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# RELATIVISTIC PARTICLES MOVING AROUND GALAXY'S CENTER

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**Abstract**. The present paper is using a general relativistic approach to deal with the well-known problem of flat galaxy's rotation curves. The steep increase of the circular velocity in the core is obtained by using the Schwarzschild interior solution. The corresponding Gordon equation is satisfied by Heun general functions. On the other hand, at large distances from the core, one can employ a newly proposed metric, describing the Schwarzschild black hole immersed in dark matter. Analytic solutions to the Gordon equation can be obtained only in particular cases.

***Key words:*** galactic rotation curves, dark matter, Gordon equation, Heun functions.

## 1. INTRODUCTION

In Newtonian gravity, the circular velocity at the galactic plane is given by the famous Keplerian formula

$$v = \sqrt{\frac{M}{r}} \tag{1}$$

where $M$ is the central mass.

More than forty years ago, it has been observed that, far from the core, instead of decreasing with the square root of the radius, the rotation curves, defined as the circular velocities around the nucleus, are universally flat [1]. More recent investigations on the orbits showed high velocities that increase with $r$, for $r < 5$ kpc and a Newtonian region, followed by a flat one, for $r > 20$ kpc [2].

Even though a star orbital speed is non-relativistic and a galaxy is a week gravity system, General Relativity is yielding relevant results on galactic scales. In order to explain the almost constant velocities at large distances from the core, different theoretical models have been formulated, the apparently more successful ones invoking the existence of dark matter [3]. A test particle orbiting the galaxy center will feel not only the matter of the galaxy, but also the cosmological background.

In the main part of our work, we are using the model proposed in [4], which is assuming that the baryonic mass of a galaxy is enclosed in its central black hole. The derived solution to the Einstein equations, sourced by a dark matter energy-momentum tensor, contains a linear contribution which compensates the Newtonian term that falls rapidly with $r$. Far from the core, this contribution is leading to the observed constant acceleration $a_0 \approx 10^{-11}\mathrm{m/s}^2$. Let us notice that $a_0$ agrees with the value of the universal constant $\gamma \approx 10^{-28}\mathrm{m}^{-1}$ given by the Mannheim–Kazanas (MK) theory [5]. The MK vacuum solution [6] was obtained more than thirty years ago, in Conformal Gravity, and has become very popular for providing relevant results on galactic scales, without the need of the dark sector.

In what it concerns the central regions, extensive observations have shown that giant stars are moving around the galactic center in closed elliptical orbits [2]. Since, in the center of our galaxy, a super-massive black hole has been detected, with the mass $M \approx 4 \times 10^6 M_S$ [2], one is expecting that, in the inner regions, the gas and stars are moving most violently.

For describing the central region, in the next section, we employ the Schwarzschild interior solution. The corresponding Gordon equation is satisfied by the Heun general functions [7, 8], whose theory can be used to derive the energy quantization law. The so-called resonant frequencies are essential characteristics of the black holes [9].

The study of bosons moving in different external configurations is an active field of research [10, 11]. Analytical solutions of the Klein–Gordon and Dirac equations are playing an important role in astrophysical analysis, since they can be used as the first step in the comparison between theory and observations [12].

## 2. SCHWARZSCHILD INTERIOR SOLUTION AND GORDON EQUATION

In General Relativity, the velocity of a particle moving around the galactic center is obtained by working out the corresponding timelike geodesics. Thus, one has to start with the spacetime line element

$$\mathrm{d}s^2 = g_{11}(\mathrm{d}r)^2 + r^2\left[(\mathrm{d}\theta)^2 + \sin^2\theta(\mathrm{d}\varphi)^2\right] - g_{00}(\mathrm{d}t)^2 = -\mathrm{d}\tau^2 \tag{2}$$

where $\tau$ is the proper time and *dot* will mean the derivative with respect to it.

