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HARMONIC MAPS AND STABILITY ON LORENTZIAN PARA SASAKIAN MANIFOLDS

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Abstract. In this paper we find some results of harmonicity and stability criteria on LP Sasakian manifolds.

Key words: harmonic maps, stability, Lorentzian para Sasakian manifolds.

1. INTRODUCTION

The theory of harmonic maps combines both global and local aspects and borrow both from Riemannian geometry and analysis. There are a lot of interesting results about harmonic maps on complex manifolds (see [8, 15]).

In the analogy to the complex case, in the last decade harmonic maps on almost contact metric manifolds were studied [1, 2, 4, 5, 7]. The identity map of a compact Riemannian manifold is a trivial example of a harmonic map but in this case, the theory of the second variation is much more complicated and interesting. For instance, the stability of the identity map on Einstein manifolds is related with the first eigenvalue of the Laplacian acting on functions [13]. In [14] and [11] we find classifications of compact simply connected irreducible Riemannian symmetric spaces for which the identity map is unstable.

By a well known result, the identity map on the euclidean sphere S^{2n+1} is unstable [13]. More generally, Gherghe, Ianus and Pastore have studied the stability of the identity map on compact Sasakian manifolds with constant $\varphi-$sectional curvature [7].

After recalling in section 2 the necessary facts about harmonic maps between general Riemannian manifolds, we give some definitions on Lorentz para Khaler manifolds and Lorentz para Sasakian manifolds. Finally in section 3, we give some results of harmonicity and stability of the identity map on Lorentz para Sasakian manifolds.

2. PRELIMINARIES

In this section, we recall some well known facts concerning harmonic maps (see [3] for more details). Let $\phi : (M, g) \longrightarrow (N, h)$ be a smooth map between two Riemannian manifolds of dimensions m and n respectively. The energy density of ϕ is a smooth function $e(\phi) : M \longrightarrow [0, \infty)$ given by

$$e(\phi)_p = \frac{1}{2} Tr_g(\phi^* h)(p) = \frac{1}{2} \sum_{i=1}^{m} h(\phi_{*p} u_i, \phi_{*p} u_i),$$

for $p \in M$ and any orthonormal basis $\{u_1, \ldots, u_m\}$ of T_pM. If M is a compact Riemannian manifold, the energy $E(\phi)$ of ϕ is the integral of its energy density:

$$E(\phi) = \int_M e(\phi)\upsilon_g,$$

where υ_g is the volume measure associated with the metric g on M. A map $\phi \in C^\infty(M,N)$ is said to be harmonic if it is a critical point of E in the set of all smooth maps between (M,g) and (N,h) i.e. for any smooth variation $\phi_t \in C^\infty(M,N)$ of ϕ $(t \in (-\varepsilon, \varepsilon))$ with $\phi_0 = \phi$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} E(\phi_t)\big|_{t=0} = 0.$$

Now, let (M,g) be a compact Riemannian manifold and $\phi : (M,g) \longrightarrow (N,h)$ be a harmonic map. We take a smooth variation $\phi_{s,t}$ with parameters $s,t \in (-\varepsilon, \varepsilon)$ such that $\phi_{0,0} = \phi$. The corresponding variation vector fields are denoted by V and W. The Hessian H_ϕ of a harmonic map ϕ is defined by

$$H_\phi(V,W) = \frac{\partial^2}{\partial s \partial t}(E(\phi_{s,t}))\big|_{(s,t)=(0,0)}.$$

The second variation formula of E is ([9], [13]):

$$H_\phi(V,W) = \int_M h(J_\phi(V),W)\upsilon_g,$$

where J_ϕ is a second order self-adjoint elliptic operator acting on the space of variation vector fields along ϕ (which can be identified with $\Gamma(\phi^{-1}(TN))$) and is defined by

$$J_\phi(V) = -\sum_{i=1}^{m} (\widetilde{\nabla}_{u_i}\widetilde{\nabla}_{u_i} - \widetilde{\nabla}_{\nabla_{u_i} u_i})V - \sum_{i=1}^{m} R^N(V, d\phi(u_i))d\phi(u_i),$$

for any $V \in \Gamma(\phi^{-1}(TN))$ and any local orthonormal frame $\{u_1, \ldots, u_m\}$ on M. Here R^N is the curvature tensor of (N,h) and $\widetilde{\bigtriangledown}$ is the pull-back connection by ϕ of the Levi-Civita connection of N.

