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ASYMPTOTIC FORMULAS FOR CONVERGENCES MOTIVATED BY KOROVKIN'S THEOREM

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Abstract. The aim of the present paper is to prove asymptotic formulas.

***Key words**:* Korovkin's theorem, uniform convergence, asymptotic approximations, differentiability at a boundary point, multiple integral.

1. INTRODUCTION

The celebrated theorem of Korovkin gives us conditions for uniform approximation of continuous functions on a compact interval via sequences of positive linear operators, see [7], [8]. More precisely, if $(T_n)_{n\in\mathbb{N}}$ is a sequence of positive linear operators that map $C([0,1])$ into itself such that the sequence $(T_n(f))_{n\in\mathbb{N}}$ converges to f uniformly on $[0,1]$ for each of the test functions $e_k(x)=x^k$, where $k=0,1,2$, then this sequence also converges to f uniformly on $[0,1]$ for every $f\in C([0,1])$. An immediate consequence of it is the fact that

$$\lim_{n\to\infty} n\int_0^1 x^n f(x)\mathrm{d}x = f(1) \tag{1}$$

for every continuous function $f:[0,1]\to\mathbb{R}$, which means the weak convergence of the sequence $nx^n\mathrm{d}x$ of measures to the Dirac measure δ_1, see [8, Exercise 1, page 54]. As the weak convergence of measures is one of the most important tools in partial differential equations, probability theory, applied and theoretic statistics etc, see [3], it is natural to investigate the extension of Korovkin's limit (1) to the case of functions of several variables. An inspection of Korovkin's argument easily yields the following generalization for several variables:

THEOREM 1. (The extension of Korovkin's theorem for several variables) *Suppose that X is a compact subset of the Euclidean space $\mathbb{R}^k$ and let $(T_n)_{n\in\mathbb{N}}$ be a sequence of linear and positive operators from $C(X)$ into itself such that*

$$\lim_{n\to\infty} T_n(f) = f \quad \textit{uniformly on } X \tag{2}$$

for each of the test functions $\mathbf{1}$, pr_1, ..., pr_k and $\sum_{i=1}^{k} pr_i^2$. Then

$$\lim_{n\to\infty} T_n(f) = f \quad \textit{uniformly on } X \tag{3}$$

for all functions $f\in C(X)$.

The family of test functions used here is built via the canonical projections on the Euclidean k-dimensional space $pr_i(x_1,...,x_k)=x_i$, $i=1$, ..., k.

For details and various generalizations see Altomare [1], Bucur and Păltineanu [4], Gal and Niculescu [5], [6] and Niculescu [9]. In this paper, we prove some possible extensions of the limit (1) to the double integrals on a triangle, see Theorem 2 and Proposition 3. We establish then a general result regarding the asymptotic evaluation for functions differentiable at $(1,0)$, see Theorem 3 and as a consequence, we obtain the asymptotic evaluation for functions differentiable at $(1,0)$ in the case of two concrete situations, Theorems 4 and 5. We define $\Delta = \left\{(x,y) \in [0,1]^2 \mid x+y \leq 1\right\}$ and denote as usual $C(\Delta) = \{f : \Delta \to \mathbb{R} \mid f \text{ is continuous}\}$ which is a real linear space and for $f \in C(\Delta)$ we define $\|f\| = \sup_{(x,y)\in\Delta} |f(x,y)|$. By $\mathbf{1}$ we denote the constant function equal with 1. In the proof of the asymptotic evaluations we need the concept of differentiability at the point $(1,0) \in \Delta$. Since $(1,0)$ is not an interior point of Δ this concept needs an explanation. Precisely it refers to differentiability at a point which is the vertex of a non-degenerate cone.

Definition 1. A function $f : \Delta \to \mathbb{R}$ is called **differentiable at** $(1,0)$ if and only if there exist A, $B \in \mathbb{R}$ such that

$$\lim_{(x,y)\to(1,0),(x,y)\in\Delta} \frac{f(x,y) - f(1,0) - A(x-1) - By}{|x-1| + |y|} = \lim_{(x,y)\to(1,0),(x,y)\in\Delta} \frac{f(x,y) - f(1,0) - A(x-1) - By}{1-x+y} = 0.$$