As in our previous investigations [13], we are working in the pseudo-orthonormal frame $E_{a(a=\overline{1,4})}$, whose corresponding dual basis is

$$\omega^1 = \sqrt{g_{11}}\,\mathrm{d}r\,, \quad \omega^2 = r\,\mathrm{d}\theta\,, \quad \omega^3 = r\sin\theta\,\mathrm{d}\varphi\,, \quad \omega^4 = \sqrt{g_{00}}\,\mathrm{d}t\,, \tag{3}$$

so that $\mathrm{d}s^2 = \eta_{ab}\omega^a\omega^b$, with $\eta_{ab} = \mathrm{diag}[1,1,1,-1]$.

From the first Cartan's equation,

$$\mathrm{d}\omega^a = \Gamma^a_{.[bc]}\,\omega^b \wedge \omega^c \tag{4}$$

with $1 \le b < c \le 4$ and $\Gamma^a_{.[bc]} = \Gamma^a_{.bc} - \Gamma^a_{.cb}$, we derive the connection coefficients

$$\begin{aligned}
\Gamma_{122} &= -\Gamma_{212} = -\frac{1}{r\sqrt{g_{11}}}\\
\Gamma_{133} &= -\Gamma_{313} = -\frac{1}{r\sqrt{g_{11}}}\\
\Gamma_{233} &= -\Gamma_{323} = -\frac{\cot\theta}{r}\\
\Gamma_{144} &= -\Gamma_{414} = \frac{1}{\sqrt{g_{11}}}\frac{g'_{00}}{2g_{00}}\,,
\end{aligned} \tag{5}$$

where $(\,)'$ means the derivative with respect to $r$. With the four velocity components,

$$u^1 = \sqrt{g_{11}}\,\dot{r}\,, \quad u^2 = r\dot{\theta}\,, \quad u^3 = r\sin\theta\,\dot{\varphi}\,, \quad u^4 = \sqrt{g_{00}}\,\dot{t}$$

the equations for the timelike geodesics

$$\frac{\mathrm{d}u^a}{\mathrm{d}\tau} + \Gamma^a_{bc}u^b u^c = 0$$

have the explicit form

$$\begin{aligned}
&\ddot{r} + \frac{g'_{11}}{2g_{11}}\dot{r}^2 - \frac{r}{g_{11}}\left[\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2\right] + \frac{g'_{00}}{2g_{11}}\,\dot{t}^2 = 0\\
&r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\sin\theta\cos\theta\,\dot{\varphi}^2 = 0\\
&r\sin\theta\,\ddot{\varphi} + 2\sin\theta\,\dot{r}\dot{\varphi} + 2r\cos\theta\,\dot{\theta}\dot{\varphi} = 0\\
&\ddot{t} + \frac{g'_{00}}{g_{00}}\,\dot{r}\dot{t} = 0
\end{aligned} \tag{6}$$

For the circular orbit in the equatorial plane, i.e. $\dot{r}=0$, $\theta=\pi/2$, $\dot{\theta}=0$, the first relation in (6) is leading to the circular velocity

$$v = r\frac{\mathrm{d}\varphi}{\mathrm{d}t} = r\frac{\dot{\varphi}}{\dot{t}} = \sqrt{\frac{rg_{00}'}{2}}\,. \tag{7}$$

From the total baryonic mass of the Milky Way, $M \approx 6\times 10^{10} M_S$, the stellar disk mass is $M_{disk} \approx 4.6\times 10^{10} M_S$, being concentrated in the region of radius $R$. The numerical value of the Schwarzschild radius is $r_S \approx 3\times 10^{14}\,\mathrm{m}$, and we consider the physical important region corresponding to $2M < r < R$. Thus, one has to employ the Schwarzschild interior solution for a spherically symmetric body [14]

$$\begin{aligned} g_{00} &= \left[\frac{3}{2}\left(1-\frac{2M}{R}\right)^{1/2} - \frac{1}{2}\left(1-\frac{2M}{R^3}r^2\right)^{1/2}\right]^2 \\ g_{11} &= \left[1-\frac{2M}{R^3}r^2\right]^{-1} \end{aligned} \tag{8}$$

where

$$M \equiv \frac{G}{c^2}M_*\,, \quad M_* = \frac{4\pi R^3}{3}\rho_*\,, \quad \rho_* = \mathrm{const.}$$

One has to impose the physical condition $M/R < 4/9$. Otherwise, the central pressure becomes infinite [15].

