THE PUBLISHING HOUSE
OF THE ROMANIAN ACADEMY

PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A
Volume xx, Number yy/2020, pp. 1–6

A result on $P_{\geq 3}$-factor uniform graphs

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Abstract: Let $k \geq 2$ be an integer, and let G be a graph. A $P_{\geq k}$-factor of a graph G is a spanning subgraph F of G such that each component of F is a path of order at least k. A graph G is a $P_{\geq k}$-factor uniform graph if G has a $P_{\geq k}$-factor including e_1 and excluding e_2 for any two distinct edges e_1 and e_2 of G. In this article, we verify that a 3-edge-connected graph G is a $P_{\geq 3}$-factor uniform graph if its sun toughness $s(G) > 1$. Furthermore, we show that the two conditions on edge-connectivity and sun toughness are sharp.

Key words: graph; edge-connectivity; sun toughness; $P_{\geq 3}$-factor; $P_{\geq 3}$-factor uniform graph.

1. INTRODUCTION

We deal with only finite, undirected and simple graphs, which have neither loops nor multiple edges. Let G be a graph. We denote by $V(G)$, $E(G)$ and $I(G)$ the vertex set, the edge set and the isolated vertex set of G, respectively, and write $i(G) = |I(G)|$. For any $v \in V(G)$, we use $d_G(v)$ to denote the degree of v in G. For any $X \subseteq V(G)$, $G[X]$ is a subgraph induced by X of G with $V(G[X]) = X$ and $E(G[X]) = \{uv \in E(G) : u, v \in X\}$, and write $G - X = G[V(G) \setminus X]$. For any $E' \subseteq E(G)$, we denote by $G - E'$ the subgraph obtained from G by deleting E'. A vertex subset X of G is independent if no two vertices in X are adjacent to each other. The number of connected components of G is denoted by $\omega(G)$. A path on n vertices is denoted by P_n and a complete graph on n vertices is denoted by K_n. Given two graphs G_1 and G_2, we use $G_1 \vee G_2$ to denote the graph obtained from $G_1 \cup G_2$ by adding all the edges joining a vertex of G_1 to a vertex of G_2.

Let $k \geq 2$ be an integer. A spanning subgraph F of a graph G is called a $P_{\geq k}$-factor of G if each component of F is a path of order at least k. A graph G is called a $P_{\geq k}$-factor covered graph if for any $e \in E(G)$, G has a $P_{\geq k}$-factor including e.

Wang [1] gave a necessary and sufficient condition for a bipartite graph having a $P_{\geq 3}$-factor. Kaneko [2] characterized a graph with a $P_{\geq 3}$-factor, which is a generalization of Wang's result. Kano, Katona and Király [3] gave a simple proof of Kaneko's result. Zhang and Zhou [4] first defined the concept of a $P_{\geq k}$-factor covered graph, and then showed a necessary and sufficient condition for a graph to be a $P_{\geq 3}$-factor covered graph. Zhou [5] obtained a new result on the existence of $P_{\geq 3}$-factor covered graphs. Gao, Wang and Chen [6] improved Zhou's previous result on $P_{\geq 3}$-factor covered graphs. Zhou [7], Zhou, Sun and Liu [8] gave some results on path factors with given properties in graphs. Zhou and Sun [9] showed a binding number condition for a graph to be $P_{\geq 3}$-factor uniform. Some other results on graph factors see [10–26].

A graph R is called a factor-critical graph if $R-\{v\}$ admits a perfect matching for every $v \in V(R)$. A graph H is defined as a sun if $H = K_1$, $H = K_2$ or H is the corona of a factor-critical graph R with order at least three, i.e., H is obtained from R by adding a new vertex $w = w(v)$ together with a new edge vw for any $v \in V(R)$. A big sun means a sun with order at least 6. We use $sun(G)$ to denote the number of sun components of G. Kaneko [2] put forward a necessary and sufficient condition for the existence of $P_{\geq 3}$-factors in graphs. Zhang and Zhou [4] generalized this result and obtained a necessary and sufficient condition for the existence of $P_{\geq 3}$-factor covered graphs.

Theorem 1 ([2]). A graph G has a $P_{\geq 3}$-factor if and only if

$$sun(G-X) \leq 2|X|$$

for all $X \subseteq V(G)$.

Theorem 2. ([4]). A connected graph G is a $P_{\geq 3}$-factor covered graph if and only if

$$sun(G-X) \leq 2|X| - \varepsilon(X)$$

for any vertex subset X of G, where $\varepsilon(X)$ is defined as follows:

$$\varepsilon(X) = \begin{cases} 2, & if\ X\ is\ not\ an\ independent\ set; \\ 1, & if\ X\ is\ a\ nonempty\ independent\ set\ and\ G-X\ admits \\ & a\ non-sun\ component; \\ 0, & otherwise. \end{cases}$$

We introduce a new parameter, i.e., sun toughness, which is denoted by $s(G)$. The sun toughness $s(G)$ of a graph G was defined as follows:

$$s(G) = \min\{\frac{|X|}{sun(G-X)} : X \subseteq V(G),\ sun(G-X) \geq 2\},$$

if G is not complete; otherwise, $s(G) = +\infty$.