Let us note that if $f : \Delta \to \mathbb{R}$ is differentiable at $(1,0)$ then: $\forall \varepsilon > 0$, $\exists \delta_\varepsilon > 0$ such that $\forall (x,y) \in \Delta - \{(1,0)\}$ with $|x-1| + |y| < \delta_\varepsilon$ we have $\frac{|f(x,y)-f(1,0)-A(x-1)-By|}{|x-1|+|y|} < \varepsilon$, or equivalent, $\forall (x,y) \in \Delta$ with $|x-1| + |y| < \delta_\varepsilon$ we have

$$|f(x,y) - f(1,0) - A(x-1) - By| \leq \varepsilon (|x-1| + |y|).$$

For $y = 0$ we get, $|f(x,0) - f(1,0) - A(x-1)| \leq \varepsilon |x-1|$, $\forall |x-1| < \delta_\varepsilon$, that is, $A = \lim_{x\to1,x<1} \frac{f(x,0)-f(1,0)}{x-1} = \frac{\partial f}{\partial x}(1,0)$. For $x = 1-y$ we get $|f(1-y,y) - f(1,0) + Ay - By| \leq 2\varepsilon |y|$, $\forall |y| < \frac{\delta_\varepsilon}{2}$. Hence there exists the directional derivative of f at $(1,0)$ along the line l: $x+y = 1$, that is,

$$\frac{\partial f}{\partial l}(1,0) := \lim_{y\to0,y>0} \frac{f(1-y,y) - f(1,0)}{y}$$

and $\frac{\partial f}{\partial l}(1,0) = B - A$, thus $B = \frac{\partial f}{\partial x}(1,0) + \frac{\partial f}{\partial l}(1,0)$.

All notation and notions used and not defined in this paper are standard, see [4].

2. THE CONVERGENCE

THEOREM 2. *Let $f : \Delta \to \mathbb{R}$ be a continuous function. Then*

$$\lim_{n\to\infty} n^2 \iint_\Delta x^n (1-y)^n f(x,y)\,\mathrm{d}x\mathrm{d}y = \frac{f(1,0)}{2}.$$

Proof. Let $L_n : C(\Delta) \to \mathbb{R}$ be the sequence of functionals defined by $L_n(f) = n^2 \iint_\Delta x^n (1-y)^n f(x,y)\,\mathrm{d}x\mathrm{d}y$. By calculations we get $L_n(\mathbf{1}) = \frac{n^2}{2(n+1)^2}$, $L_n(pr_1) = \frac{n^2}{(n+2)(2n+3)}$, $L_n\left(pr_1^2\right) = \frac{n^2}{2(n+3)(n+2)}$, $L_n(pr_2) = \frac{n^2}{2(n+1)^2(2n+3)}$, $L_n\left(pr_2^2\right) = \frac{n^2}{2(n+1)^2(n+2)(2n+3)}$. We deduce that $\lim_{n\to\infty} L_n(f) = \frac{f(1,0)}{2}$, $\forall f \in \left\{\mathbf{1}, pr_1, pr_2, pr_1^2 + pr_2^2\right\}$. According to Theorem 1 one can conclude that $\lim_{n\to\infty} L_n(f) = \frac{f(1,0)}{2}$ for all $f \in C(\Delta)$. □

3. A GENERAL ASYMPTOTIC EVALUATION

The aim of the present paper is to provide an asymptotic evaluation for the convergence established in Theorem 2. This is done in the context of functions differentiable at $(1,0)$ and makes the objective of Theorem 4 below. For this we prove the following general result.

THEOREM 3. *Let $K_n : \Delta \to [0,\infty)$ be a sequence of continuous functions and $L_n : C(\Delta) \to \mathbb{R}$ the functional defined by*

$$L_n(f) = \iint_\Delta K_n(x,y) f(x,y)\,\mathrm{d}x\mathrm{d}y.$$

Suppose that there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $\iint_\Delta K_n(x,y)\left((1-x)+y\right)\mathrm{d}x\mathrm{d}y > 0$. Then the following assertions are equivalent:

(i)

$$\lim_{n\to\infty} \frac{\iint_\Delta \left((1-x)+y\right)^2 K_n(x,y)\,\mathrm{d}x\mathrm{d}y}{\iint_\Delta \left((1-x)+y\right) K_n(x,y)\,\mathrm{d}x\mathrm{d}y} = 0.$$