The index of a harmonic map $\phi : (M,g) \longrightarrow (N,h)$ is defined as the dimension of the largest subspace of $\Gamma(\phi^{-1}(TN))$ on which the Hessian H_ϕ is negative definite. A harmonic map ϕ is said to be stable if the index of ϕ is zero and otherwise is said to be unstable.

The operator $\overline{\triangle}_\phi$ defined by

$$\overline{\triangle}_\phi V = -\sum_{i=1}^{m} (\widetilde{\nabla}_{u_i}\widetilde{\nabla}_{u_i} - \widetilde{\nabla}_{\nabla_{u_i} u_i})V, \qquad V \in \Gamma(\phi^{-1}(TN))$$

is called the rough Laplacian.

Due to the Hodge de Rham Kodaira theory, the spectrum of J_ϕ consists of a discrete set of an infinite number of eigenvalues with finite muiltiplicities and without accumulation points.

Libermann introduced an almost para-Hermitian manifold $M(J,h)$, as a smooth manifold of dimension 2m endowed with an almost para-complex structure J such that $J^2 = I$ and a pseudo-Riemannian metric $h(JX,Y) = -h(X,JY)$, The fundamental 2-form of almost para-Hermitian manifold is defined by $\phi(X,Y) = h(JX,Y), \forall X,Y \in \Gamma(TM)$. An almost para-Hermitian manifold is called para-Kahler if $\nabla J = 0$, see detail [12].

A differentiable manifold M^m with structure φ, a $(1,1)$-tensor field , a vector field ξ and a 1-form η is LP Sasakian manifold such that

$$\varphi^2 = I + \eta \otimes \xi \quad \text{and} \quad \eta(\xi) = -1.$$

We also have $\varphi(\xi) = 0, \ \eta \circ \varphi = 0$ and $\mathrm{Rank}\phi = m-1$.

On any LP Sasakian manifold, we can define a compatible metric that is a metric g such that

$$g(\varphi X, \varphi Y) = g(X,Y) + \eta(X)\eta(Y).$$

And

$$g(X,\xi) = \eta(X), \qquad \nabla_X \xi = \phi X, \qquad (\nabla_X \phi)Y = [g(X,Y) + \eta(X)\eta(Y)]\,\xi + [X + \eta(X)\xi]\,\eta(Y)$$

for any vector fields X, Y on M. In this case the manifold will be called LP Sasakian manifold. Further the following relations hold on LP Sasakian manifolds [10];

$$\begin{aligned} R(\xi,X)Y &= g(X,Y)\xi - \eta(Y)X, \\ R(X,Y)\xi &= \eta(Y)X - \eta(X)Y, \\ R(\xi,Y)\xi &= X + \eta(X)\xi, \end{aligned}$$

for any vector fields X, Y on M. $R(X,Y)Z$ is the Riemannian curvature tensor.

3. MAIN RESULTS

THEOREM 1. *Let M_1 and M_2 be two LP Sasakian manifolds. Suppose that $F : M_1 \to M_2$ is a (ϕ_1,ϕ_2)-holomorphic map such that $\xi_2 \in (imdF)^{\perp}$. Then $\tau(F) \in D$.*

Proof. Let $\{e_1, e_2, \ldots, e_{n-1}, \xi_1\}$ be orthonormal basis on TM_1. Then $\{\phi_1 e_1, \phi_1 e_2, \ldots, \phi_1 e_{n-1}, \xi_1\}$ are also orthonormal basis on TM_1. We can write

$$\tau(F) = \sum_{i=1}^{n-1} \nabla \mathrm{d}F(\phi_1 e_i, \phi_1 e_i) + \nabla \mathrm{d}F(\xi_1, \xi_1)$$