The velocity defined in (7) has the explicit form

$$v^2 = ax^2\left[\frac{3}{2}\frac{\sqrt{1-2a}}{\sqrt{1-2ax^2}} - \frac{1}{2}\right] \tag{9}$$

where we have introduced the dimensionless notations $a = M/R$ and $x = r/R$. The above formula shows that the inner rotation curves are rising with $r$. The calculations have been done for $2ax^2 < 1$, which agrees with the condition $M/R < 4/9$.

To first order in $ax^2$, the relation (9) can be approximated to

$$v^2 \approx ax^2 = \frac{Mr^2}{R^3} = \frac{M(r)}{r}$$

pointing out the expected dependence $M(r) \sim r^3$ of the innermost enclosed mass. Such expressions of the enclosed mass in the Galactic Center, as a function of $r$, are usually obtained from the measured rotation curves [2].

The dynamics of a massive scalar field of mass $m_0$ and energy $\omega$ is described by the Gordon equation

$$\frac{1}{\sqrt{g_{00}g_{11}}}\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\sqrt{\frac{g_{00}}{g_{11}}}\frac{\partial\Phi}{\partial r}\right] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial\Phi}{\partial\theta}\right] + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2} - \frac{1}{g_{00}}\frac{\partial^2\Phi}{\partial t^2} - m_0^2\Phi = 0\,. \tag{10}$$

With the variables separation

$$\Phi = F(r)Y_l^m(\theta,\varphi)e^{-i\omega t}, \tag{11}$$

where $Y_l^m$ are the spherical functions, this is leading to the following differential equation for the radial function

$$\frac{1}{\sqrt{g_{00}g_{11}}}\frac{\mathrm{d}}{\mathrm{d}r}\left[r^2\sqrt{\frac{g_{00}}{g_{11}}}\frac{\mathrm{d}F}{\mathrm{d}r}\right] + \left[-l(l+1) + \frac{\omega^2 r^2}{g_{00}} - m_0^2 r^2\right]F = 0 \tag{12}$$

With the change of function

$$u(r) = r\sqrt{h(r)}F(r)\,,$$

where

$$h(r) = \sqrt{\frac{g_{00}}{g_{11}}}$$

the radial equation (12) turns into the Schrodinger form

$$\frac{\mathrm{d}^2 u}{\mathrm{d}r^2} + \left\{ g_{11} \left[ \frac{\omega^2}{g_{00}} - \frac{l(l+1)}{r^2} - m_0^2 \right] - \frac{h''}{2h} - \frac{h'}{rh} + \left( \frac{h'}{2h} \right)^2 \right\} u = 0 \,. \tag{13}$$

For the metric functions (8), the above equation can be solved only by using a numerical approach.

To first order in $a = M/R$, the expressions (8) can be approximated by the simple relations

$$g_{00} \approx 1 - 3a + ax^2 \,, \quad g_{11} \approx 1 + 2ax^2 \tag{14}$$

with $x = r/R$, and the equation (12) becomes

$$\left[ 1 + \frac{3a}{2}(1-x^2) \right] \frac{\mathrm{d}}{\mathrm{d}x} \left\{ x^2 \left[ 1 - \frac{a}{2}(3+x^2) \right] \frac{\mathrm{d}F}{\mathrm{d}x} \right\} + \left\{ -l(l+1) + \omega^2 R^2 x^2 \left[ 1 + a(3-x^2) \right] - m_0^2 R^2 x^2 \right\} F = 0 \,. \tag{15}$$