(ii) For every $f \in C(\Delta)$ differentiable at $(1,0)$ we have

$$\lim_{n\to\infty} \frac{L_n(f) - f(1,0)\lambda_n + \frac{\partial f}{\partial x}(1,0)\alpha_n - \left(\frac{\partial f}{\partial x}(1,0) + \frac{\partial f}{\partial l}(1,0)\right)\beta_n}{\alpha_n + \beta_n} = 0$$

where $\lambda_n = \iint_\Delta K_n(x,y)\,\mathrm{d}x\mathrm{d}y$, $\alpha_n = \iint_\Delta (1-x) K_n(x,y)\,\mathrm{d}x\mathrm{d}y$, $\beta_n = \iint_\Delta y K_n(x,y)\,\mathrm{d}x\mathrm{d}y$.

Proof. (i)$\Rightarrow$(ii). Since f is differentiable at $(1,0)$, $\lim\limits_{(x,y)\to(1,0),(x,y)\in\Delta} \frac{g(x,y)}{|x-1|+|y|} = 0$, where $g : \Delta \to \mathbb{R}$, $g(x,y) = f(x,y) - f(1,0) - A(x-1) - By$, see the Definition 1. Let $\varepsilon > 0$. Then there exists $\delta_\varepsilon > 0$ such that $\forall (x,y) \in \Delta$ with $|x-1|+|y| < \delta_\varepsilon$ we have $|g(x,y)| \leq \frac{\varepsilon}{2}(|x-1|+|y|)$. We prove that $\forall (x,y) \in \Delta$ we have

$$|g(x,y)| \leq \frac{\varepsilon}{2}(|x-1|+|y|) + \eta_\varepsilon (|x-1|+y)^2. \tag{4}$$

where $\eta_\varepsilon = \frac{\|g\|}{\delta_\varepsilon^2}$. This argument has its origins in the Korovkin's proof of his theorem, see [8, page 13], or [10, Lemma 1]. Indeed, let $(x,y) \in \Delta$. We can have the situations: a) $|x-1|+|y| < \delta_\varepsilon$. In this case $|g(x,y)| \leq \frac{\varepsilon}{2}(|x-1|+|y|) \leq \frac{\varepsilon}{2}(|x-1|+|y|) + \frac{\|g\|}{\delta_\varepsilon^2}(|x-1|+|y|)^2$. b) $|x-1|+|y| \geq \delta_\varepsilon$. In this case $1 \leq \frac{(|x-1|+|y|)^2}{\delta_\varepsilon^2}$ and then

$$|g(x,y)| \leq \|g\| \leq \|g\| \frac{(|x-1|+|y|)^2}{\delta_\varepsilon^2} \leq \frac{\varepsilon}{2}(|x-1|+|y|) + \frac{\|g\|}{\delta_\varepsilon^2}(|x-1|+|y|)^2.$$

Let $n \geq n_0$. From (4) multiplying with $K_n(x,y) \geq 0$ and then, by integration we get

$$|L_n(f) - f(1,0)\lambda_n + A\alpha_n - B\beta_n| \leq$$

$$\iint_\Delta |K_n(x,y) f(x,y) - f(1,0) K_n(x,y) + A(1-x) K_n(x,y) - ByK_n(x,y)|\,\mathrm{d}x\mathrm{d}y$$
$$\leq \frac{\varepsilon}{2}\iint_\Delta K_n(x,y)\left((1-x)+y\right)\mathrm{d}x\mathrm{d}y + \eta_\varepsilon \iint_\Delta K_n(x,y)\left((1-x)+y\right)^2 \mathrm{d}x\mathrm{d}y.$$

Thus

$$\left|\frac{L_n(f) - f(1,0)\lambda_n + A\alpha_n - B\beta_n}{\alpha_n + \beta_n}\right| \leq \frac{\varepsilon}{2} + \eta_\varepsilon \frac{\iint_\Delta K_n(x,y)\left((1-x)+y\right)^2 \mathrm{d}x\mathrm{d}y}{\iint_\Delta K_n(x,y)\left((1-x)+y\right)\mathrm{d}x\mathrm{d}y}.$$