since $\nabla \mathrm{d}F(\xi_1,\xi_1) = 0$

$$\begin{aligned} \tau(F) &= \sum_{i=1}^{n-1} \nabla \mathrm{d}F(\phi_1 e_i, \phi_1 e_i) \\ &= \sum_{i=1}^{n-1} \phi_2[(\nabla_{\mathrm{d}Fe_i}\phi_2)\mathrm{d}Fe_i] + \phi_2^2(\nabla_{\mathrm{d}Fe_i}\mathrm{d}Fe_i) - \phi_2[\mathrm{d}F(\nabla_{e_i}\phi_1)e_i] - \phi_2^2(\mathrm{d}F\nabla_{e_i}e_i) \\ &= \sum_{i=1}^{n-1} \phi_2^2 \nabla \mathrm{d}F(e_i,e_i) + \phi_2[(\nabla_{\mathrm{d}Fe_i}\phi_2)\mathrm{d}Fe_i] - \phi_2 \mathrm{d}F[(\nabla_{e_i}\phi_1)e_i] \\ &= \sum_{i=1}^{n-1} \phi_2^2 \nabla \mathrm{d}F(e_i,e_i) + \phi_2\,[g(\mathrm{d}Fe_i,\mathrm{d}Fe_i)\xi_2 + \eta_2(\mathrm{d}Fe_i)\mathrm{d}Fe_i + 2\eta_2(\mathrm{d}Fe_i)\eta_2(\mathrm{d}Fe_i)\xi_2] \\ &\quad + \phi_2\,\mathrm{d}F\,[g(e_i,e_i)\xi_1 + \eta_1(e_i)e_i + 2\eta_1(e_i)\eta_1(e_i)\xi_1], \end{aligned}$$

since $\mathrm{d}F\xi_1 = 0$ and $\eta_1(\mathrm{d}Fe_i) = 0, \forall i$.

$$\begin{aligned} \tau(F) &= \sum_{i=1}^{n-1} \phi_2^2 \nabla \mathrm{d}F(e_i,e_i) \\ \tau(F) &= \phi_2^2 \tau(F) \\ \tau(F) &= \tau(F) + \eta_2(\tau(F))\xi_2 \\ \eta_2(\tau(F))\xi_2 &= 0. \end{aligned}$$

This shows that $\tau(F) \in D$.

THEOREM 2. *Any (ϕ, J)-holomorphic map from LP Sasakian manifold $M(f, \xi, \eta, g)$ to a para Kähler manifold $N(J,h)$ is a harmonic map.*

Proof. For such a map we extend formula in [6] for Lorentz para Sasakian and para *Kähler* manifolds as:

$$J(\tau(F)) = F_*(\mathrm{div}\phi) - tr_g\beta, \tag{1}$$

where $\beta(X,Y) = (\widetilde{\nabla}_X J)F_*Y$, $\widetilde{\nabla}$ being the connection induced in the pull-back bundle F^*TN.

Let $\{e_1, ..., e_{m-1}, \xi\}$ be a local orthonormal basis on TM, Then we have

$$\begin{aligned}
\mathrm{div}\phi &= \sum_{i=1}^{m}(\nabla_{e_i}\phi)e_i \\
&= \sum_{i=1}^{m-1}(\nabla_{e_i}\phi)e_i + (\nabla_\xi\phi)\xi \\
&= \sum_{i=1}^{m-1}[g(e_i,e_i)\xi + \eta(e_i)e_i + 2\eta(e_i)\eta(e_i)\xi] + g(\xi,\xi)\xi + \eta(\xi)\xi + 2\eta(\xi)\eta(\xi)\xi \\
&= (m-1)\xi.
\end{aligned}$$

But, as F is (ϕ, J)-holomorphic, then $F_*(\xi) = 0$ and we get $F_*(\mathrm{div}\phi) = 0$. Finally, as N is para Kähler manifold we have $\nabla J = 0$, and thus the second term of the formula 1 vanishes. Therefore $\tau(F) = 0$ and thus F is harmonic.

THEOREM 3. *Let N(J,h) be a para Kähler manifold, $M(\phi, \xi, \eta, g)$ be a LP Sasakian manifold and $F : N \longrightarrow M$ be a (J, ϕ)-holomorphic map. Then F is harmonic if and only if F is constant.*

Proof. For such a map we extend formula in [6] for Lorentz para Sasakian and para *Kähler* manifolds as:

$$\phi(\tau(F)) = F_*(\mathrm{div}J) - \mathrm{tr}_h\beta,$$

where $\beta(X,Y) = (\widetilde{\nabla}_X\phi)(F_*Y)$.