Using Maple, one may find the solution of the above equation expressed in terms of the Heun general function [13]

$$F = C r^l \, \mathrm{HeunG} \left[ s, q, \alpha, \beta, \gamma, \delta, \frac{3Mr^2}{2R^3} \right], \tag{16}$$

where $C$ is an integration constant and the Heun's parameters are

$$\begin{aligned} & s = 3(1-2a) \,, \quad q \approx -\frac{(\omega^2 - m_0^2)R^2}{2a} + l\left( l + \frac{3}{2} \right) \,, \quad \alpha \approx \frac{\omega R}{\sqrt{3a}} + \frac{2l+3}{4} \\ & \beta \approx -\frac{\omega R}{\sqrt{3a}} + \frac{2l+1}{4} \,, \quad \gamma \approx l + \frac{3}{2} \,, \quad \delta \approx 0 \,. \end{aligned} \tag{17}$$

One may look for a polynomial form of the Heun functions by imposing the condition $\beta = -n$ [7, 8]. This leads to the non-trivial energy spectrum

$$\omega_{nl} = \sqrt{\frac{3M}{R^3}} \left( n + \frac{2l+1}{4} \right), \tag{18}$$

which is depending only on the star's density. These so-called resonant frequencies are a very useful tool to determine the essential characteristics of the black hole.

The theory of the general Heun equation is quite complicated and there are many open questions related to the normalization procedure and the behavior of the solutions around the singular points. That is why, it is very useful to express the analytical solution of the Heun general equation in terms of more familiar functions. One may notice that the Schrodinger-type equation 13), with the relations 14, has the simple expression

$$u'' + \left\{ \omega^2 - m_0^2 - \frac{l(l+1)}{r^2} + a \left[ (\omega^2 - 2m_0^2) r^2 + 3\omega^2 + \frac{3}{2} - 2l(l+1) \right] \right\} u = 0 \tag{19}$$

whose solution is

$$u(r) = r^{-l} \exp \left[ -\sqrt{a(2m_0^2 - \omega^2)} \frac{r^2}{2} \right] U,$$

where $U$ is the confluent hypergeometric function. A closer look at the effective potential

$$V_{eff} = \frac{l(l+1)}{r^2} - a \left[ (\omega^2 - 2m_0^2) r^2 + 3\omega^2 + \frac{3}{2} - 2l(l+1) \right]$$

can lead to important information on the black hole stability against a massive scalar field perturbation.

## 3. THE SCHWARZSCHILD BLACK HOLE IMMERSED IN DARK MATTER

Let us consider the region outside the core, where all the baryonic mass is concentrated. For the Milky Way, the radial coordinate would be in the range $r > 5$ kpc.

The difference between the galaxy mass given by the luminosity and the mass predicted by the rotation velocities is a strong evidence for the existence of spherical haloes of dark matter surrounding the spiral galaxies.

In order to flatten out the expression (9), one should correct the Schwarzschild external metric, by considering additional contributions. One way is to work with the simple model for galaxies proposed in [4].

Thus, for the static spherically symmetric black hole immersed in dark matter, the metric functions $g_{00}(r)$ and $g_{11}(r)$ in the line element (2) are given by:

$$g_{00} = 1 - \frac{2M}{r} + V(r)\,, \quad g_{11}(r) = \left[1 - \frac{2M}{r} + H(r)\right]^{-1},$$

where $M$ is the baryonic mass of the galaxy, enclosed inside the black hole.

Since the contributions $V$ and $H$ are significant only on large scales, one may use the Taylor expansions around the black hole (Schwarzschild) horizon $r_S = 2M$ and the metric functions can be approximated to the expressions [4]

$$\begin{aligned} g_{00} &= 1 - \frac{2M}{r} + 2a_0(r-2M) = \frac{(r-2M)(1+2a_0 r)}{r} \\ g_{11} &= \left[1 - \frac{2M}{r} - 2a_0(r-2M)\right]^{-1} = \frac{r}{(r-2M)(1-2a_0 r)}\,. \end{aligned} \tag{20}$$

Additional quadratic contributions in the metric functions can be induced by inhomogeneities in the cosmic background and are usually associated with distances larger than 1 Mpc.