A graph G is defined as a $P_{\geq k}$-factor uniform graph if G admits a $P_{\geq k}$-factor containing e_1 and excluding e_2 for any two distinct edges e_1 and e_2 of G, which is an extension of the concept of a $P_{\geq k}$-factor covered graph. In this paper, we investigate the $P_{\geq 3}$-factor uniform graph and obtain a sun toughness condition for the existence of $P_{\geq 3}$-factor uniform graphs.

Theorem 3. Let G be a 3-edge-connected graph. Then G is a $P_{\geq 3}$-factor uniform graph if its sun toughness $s(G) > 1$.

2. THE PROOF OF THEOREM 3

Proof of Theorem 3. Since G is 3-edge-connected, we have $|V(G)| \geq 4$. If G is a complete graph, then it is easily seen that G is a $P_{\geq 3}$-factor uniform graph by $|V(G)| \geq 4$. Next, we consider that G is a non-complete graph.

Note that G is 3-edge-connected. Thus, we know that $G' = G - e$ is connected for all $e = xy \in E(G)$. In order to justify Theorem 3, we only need to verify that G' is $P_{\geq 3}$-factor covered. On the contrary, suppose that G' is not $P_{\geq 3}$-factor covered. Then it follows from Theorem 2 that there exists some vertex subset X of G' such that

$$sun(G'-X) \geq 2|X| - \varepsilon(X) + 1. \tag{1}$$

Claim 1. $|X| = 2$.

Proof. If $|X| = 0$, then it follows from (1) that

$$sun(G') \geq 1. \tag{2}$$

Since G is 3-edge-connected and $G' = G - e$, we have

$$sun(G') \leq \omega(G') = 1. \tag{3}$$

According to (2) and (3), we get

$$sun(G') = \omega(G') = 1.$$

Note that $|V(G')| = |V(G)| \geq 4$. Therefore, $G' \neq K_1$ and $G' \neq K_2$. Thus, G' is a big sun. Obviously, there are at least three vertices with degree 1 in G', and so there is at least one vertex with degree 1 in $G = G' + e$. This contradicts that G is 3-edge-connected.

If $|X| = 1$, then by (1) and $\varepsilon(X) \leq 1$ we get $sun(G' - X) \geq 2$. Let C be any sun component of G'. If $C = K_1$, then for $x \in V(C)$ we have $d_{G'}(x) = 0$, and so $d_G(x) \leq 2$ by $|X| = 1$ and $G = G' + e$. This contradicts that G is 3-edge-connected. If $C = K_2$ or C is a big sun component of G', then there exist at least two vertices u and v with $d_{G'}(u) = d_{G'}(v) = 1$. Combining this with $|X| = 1$ and $G = G' + e$, it is easily seen that $d_G(u) \leq 2$ or $d_G(v) \leq 2$. This contradicts that G is 3-edge-connected.

If $|X| \geq 3$, then by (1) and $\varepsilon(X) \leq 2$ we obtain $sun(G' - X) \geq 2|X| - \varepsilon(X) + 1 \geq 2|X| - 1 \geq 5$. Combining this with $sun(G' - X) \leq sun(G - X) + 2$, we have $sun(G - X) \geq 3$. Using the definition of $s(G)$, we obtain

$$\begin{aligned} s(G) &\leq \frac{|X|}{sun(G-X)} \leq \frac{|X|}{sun(G'-X)-2} \\ &\leq \frac{|X|}{2|X|-3} \leq \frac{3}{6-3} = 1, \end{aligned}$$

which contradicts that $s(G) > 1$. Therefore, $|X| = 2$. Claim 1 is justified. □

In light of (1), $\varepsilon(X) \leq |X|$ and Claim 1, we obtain

$$sun(G' - X) \geq 2|X| - \varepsilon(X) + 1 \geq |X| + 1 = 3. \tag{4}$$

It follows from (4) and $G' = G - e$ that

$$3 \leq sun(G' - X) = sun(G - e - X) \leq sun(G - X) + 2, \tag{5}$$

which implies

$$sun(G - X) \geq 1.$$

Next, we consider two cases in light of the value of $sun(G - X)$.

Case 1. $sun(G - X) \geq 2$.

Using Claim 1, $s(G) > 1$ and the concept of $s(G)$, we have

$$\begin{aligned} 1 &< s(G) \leq \frac{|X|}{sun(G-X)} \\ &\leq \frac{|X|}{2} = 1, \end{aligned}$$

a contradiction.

Case 2. $sun(G - X) = 1$.