By (i) there exists $n_\varepsilon \geq n_0$ such that $\forall n \geq n_\varepsilon$ we have $\frac{\iint_\Delta K_n(x,y)((1-x)+y)^2 \mathrm{d}x\mathrm{d}y}{\iint_\Delta K_n(x,y)((1-x)+y)\mathrm{d}x\mathrm{d}y} < \frac{\varepsilon}{2(\eta_\varepsilon+1)}$ and hence $\forall n \geq n_\varepsilon$ we have $\left|\frac{L_n(f) - f(1,0)\lambda_n + A\alpha_n - B\beta_n}{\alpha_n + \beta_n}\right| < \varepsilon$. Thus (ii) is proved. (ii)$\Rightarrow$(i). It follows from (ii) applied for the function $f : \Delta \to \mathbb{R}$, $f(x,y) = \left((1-x)+y\right)^2$; $\frac{\partial f}{\partial x}(1,0) = 0$, $\frac{\partial f}{\partial l}(1,0) = 0$. $\square$

4. SOME EXAMPLES

We need latter the following calculations.

PROPOSITION 1. *The following formulas hold true:*

$$\iint_{\Delta} x^n(1-y)^n(1-x)\mathrm{d}x\mathrm{d}y = \frac{3n+4}{2(n+1)^2(n+2)(2n+3)};$$

$$\iint_{\Delta} x^n(1-y)^n y\,\mathrm{d}x\mathrm{d}y = \frac{1}{2(n+1)^2(2n+3)};$$

$$\iint_{\Delta} x^n(1-y)^n(1-x)^2\mathrm{d}x\mathrm{d}y = \frac{7n+9}{2(n+1)^2(n+2)(n+3)(2n+3)};$$

$$\iint_{\Delta} x^n(1-y)^n(1-x)y\,\mathrm{d}x\mathrm{d}y = \frac{1}{2(n+1)^2(n+2)^2};$$

$$\iint_{\Delta} x^n(1-y)^n y^2\mathrm{d}x\mathrm{d}y = \frac{1}{2(n+1)^2(n+2)(2n+3)}.$$

Proof. We will use the well-known equality

$$\iint_{\Delta} f(x,y)\,\mathrm{d}x\mathrm{d}y = \int_0^1 \mathrm{d}x \int_0^{1-x} f(x,y)\,\mathrm{d}y \tag{5}$$

for continuous functions $f:\Delta\to\mathbb{R}$, see [2, page 247]. For more details regarding the multiple Riemann integral we recommend the reader the excellent treatment of this concept in the book of Boboc, see [2]. The first two equalities follows by direct calculations from the relation (5) and we omit it. For the last equalities we will use the equality (5) and integration by parts. We have

$$\iint_{\Delta} x^n(1-y)^n(1-x)^2\mathrm{d}x\mathrm{d}y = \frac{1}{n+1}\int_0^1 x^n\left(1-x^{n+1}\right)(1-x)^2\mathrm{d}x$$

$$= \frac{1}{(n+1)^2}\int_0^1\left(x^{n+1}-\frac{x^{2n+2}}{2}\right)'(1-x)^2\mathrm{d}x = \frac{2}{(n+1)^2}\int_0^1\left(x^{n+1}-\frac{x^{2n+2}}{2}\right)(1-x)\,\mathrm{d}x$$

$$= \frac{2}{(n+1)^2}\int_0^1\left(\frac{x^{n+2}}{n+2}-\frac{x^{2n+3}}{2(2n+3)}\right)'(1-x)\,\mathrm{d}x = \frac{2}{(n+1)^2}\int_0^1\left(\frac{x^{n+2}}{n+2}-\frac{x^{2n+3}}{2(2n+3)}\right)\mathrm{d}x$$

$$= \frac{2}{(n+1)^2}\left(\frac{1}{(n+2)(n+3)}-\frac{1}{4(n+2)(2n+3)}\right) = \frac{7n+9}{2(n+1)^2(n+2)(n+3)(2n+3)}.$$

We have

$$\iint_{\Delta} x^n(1-y)^n(1-x)\,y\,\mathrm{d}x\mathrm{d}y = \iint_{\Delta} x^n(1-x)(1-y)^n\mathrm{d}x\mathrm{d}y - \iint_{\Delta} x^n(1-x)(1-y)^{n+1}\mathrm{d}x\mathrm{d}y$$

$$= \frac{1}{n+1}\int_0^1\left(x^n-x^{n+1}\right)\left(1-x^{n+1}\right)\mathrm{d}x - \frac{1}{n+2}\int_0^1\left(x^n-x^{n+1}\right)\left(1-x^{n+2}\right)\mathrm{d}x.$$

