A result on $P_{2,3}$-factor uniform graphs

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Abstract: Let $k \geq 2$ be an integer, and let $G$ be a graph. A $P_{2,3}$-factor of a graph $G$ is a spanning subgraph $F$ of $G$ such that each component of $F$ is a path of order at least $k$. A graph $G$ is a $P_{2,3}$-factor uniform graph if $G$ has a $P_{2,3}$-factor including $e_1$ and excluding $e_2$ for any two distinct edges $e_1$ and $e_2$ of $G$. In this article, we verify that a 3-edge-connected graph $G$ is a $P_{2,3}$-factor uniform graph if its sun toughness $s(G) > 1$. Furthermore, we show that the two conditions on edge-connectivity and sun toughness are sharp.

Key words: graph; edge-connectivity; sun toughness; $P_{2,3}$-factor; $P_{2,3}$-factor uniform graph.

1. INTRODUCTION

We deal with only finite, undirected and simple graphs, which have neither loops nor multiple edges. Let $G$ be a graph. We denote by $V(G)$, $E(G)$ and $I(G)$ the vertex set, the edge set and the isolated vertex set of $G$, respectively, and write $d_G(v) = |I(G)|$. For any $v \in V(G)$, we use $d_G(v)$ to denote the degree of $v$ in $G$. For any $X \subseteq V(G)$, $G[X]$ is a subgraph induced by $X$ of $G$ with $V(G[X]) = X$ and $E(G[X]) = \{uv \in E(G) : u, v \in X\}$, and write $G - X = G[V(G) \setminus X]$. For any $E' \subseteq E(G)$, we denote by $G - E'$ the subgraph obtained from $G$ by deleting $E'$. A vertex subset $X$ of $G$ is independent if no two vertices in $X$ are adjacent to each other. The number of connected components of $G$ is denoted by $\omega(G)$. A path on $n$ vertices is denoted by $P_n$ and a complete graph on $n$ vertices is denoted by $K_n$. Given two graphs $G_1$ and $G_2$, we use $G_1 \lor G_2$ to denote the graph obtained from $G_1 \cup G_2$ by adding all the edges joining a vertex of $G_1$ to a vertex of $G_2$.

Let $k \geq 2$ be an integer. A spanning subgraph $F$ of a graph $G$ is called a $P_{2,k}$-factor of $G$ if each component of $F$ is a path of order at least $k$. A graph $G$ is called a $P_{2,k}$-factor covered graph if for any $e \in E(G)$, $G$ has a $P_{2,k}$-factor including $e$.

Wang [1] gave a necessary and sufficient condition for a bipartite graph having a $P_{2,3}$-factor. Kaneko [2] characterized a graph with a $P_{2,3}$-factor, which is a generalization of Wang’s result. Kano, Katona and Király [3] gave a simple proof of Kaneko’s result. Zhang and Zhou [4] first defined the concept of a $P_{2,k}$-factor covered graph, and then showed a necessary and sufficient condition for a graph to be a $P_{2,3}$-factor covered graph. Zhou [5] obtained a new result on the existence of $P_{2,3}$-factor covered graphs. Some other results on graph factors see [6, 21].

A graph $R$ is called a factor-critical graph if $R - \{v\}$ admits a perfect matching for every $v \in V(R)$. A graph $H$ is defined as a sun if $H = K_1, H = K_2$ or $H$ is the corona of a factor-critical graph $R$ with order at least three, i.e., $H$ is obtained from $R$ by adding a new vertex $w = w(v)$ together with a new edge $vw$ for any $v \in V(R)$. A big

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sun means a sun with order at least 6. We use \( \text{sun}(G) \) to denote the number of sun components of \( G \). Kaneko \[2\] put forward a necessary and sufficient condition for the existence of \( P_{\geq 3} \)-factors in graphs. Zhang and Zhou \[4\] generalized this result and obtained a necessary and sufficient condition for the existence of \( P_{\geq 3} \)-factor covered graphs.

**Theorem 1** (\[2\]). A graph \( G \) has a \( P_{\geq 3} \)-factor if and only if
\[
\text{sun}(G - X) \leq 2|X|
\]
for all \( X \subseteq V(G) \).

**Theorem 2.** (\[4\]). A connected graph \( G \) is a \( P_{\geq 3} \)-factor covered graph if and only if
\[
\text{sun}(G - X) \leq 2|X| - \varepsilon(X)
\]
for any vertex subset \( X \) of \( G \), where \( \varepsilon(X) \) is defined as follows:
\[
\varepsilon(X) = \begin{cases} 
2, & \text{if } X \text{ is not an independent set;} \\
1, & \text{if } X \text{ is a nonempty independent set and } G - X \text{ admits a non- } \text{sun component;} \\
0, & \text{otherwise.}
\end{cases}
\]

We introduce a new parameter, i.e., sun toughness, which is denoted by \( s(G) \). The sun toughness \( s(G) \) of a graph \( G \) was defined as follows:
\[
s(G) = \min \left\{ \frac{|X|}{\text{sun}(G - X)} : X \subseteq V(G), \text{sun}(G - X) \geq 2 \right\},
\]
if \( G \) is not complete; otherwise, \( s(G) = +\infty \).