The equations for the timelike geodesics being given in (6), one may compute the velocity, using the general expression (7). This is given by the formula

$$v = \sqrt{\frac{M}{r} + a_0 r}\,, \tag{21}$$

where the universal constant acceleration $a_0 \approx 10^{-11}\mathrm{m/s}^2$ can be estimated from the Milky Way data [2].

As a function of $r$, the above relation is pointing out the minimum velocity $v_{min}^4 = 4a_0 M$, which agrees with the Tully-Fisher relation [16], between the baryonic mass of the galaxy and the constant velocity in the flat part of the rotation curve.

Let us mention that the result (21) can be obtained in modified theories of gravitation, without the need of the dark sector, as for example by working with the particular expression of the Mannheim–Kazanas (MK) metric [5, 6]

$$g_{00} = g_{11}^{-1} = 1 - \frac{2M}{r} + \gamma r$$

The universal constant $\gamma \approx 3 \times 10^{-28}\,\mathrm{m}^{-1}$ may be related to $a_0$ by $\gamma = 2a_0/c^2$. The unique horizon of the MK metric is given by the positive root of the equation $g_{00} = 0$, and has the expression

$$r_h = \frac{\sqrt{1+8\gamma M} - 1}{2\gamma}\,.$$

For $r_* = \sqrt{2M/\gamma}$, where the velocity has the minimum value $v_{min}^4 = 2\gamma M$, the MK metric function is $g_{00} = 1$, pointing out the existence of a Minkowski region.

The main problem with the MK metric is the fact that the Einstein's tensor components, namely

$$G_{11} = \frac{2\gamma}{r} = -G_{44}\ , \quad G_{22} = G_{33} = \frac{\gamma}{r} \tag{22}$$

are leading to a negative energy density, $T_{44} = G_{44}$, corresponding to an exotic type of matter surrounding the astrophysical object.

From this point of view, the metric functions (20) have the advantage of yielding a positive energy density

$$\rho = \frac{4a_0(r-M)}{r^2}\,. \tag{23}$$

Moreover, all the energy conditions are satisfied, outside the black hole horizon which coincides with the Schwarzschild one $r_S = 2M$. For a detailed discussion, we recommend [4]. To first order in $a_0$, the radial pressure $P_r = G_{11}$, with $G_{11}$ computed in the tetradic frame whose dual basis is (3), has the expression

$$P_r \approx -\frac{4M(M-a_0r^2)}{r^2(r-2M)^2}\,,$$

while the other two pressures, characterizing an anisotropic fluid, are:

$$P_\theta = P_\varphi \approx \frac{4M^2}{r(r-2M)^3}\,.$$

All the pressures are vanishing for outer regions of galaxies and one may notice the particular value $r_* = \sqrt{M/a_0}$, where the radial pressure turns from negative to positive values.

The Gordon equation (10), with the variables separation (11), leads to the radial equation (12). The absolute value of the numerical solution is represented in Fig. 1, for $2M < r < 1/(2a_0)$. One may notice the existence of zero probability regions, starting with the one on the horizon. Also, there is a maximum probability just after the Schwarzschild horizon and a series of decreasing maxima as $r$ is increasing.