Integrating by parts

$$\frac{1}{n+1}\int_0^1\left(x^n-x^{n+1}\right)\left(1-x^{n+1}\right)\mathrm{d}x = \frac{1}{n+1}\int_0^1\left(\frac{x^{n+1}}{n+1}-\frac{x^{n+2}}{n+2}\right)'\left(1-x^{n+1}\right)\mathrm{d}x$$

$$= \int_0^1\left(\frac{x^{n+1}}{n+1}-\frac{x^{n+2}}{n+2}\right)x^n\mathrm{d}x = \frac{1}{2(n+1)^2} - \frac{1}{(n+2)(2n+3)} = \frac{3n+4}{2(n+1)^2(n+2)(2n+3)}$$

and similar

$$\frac{1}{n+2}\int_0^1 \left(x^n - x^{n+1}\right)\left(1-x^{n+2}\right)\mathrm{d}x = \frac{1}{n+2}\int_0^1 \left(\frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2}\right)'\left(1-x^{n+2}\right)\mathrm{d}x$$

$$= \int_0^1 \left(\frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2}\right)x^{n+1}\mathrm{d}x = \frac{1}{(n+1)(2n+3)} - \frac{1}{2(n+2)^2} = \frac{3n+5}{2(n+1)(n+2)^2(2n+3)}.$$

It follows that $\iint_\Delta x^n(1-y)^n(1-x)y\,\mathrm{d}x\mathrm{d}y = \frac{1}{2(n+1)^2(n+2)^2}$. We have also

$$\iint_\Delta x^n(1-y)^n y^2\mathrm{d}x\mathrm{d}y = \iint_\Delta x^n\left((1-y)^{n+2} - 2(1-y)^{n+1} + (1-y)^n\right)\mathrm{d}x\mathrm{d}y$$

$$= \int_0^1 x^n\left(\frac{1-x^{n+3}}{n+3} - \frac{2\left(1-x^{n+2}\right)}{n+2} + \frac{1-x^{n+1}}{n+1}\right)\mathrm{d}x = \frac{1}{2(n+1)^2(n+2)(2n+3)}.$$

□

We are now in the position to prove the asymptotic evaluation of the sequence from Theorem 2.

THEOREM 4. *Let $L_n : C(\Delta) \to \mathbb{R}$ be the functional defined by*

$$L_n(f) = n^2\iint_\Delta x^n(1-y)^n f(x,y)\,\mathrm{d}x\mathrm{d}y.$$

Then for every $f \in C(\Delta)$ differentiable at $(1,0)$ we have

$$\lim_{n\to\infty} n\left[L_n(f) - \frac{f(1,0)}{2}\right] = -f(1,0) - \frac{1}{2}\frac{\partial f}{\partial x}(1,0) + \frac{1}{4}\frac{\partial f}{\partial l}(1,0).$$

Proof. Let $K_n(x,y) = n^2x^n(1-y)^n$. From Proposition 1 we deduce $\lim_{n\to\infty} n^2\iint_\Delta K_n(x,y)\left((1-x)+y\right)^2\mathrm{d}x\mathrm{d}y =$ 3 and $\lim_{n\to\infty} n\iint_\Delta K_n(x,y)\left((1-x)+y\right)\mathrm{d}x\mathrm{d}y = 1$. The condition (i) in Theorem 3 is satisfied and hence for every $f \in C(\Delta)$ differentiable at $(1,0)$ we have

$$\lim_{n\to\infty}\frac{L_n(f) - f(1,0)\lambda_n + \frac{\partial f}{\partial x}(1,0)\alpha_n - \left(\frac{\partial f}{\partial x}(1,0) + \frac{\partial f}{\partial l}(1,0)\right)\beta_n}{\alpha_n+\beta_n} = 0.$$

Since $\lim_{n\to\infty} n\alpha_n = \frac{3}{4}$, $\lim_{n\to\infty} n\beta_n = \frac{1}{4}$ we get

$$\lim_{n\to\infty} n\left[L_n(f) - f(1,0)\lambda_n\right] = -\frac{3}{4}\frac{\partial f}{\partial x}(1,0) + \frac{1}{4}\left(\frac{\partial f}{\partial x}(1,0) + \frac{\partial f}{\partial l}(1,0)\right) = -\frac{1}{2}\frac{\partial f}{\partial x}(1,0) + \frac{1}{4}\frac{\partial f}{\partial l}(1,0).$$