Suppose that M is a Kähler manifold of real dimension $2n$. Then we have:

$$\mathrm{div}J = \sum_{i=1}^{2n}(\nabla_{e_i}J)e_i = 0,$$

where $\{e_i\}_{i=1...2n}$ is an orthonormal local basis on TN. Now, using the formula (**??**) we obtain

$$\begin{aligned}
Tr_h\beta &= \sum_{i=1}^{2n}(\widetilde{\nabla}_{e_i}\phi)(F_*e_i) = \sum_{i=1}^{2n}(\nabla_{\mathrm{d}Fe_i}\phi)(\mathrm{d}Fe_i) \\
&= \sum_{i=1}^{2n}\{g(\mathrm{d}Fe_i, \mathrm{d}Fe_i)\xi + \eta(\mathrm{d}Fe_i)\mathrm{d}Fe_i + 2\eta(\mathrm{d}Fe_i)\eta(\mathrm{d}Fe_i)\xi\}.
\end{aligned}$$

As F is a (J, f)-holomorphic map, we have $\eta(F_*e_i) = \eta(F_*J^2e_i) = \eta(fF_*Je_i) = 0$ and thus

$$\begin{aligned}
\phi(\tau(F)) &= -\sum_{i=1}^{2n} g(F_*e_i, F_*e_i)\xi. \\
\phi^2(\tau(F)) &= 0.
\end{aligned}$$

$$
\begin{aligned}
\tau(F) &= -\eta(\tau(F))\xi \\
&= -g(\tau(F),\xi)\xi \\
&= -g(\nabla_{dFe_i}\mathrm{d}Fe_i - \mathrm{d}F\nabla_{e_i}e_i,\xi)\xi \\
&= -g(\nabla_{dFe_i}\mathrm{d}Fe_i,\xi) + g(\mathrm{d}F\nabla_{e_i}e_i,\xi)\xi \\
&= -g(\nabla_{dFe_i}\mathrm{d}FJ^2e_i,\xi) + g(\mathrm{d}FJ^2(\nabla_{e_i}e_i),\xi)\xi \\
&= -g(\nabla_{dFe_i}\phi\mathrm{d}FJe_i,\xi) + g(\phi\mathrm{d}FJ(\nabla_{e_i}e_i),\xi)\xi \\
&= -g((\nabla_{\mathrm{d}Fe_i}\phi)\mathrm{d}FJe_i,\xi) - g(\phi(\nabla_{\mathrm{d}Fe_i}\mathrm{d}FJe_i),\xi)\xi \\
&= -(g(\mathrm{d}Fe_i,\mathrm{d}FJe_i)\xi,\xi)\xi = g(\mathrm{d}Fe_i,\phi\mathrm{d}Fe_i).\xi
\end{aligned}
$$

$\tau(F)=0$ implies F is a constant map.

THEOREM 4. *Let M be a compact Lorentzian para Sasakian manifold. If $m \leq 3$, then the identity map 1_M is weakly stable.*

Proof. Let M be a compact Lorentzian para Sasakian manifold $M(\varphi, \xi, \eta, g)$. We consider the identity map on such a manifold $(F = 1_M)$. In this case see [15], the second variation formula is

$$H_{1_M}(V,V) = \int_M h(\overline{\triangle}V,V)\upsilon_g - \sum_{i=1}^{m}\int_M h(R(V,u_i)u_i,V)\upsilon_g,$$

where $V \in \Gamma(TM)$ and $\{e_1,...,e_{m-1},\xi\}$ is a local orthonormal frame on TM.

Let $\{e_1,...,e_{m-1},\xi\}$ be an orthonormal local frame. Then we have

$$R(e_i,V)e_i = g(V,e_i)e_i - g(e_i,e_i)V, \tag{2}$$

$$R(\xi,V)\xi = V + \eta(V)\xi. \tag{3}$$

From the above relations, we get

$$\sum_{i=1}^{m} R(e_i,V)e_i = -(m-3)V$$

and thus

$$\sum_{i=1}^{m} g(R(e_i,V)e_i,V) = -(m-3)g(V,V).$$

It is not very difficult to prove that

$$\int_M h(\overline{\triangle}V,V)\upsilon_g = \int_M h(\widetilde{\nabla}V,\widetilde{\nabla}V)\upsilon_g, \qquad V \in \Gamma(TM),$$

Now the second varition formula becomes

$$H_{1_M}(V,V) = \int_M h(\widetilde{\nabla}V,\widetilde{\nabla}V) - \int_M (m-3)g(V,V)\upsilon_g$$

and thus the identity map 1_M is weakly stable if $m \leq 3$.

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