuber are obtained.

The proposed technique made it possible to construct and extend the set of exact analytical solutions of individual classes of static boundary value problems of the spatial theory of elasticity, which describe the stress distribution. The structure of the corresponding external loads on the lateral surface of a given single-connected elastic body is specified. The correctness of the corresponding main boundary value problems for the spatial body is substantiated.

As an example of the application of the described technique, the prismatic elastic body is considered. The analysis of the structure of the external load is carried out. This illustration of the application of the results obtained in this work is important for describing the stress-strain state of structural elements of industrial equipment in order to synthesize and optimize the parameters of their reliable operation.

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