Also $\lambda_n = n^2\iint_\Delta x^n(1-y)^n\mathrm{d}x\mathrm{d}y = \frac{n^2}{2(n+1)^2}$, $\lim_{n\to\infty} n\left(\lambda_n - \frac{1}{2}\right) = -1$. Then passing to the limit in the equality

$$n\left[L_n(f) - \frac{f(1,0)}{2}\right] = n\left[L_n(f) - f(1,0)\lambda_n\right] + f(1,0)\,n\left(\lambda_n - \frac{1}{2}\right)$$

we get the limit from the statement. □

We remark the asymetric form of the asymptotic evaluation in Theorem 4. Our next objective is to give a new example of the sequences of the operators as those from Theorems 2 and 4. We need the folowing result.

PROPOSITION 2. *Let $(x_n)_{n\geq 1}$ be a sequence of the real numbers and $\alpha > 1$ such that $\lim\limits_{n\to\infty} n^{\alpha} x_n = b \in \mathbb{R}$. Then $\lim\limits_{n\to\infty} n^{\alpha-1}(x_n + \cdots + x_{2n-1}) = \frac{(2^{\alpha-1}-1)b}{2^{\alpha-1}(\alpha-1)}$.*

Proof. Let $\varepsilon > 0$. There exists $n_\varepsilon \in \mathbb{N}$ such that $\forall n \geq n_\varepsilon$ we have $|n^{\alpha} x_n - b| < \varepsilon$, or $\left|x_n - \frac{b}{n^{\alpha}}\right| < \frac{\varepsilon}{n^{\alpha}}$. Let $n \geq n_\varepsilon$. For every $k = 0, ..., n-1$ we have $n+k \geq n_\varepsilon$ and hence $\left|x_{n+k} - \frac{b}{(n+k)^{\alpha}}\right| < \frac{\varepsilon}{(n+k)^{\alpha}}$. We deduce that

$$\left|\sum_{k=0}^{n-1} x_{n+k} - b\sum_{k=0}^{n-1}\frac{1}{(n+k)^{\alpha}}\right| \leq \sum_{k=0}^{n-1}\left|x_{n+k} - \frac{b}{(n+k)^{\alpha}}\right| < \varepsilon \sum_{k=0}^{n-1}\frac{1}{(n+k)^{\alpha}}$$

or $\left|\frac{\sum_{k=0}^{n-1} x_{n+k}}{\sum_{k=0}^{n-1}\frac{1}{(n+k)^{\alpha}}} - b\right| < \varepsilon$. It follows that $\lim\limits_{n\to\infty}\frac{\sum_{k=0}^{n-1} x_{n+k}}{\sum_{k=0}^{n-1}\frac{1}{(n+k)^{\alpha}}} = b$, or, $\lim\limits_{n\to\infty}\frac{n^{\alpha-1}\sum_{k=0}^{n-1} x_{n+k}}{\frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{\left(1+\frac{k}{n}\right)^{\alpha}}} = b$. Since $\lim\limits_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{\left(1+\frac{k}{n}\right)^{\alpha}} = \int_0^1 \frac{\mathrm{d}x}{(x+1)^{\alpha}} = \frac{2^{\alpha-1}-1}{2^{\alpha-1}(\alpha-1)}$ we get the limit from the statement. □

PROPOSITION 3. *Let $f : \Delta \to \mathbb{R}$ be a continuous function. Then*

$$\lim_{n\to\infty} n \iint_{\Delta} x^n (1-y)^n \left(\sum_{k=0}^{n-1} x^k (1-y)^k\right) f(x,y)\,\mathrm{d}x\mathrm{d}y = \frac{f(1,0)}{4}.$$

Proof. It follows from Theorem 2 and Proposition 2 ($\alpha = 2$). □

THEOREM 5. *Let $A_n : C(\Delta) \to \mathbb{R}$ be the functional defined by*

$$A_n(f) = n \iint_{\Delta} x^n (1-y)^n \left(\sum_{k=0}^{n-1} x^k (1-y)^k\right) f(x,y)\,\mathrm{d}x\mathrm{d}y.$$

Then for every $f \in C(\Delta)$ differentiable at $(1,0)$ we have

$$\lim_{n\to\infty} n\left[A_n(f) - \frac{f(1,0)}{4}\right] = -\frac{3f(1,0)}{16} - \frac{3}{16}\frac{\partial f}{\partial x}(1,0) + \frac{3}{32}\frac{\partial f}{\partial l}(1,0).$$