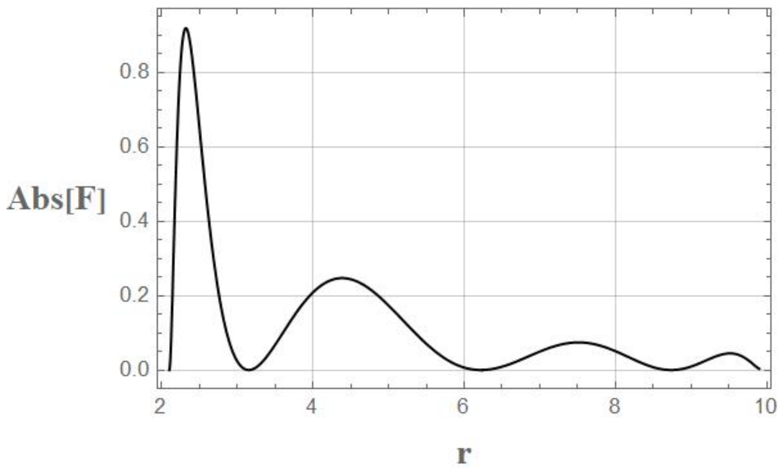


Fig. 1 – The absolute value of the solution of (12) as a function of $r$.

This behavior is similar to the case of the Schwarzschild metric, for which the Gordon equation can be exactly solved. The solution, valid for $r \in [2M, \infty)$, is given by the Heun confluent functions [9]

$$F(r) \sim e^{\frac{\alpha x}{2}}\, x^{\frac{\beta}{2}}\, \mathrm{HeunC}\left[\alpha, \beta, \gamma, \delta, \eta, x\right]$$

with the variable $x = 1 - r/(2M)$ and the parameters

$$\alpha = -4\mathrm{i}M\sqrt{\omega^2 - m_0^2}, \quad \beta = \pm 4\mathrm{i}M\omega, \quad \gamma = 0, \quad \delta = 4M^2\left(m_0^2 - 2\omega^2\right), \quad \eta = -\delta - \ell(\ell+1)$$

The energy levels and the zero probability regions are obtained once we impose the condition [7, 8]

$$\frac{\delta}{\alpha} = -\left[n + 1 + \frac{\beta}{2}\right]$$

for which the Heun confluent functions get a polynomial form. In the massless case, one obtains the equally-spaced Schwarzschild imaginary spectrum

$$\omega_n = -\frac{\mathrm{i}\,(n+1)}{4M}.$$

## 4. CONCLUDING REMARKS

In the present paper, we have considered two main regions of the galaxy, considered as a continuous gravitationally bound object. The first one is the internal region, $2M < r < R$, while the second one is outside the core, on the scale of kilo-parsecs.

For $r < R$, the geodesic equations can be derived in the same way as for the exterior Schwarzschild solution. The energy and angular momentum are conserved quantities and one may fix the direction of the angular momentum along the polar axis by setting $\theta = \pi/2$. We have assumed a constant density of the baryonic matter in the inner region of radius $R$ and we have used the Schwarzschild interior metric (8). The rising Milky Ways rotation curves, as the ones given in (9), have been experimentally seen for $r < 2$ kpc [17] .

The solution to the Gordon equation (10) is expressed in terms of the Heun general functions whose theory has been used to obtain the energy levels $\omega_{nl}$ given in (18). The so-called quasinormal frequencies are very important since they are offering important information on the black hole's characteristics [9]. In general, the discrete spectra is complex, with the real part representing the actual frequency of the oscillation and the imaginary part representing the damping.

As it is known, the effects of dark matter should be taken into account outside the core, on the scale of kilo-parsecs. For these regions, we have used the expressions (20), derived in [4]. The linear term in $r$ is very small but it will weakly break the asymptotic flatness of the Schwarzschild metric. For large distances, this term will produce a constant inward acceleration which may be comparable to the Newtonian one. Similarly to the MK metric [6], the expression of $g_{00}$ in (20) is leading to flat galactic rotational curves, for large $r$ values. But contrary to the MK solution which was derived in Conformal Gravity and is supported by exotic matter with negative energy density, the metric functions (20) are compatible with the positive energy density given in (23). In the outer regions, the galactic components behave and interact as a pressureless fluid.

The absolute value of the numerical solution to the radial equation (12), with the metric functions (20), is represented in the figure 1. Similarly to the Schwarzschild case, this has an oscillatory behavior, with a prominent maximum just outside the Schwarzschild horizon and is vanishing both on the Schwarzschild and on the cosmological horizons.

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