A graph \( G \) is defined as a \( P_{\geq k} \)-factor uniform graph if \( G \) admits a \( P_{\geq k} \)-factor containing \( e_1 \) and excluding \( e_2 \) for any two distinct edges \( e_1 \) and \( e_2 \) of \( G \), which is an extension of the concept of a \( P_{\geq k} \)-factor covered graph. In this paper, we investigate the \( P_{\geq 3} \)-factor uniform graph and obtain a sun toughness condition for the existence of \( P_{\geq 3} \)-factor uniform graphs.

**Theorem 3.** Let \( G \) be a 3-edge-connected graph. Then \( G \) is a \( P_{\geq 3} \)-factor uniform graph if its sun toughness \( s(G) > 1 \).

### 2. THE PROOF OF THEOREM 3

**Proof of Theorem 3.** Since \( G \) is 3-edge-connected, we have \( |V(G)| \geq 4 \). If \( G \) is a complete graph, then it is easily seen that \( G \) is a \( P_{\geq 3} \)-factor uniform graph by \( |V(G)| \geq 4 \). Next, we consider that \( G \) is a non-complete graph.

Note that \( G \) is 3-edge-connected. Thus, we know that \( G' = G - e \) is connected for all \( e = xy \in E(G) \). In order to justify Theorem 3, we only need to verify that \( G' \) is \( P_{\geq 3} \)-factor covered. On the contrary, suppose that \( G' \) is not \( P_{\geq 3} \)-factor covered. Then it follows from Theorem 2 that there exists some vertex subset \( X \) of \( G' \) such that
\[
\text{sun}(G' - X) \geq 2|X| - \varepsilon(X) + 1.
\]

**Claim 1.** \( |X| = 2 \).

**Proof.** If \( |X| = 0 \), then it follows from (1) that
\[
\text{sun}(G') \geq 1.
\]

Since $G$ is 3-edge-connected and $G' = G - e$, we have
\[ \text{sun}(G') \leq \omega(G') = 1. \] (3)

According to (2) and (3), we get
\[ \text{sun}(G') = \omega(G') = 1. \]

Note that $|V(G')| = |V(G)| \geq 4$. Therefore, $G' \neq K_1$ and $G' \neq K_2$. Thus, $G'$ is a big sun. Obviously, there are at least three vertices with degree 1 in $G'$, and so there is at least one vertex with degree 1 in $G = G' + e$. This contradicts that $G$ is 3-edge-connected.

If $|X| = 1$, then by (1) and $\epsilon(X) \leq 1$ we get $\text{sun}(G' - X) \geq 2$. Let $C$ be any sun component of $G'$. If $C = K_1$, then for $x \in V(C)$ we have $d_G(x) = 0$, and so $d_G(x) \leq 2$ by $|X| = 1$ and $G = G' + e$. This contradicts that $G$ is 3-edge-connected. If $C = K_2$ or $C$ is a big sun component of $G'$, then there exist at least two vertices $u$ and $v$ with $d_G(u) = d_G(v) = 1$. Combining this with $|X| = 1$ and $G = G' + e$, it is easily seen that $d_G(u) \leq 2$ or $d_G(v) \leq 2$. This contradicts that $G$ is 3-edge-connected.

If $|X| \geq 3$, then by (1) and $\epsilon(X) \leq 2$ we obtain $\text{sun}(G' - X) \geq 2|X| - \epsilon(X) + 1 \geq 2|X| - 1 \geq 5$. Combining this with $\text{sun}(G' - X) \leq \text{sun}(G - X) + 2$, we have $\text{sun}(G - X) \geq 3$. Using the definition of $s(G)$, we obtain
\[ s(G) \leq \frac{|X|}{\text{sun}(G - X) - 2} \leq \frac{|X|}{2|X| - 3} \leq \frac{3}{6 - 3} = 1, \]
which contradicts that $s(G) > 1$. Therefore, $|X| = 2$. Claim 1 is justified. \hfill \Box

In light of (1), $\epsilon(X) \leq |X|$ and Claim 1, we obtain
\[ \text{sun}(G' - X) \geq 2|X| - \epsilon(X) + 1 \geq |X| + 1 = 3. \] (4)

It follows from (4) and $G' = G - e$ that
\[ 3 \leq \text{sun}(G' - X) = \text{sun}(G - e - X) \leq \text{sun}(G - X) + 2, \] (5)
which implies
\[ \text{sun}(G - X) \geq 1. \]

Next, we consider two cases in light of the value of $\text{sun}(G - X)$.

**Case 1.** $\text{sun}(G - X) \geq 2$.

Using Claim 1, $s(G) > 1$ and the concept of $s(G)$, we have
\[ 1 < s(G) \leq \frac{|X|}{\text{sun}(G - X)} \leq \frac{|X|}{2} = 1, \]
a contradiction.