Proof. Let $V_n : C(\Delta) \to \mathbb{R}$ be defined by $V_n(f) = \iint_{\Delta} x^n (1-y)^n f(x,y)\,\mathrm{d}x\mathrm{d}y$. Let $f \in C(\Delta)$ be differentiable at $(1,0)$. From Theorem 4, $\lim\limits_{n\to\infty} n\left[n^2 V_n(f) - a\right] = b$ with $a = \frac{f(1,0)}{2}$, $b = -f(1,0) - \frac{1}{2}\frac{\partial f}{\partial x}(1,0) + \frac{1}{4}\frac{\partial f}{\partial l}(1,0)$. If, for every $n \in \mathbb{N}$ we define $R_n = n^2 V_n(f) - a - \frac{b}{n}$ then, $n^2 V_n(f) = a + \frac{b}{n} + R_n$ and $\lim\limits_{n\to\infty} nR_n = 0$. From $V_n(f) = \frac{a}{n^2} + \frac{b}{n^3} + \frac{R_n}{n^2}$, by summation we get $\sum_{k=n}^{2n-1} V_k(f) = a\sum_{k=n}^{2n-1}\frac{1}{k^2} + b\sum_{k=n}^{2n-1}\frac{1}{k^3} + E_n$, $E_n = \sum_{k=n}^{2n-1} r_k$, where $r_n = \frac{R_n}{n^2}$. Since $\lim\limits_{n\to\infty} n^3 r_n = \lim\limits_{n\to\infty} nR_n = 0$ from Proposition 2, $\lim\limits_{n\to\infty} n^2 E_n = 0$, that is $E_n = o\left(\frac{1}{n^2}\right)$. Hence

$$\sum_{k=n}^{2n-1} V_k(f) = a\sum_{k=n}^{2n-1}\frac{1}{k^2} + b\sum_{k=n}^{2n-1}\frac{1}{k^3} + o\left(\frac{1}{n^2}\right).$$

Since $\sum_{k=n}^{2n-1}\frac{1}{k^3} = \frac{1}{n^3}\sum_{k=0}^{n-1}\frac{1}{\left(1+\frac{k}{n}\right)^3} = \frac{1}{n^2}t_n$ and $\lim\limits_{n\to\infty} t_n = \int_0^1 \frac{\mathrm{d}x}{(x+1)^3} = \frac{3}{8}$ we have $\sum_{k=n}^{2n-1}\frac{1}{k^3} = \frac{3}{8n^2} + o\left(\frac{1}{n^2}\right)$. Let us note the equality $\sum_{k=n}^{2n-1}\frac{1}{k^2} = \frac{1}{n^2}\sum_{k=0}^{n-1}\frac{1}{\left(1+\frac{k}{n}\right)^2} = \frac{1}{n}s_n$, where $s_n = \frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{\left(1+\frac{k}{n}\right)^2}$. As is well-known, for functions $\varphi \in C^1$ the following evaluation holds $\frac{1}{n}\sum_{k=0}^{n-1}\varphi\left(\frac{k}{n}\right) = \int_0^1 \varphi(x)\,\mathrm{d}x - \frac{\varphi(1)-\varphi(0)}{2n} + o\left(\frac{1}{n}\right)$, which for $\varphi(x) = \frac{1}{(x+1)^2}$, gives us that

$s_n = \frac{1}{2} + \frac{3}{8n} + o\left(\frac{1}{n}\right)$ and hence $\sum\limits_{k=n}^{2n-1} \frac{1}{k^2} = \frac{1}{2n} + \frac{3}{8n^2} + o\left(\frac{1}{n^2}\right)$. We get $\sum\limits_{k=n}^{2n-1} V_k(f) = \frac{a}{2n} + \left(\frac{3a}{8} + \frac{3b}{8}\right)\frac{1}{n^2} + o\left(\frac{1}{n^2}\right)$, or $n\sum\limits_{k=n}^{2n-1} V_k(f) = \frac{a}{2} + \left(\frac{3a}{8} + \frac{3b}{8}\right)\frac{1}{n} + o\left(\frac{1}{n}\right)$. Since $A_n(f) = n\sum\limits_{k=n}^{2n-1} V_k(f)$ the limit from the statement follows. □

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