We denote by C_1 the sun component of $G - X$. From (5), we get that $sun(G' - X) = 3$. Combining this with $G' = G - e$, we know that C_1 is also a sun component of $G' - X$, and we denote by C_2 and C_3 the other two sun components of $G' - X$. Obviously, one vertex of e belongs to $V(C_2)$ and the other vertex of e belongs to $V(C_3)$. Note that $e = xy$, and let $x \in V(C_2)$ and $y \in V(C_3)$.

Subcase 2.1. $C_2 \neq K_1$ or $C_3 \neq K_1$.

Without loss of generality, let $C_2 \neq K_1$. Then $C_2 = K_2$ or C_2 is a big sun.

If $C_2 = K_2$, then $sun(G - X \cup \{x\}) = sun(G' - X \cup \{x\}) = 3$. In view of $s(G) > 1$, Claim 1 and the concept of $s(G)$, we get

$$\begin{aligned} 1 &< s(G) \leq \frac{|X \cup \{x\}|}{sun(G - X \cup \{x\})} \\ &= \frac{|X| + 1}{3} = 1, \end{aligned}$$

which is a contradiction.

If C_2 is a big sun. Then we write R_0 for the factor-critical graph in C_2. Thus, $d_{C_2}(u) = 1$ for any $u \in V(C_2) \setminus V(R_0)$ and $|V(R_0)| = \frac{|V(C_2)|}{2} \geq 3$. Note that $y \in V(C_3)$. If $x \in V(R_0)$, then we have

$$sun(G - X \cup \{x\}) = sun(G' - X \cup \{x\}) = 3.$$

In terms of Claim 1, $s(G) > 1$ and the concept of $s(G)$, we get

$$\begin{aligned} 1 &< s(G) \leq \frac{|X \cup \{x\}|}{sun(G - X \cup \{x\})} \\ &= \frac{1 + |X|}{3} = 1, \end{aligned}$$

a contradiction. If $x \in V(C_2) \setminus V(R_0)$, then $\exists x_0 \in V(R_0)$ such that $xx_0 \in E(C_2)$. Thus, we obtain

$$sun(G - X \cup (V(R_0) \setminus \{x_0\}) \cup \{x\}) = sun(G' - X \cup (V(R_0) \setminus \{x_0\}) \cup \{x\}) = |V(R_0)| + 2.$$

Combining this with Claim 1 and the concept of $s(G)$, we get

$$\begin{aligned} s(G) &\leq \frac{|X \cup (V(R_0) \setminus \{x_0\}) \cup \{x\}|}{sun(G - X \cup (V(R_0) \setminus \{x_0\}) \cup \{x\})} \\ &= \frac{|X| + |V(R')|}{|V(R_0)| + 2} = \frac{2 + |V(R')|}{|V(R_0)| + 2} = 1, \end{aligned}$$

which contradicts that $s(G) > 1$.

Subcase 2.2 $C_2 = K_1$ and $C_3 = K_1$.

Apparently, $C_2 \cup C_3 + e = K_2$, which is a sun component of $G - X$. Thus, $sun(G - X) = 2$. This contradicts that $sun(G - X) = 1$. Theorem 3 is testified. □

3. REMARKS

Remark 1. We point out here that the sun toughness condition stated in Theorem 3 is sharp, that is, we cannot replace $s(G) > 1$ by $s(G) \geq 1$. Let $G = H \vee (K_2 \cup P_4)$, where $H = K_2$ and $P_4 = v_0v_1v_2v_3$. We easily calculate that $s(G) = \frac{|V(H) \cup \{v_1\}|}{sun(G - V(H) \cup \{v_1\})} = 1$ and G is 3-edge-connected. We write $e = v_1v_2$ and $G' = G - e$. Set $X = V(H) \subseteq V(G')$. Then $\varepsilon(X) = 2$ and $sun(G' - X) = 3 > 2 = 2|X| - \varepsilon(X)$. Using Theorem 2, G' is not $P_{\geq 3}$-factor covered, and so G is not $P_{\geq 3}$-factor uniform.

Remark 2. Now, we show that the edge-connectivity in Theorem 3 is sharp, that is, we cannot replace 3-edge-connected by 2-edge-connected. Let $G = K_1 \vee (K_2 \cup K_4)$. We easily see that G is 2-edge-connected and

$s(G) = \frac{3}{2} > 1$. Let $G' = G - e$ for $e \in E(K_2)$. We choose $X = V(K_1)$, and so $\varepsilon(X) = 1$. Thus, we have $sun(G' - X) = 2 > 1 = 2|X| - \varepsilon(X)$. In light of Theorem 2, G' is not $P_{\geq 3}$-factor covered, and so G is not $P_{\geq 3}$-factor uniform.

ACKNOWLEDGEMENTS

The authors would like to express their sincere gratitude to the referees and editors for reading this paper carefully and presenting some critical comments which leads to a number of improvements. This work is supported by 333 Project of Jiangsu Province and Six Big Talent Peak of Jiangsu Province, China (Grant No. JY–022).

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Received on January 9, 2020