**Case 2.** $\text{sun}(G - X) = 1$.

We denote by $C_1$ the sun component of $G - X$. From (5), we get that $\text{sun}(G' - X) = 3$. Combining this with $G' = G - e$, we know that $C_1$ is also a sun component of $G' - X$, and we denote by $C_2$ and $C_3$ the other two sun components of $G' - X$. Obviously, one vertex of $e$ belongs to $V(C_2)$ and the other vertex of $e$ belongs to $V(C_3)$. Note that $e = xy$, and let $x \in V(C_2)$ and $y \in V(C_3)$.

**Subcase 2.1.** $C_2 \neq K_1$ or $C_3 \neq K_1$.

Without loss of generality, let $C_2 \neq K_1$. Then $C_2 = K_2$ or $C_2$ is a big sun.

If $C_2 = K_2$, then $\text{sun}(G - X \cup \{x\}) = \text{sun}(G' - X \cup \{x\}) = 3$. In view of $s(G) > 1$, Claim 1 and the concept
of \( s(G) \), we get

\[
1 < s(G) \leq \frac{|X \cup \{x\}|}{\text{sun}(G - X \cup \{x\})} = \frac{|X| + 1}{3} = 1,
\]

which is a contradiction.

If \( C_2 \) is a big sun. Then we write \( R_0 \) for the factor-critical graph in \( C_2 \). Thus, \( d_{C_2}(u) = 1 \) for any \( u \in V(C_2) \setminus V(R_0) \) and \( |V(R_0)| = \frac{|V(C_2)|}{2} \geq 3 \). Note that \( y \in V(C_3) \). If \( x \in V(R_0) \), then we have

\[
\text{sun}(G - X \cup \{x\}) = \text{sun}(G' - X \cup \{x\}) = 3.
\]

In terms of Claim 1, \( s(G) > 1 \) and the concept of \( s(G) \), we get

\[
1 < s(G) \leq \frac{|X \cup \{x\}|}{\text{sun}(G - X \cup \{x\})} = \frac{1 + |X|}{3} = 1,
\]

a contradiction. If \( x \in V(C_2) \setminus V(R_0) \), then \( \exists x_0 \in V(R_0) \) such that \( xx_0 \in E(C_2) \). Thus, we obtain

\[
\text{sun}(G - X \cup (V(R_0) \setminus \{x_0\}) \cup \{x\}) = \text{sun}(G' - X \cup (V(R_0) \setminus \{x_0\}) \cup \{x\}) = |V(R_0)| + 2.
\]

Combining this with Claim 1 and the concept of \( s(G) \), we get

\[
s(G) \leq \frac{|X \cup (V(R_0) \setminus \{x_0\}) \cup \{x\}|}{\text{sun}(G - X \cup (V(R_0) \setminus \{x_0\}) \cup \{x\})} = \frac{|X| + |V(R')|}{|V(R_0)| + 2} = \frac{2 + |V(R')|}{|V(R_0)| + 2} = 1,
\]

which contradicts that \( s(G) > 1 \).

**Subcase 2.2** \( C_2 = K_1 \) and \( C_3 = K_1 \).

Apparently, \( C_2 \cup C_3 + e = K_2 \), which is a sun component of \( G - X \). Thus, \( \text{sun}(G - X) = 2 \). This contradicts that \( \text{sun}(G - X) = 1 \). Theorem 3 is testified. \( \square \)

### 3. REMARKS

**Remark 1.** We point out here that the sun toughness condition stated in Theorem 3 is sharp, that is, we cannot replace \( s(G) > 1 \) by \( s(G) \geq 1 \). Let \( G = H \cup (K_2 \cup P_4) \), where \( H = K_2 \) and \( P_4 = v_0v_1v_2v_3 \). We easily calculate that \( s(G) = \frac{|V(H) \cup \{v_1\}|}{\text{sun}(G - V(H) \cup \{v_1\})} = 1 \) and \( G \) is 3-edge-connected. We write \( e = v_1v_2 \) and \( G' = G - e \). Set \( X = V(H) \subseteq V(G') \). Then \( e(X) = 2 \) and \( \text{sun}(G' - X) = 3 > 2 = 2|X| - e(X) \). Using Theorem 2, \( G' \) is not \( P_{2,3} \)-factor covered, and so \( G \) is not \( P_{2,3} \)-factor uniform.

**Remark 2.** Now, we show that the edge-connectivity in Theorem 3 is sharp, that is, we cannot replace 3-edge-connected by 2-edge-connected. Let \( G = K_4 \cup (K_2 \cup K_4) \). We easily see that \( G \) is 2-edge-connected and \( s(G) = \frac{\frac{3}{2}}{2} > 1 \). Let \( G' = G - e \) for \( e \in E(K_2) \). We choose \( X = V(K_1) \), and so \( e(X) = 1 \). Thus, we have \( \text{sun}(G' - X) = 2 > 1 = 2|X| - e(X) \). In light of Theorem 2, \( G' \) is not \( P_{2,3} \)-factor covered, and so \( G \) is not \( P_{2,3} \)-factor uniform.
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