A result on \( P_{\geq 3} \)-factor uniform graphs

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Abstract: Let \( k \geq 2 \) be an integer, and let \( G \) be a graph. A \( P_{2k} \)-factor of a graph \( G \) is a spanning subgraph \( F \) of \( G \) such that each component of \( F \) is a path of order at least \( k \). A graph \( G \) is a \( P_{2k} \)-factor uniform graph if \( G \) has a \( P_{2k} \)-factor including \( e_1 \) and excluding \( e_2 \) for any two distinct edges \( e_1 \) and \( e_2 \) of \( G \). In this article, we verify that a 3-edge-connected graph \( G \) is a \( P_{3} \)-factor uniform graph if its sun toughness \( s(G) > 1 \). Furthermore, we show that the two conditions on edge-connectivity and sun toughness are sharp.

Key words: graph; edge-connectivity; sun toughness; \( P_{3} \)-factor; \( P_{3} \)-factor uniform graph.

1. INTRODUCTION

We deal with only finite, undirected and simple graphs, which have neither loops nor multiple edges. Let \( G \) be a graph. We denote by \( V(G) \), \( E(G) \) and \( I(G) \) the vertex set, the edge set and the isolated vertex set of \( G \), respectively, and write \( i(G) = |I(G)| \). For any \( v \in V(G) \), we use \( d_G(v) \) to denote the degree of \( v \) in \( G \). For any \( X \subseteq V(G) \), \( G[X] \) is a subgraph induced by \( X \) of \( G \) with \( V(G[X]) = X \) and \( E(G[X]) = \{ uv \in E(G) : u, v \in X \} \), and write \( G - X = G[V(G) \setminus X] \). For any \( E' \subseteq E(G) \), we denote by \( G - E' \) the subgraph obtained from \( G \) by deleting \( E' \). A vertex subset \( X \) of \( G \) is independent if no two vertices in \( X \) are adjacent to each other. The number of connected components of \( G \) is denoted by \( \omega(G) \). A path on \( n \) vertices is denoted by \( P_n \) and a complete graph on \( n \) vertices is denoted by \( K_n \). Given two graphs \( G_1 \) and \( G_2 \), we use \( G_1 \lor G_2 \) to denote the graph obtained from \( G_1 \lor G_2 \) by adding all the edges joining a vertex of \( G_1 \) to a vertex of \( G_2 \).

Let \( k \geq 2 \) be an integer. A spanning subgraph \( F \) of a graph \( G \) is called a \( P_{2k} \)-factor of \( G \) if each component of \( F \) is a path of order at least \( k \). A graph \( G \) is called a \( P_{2k} \)-factor covered graph if for any \( e \in E(G) \), \( G \) has a \( P_{2k} \)-factor including \( e \).

A graph $R$ is called a factor-critical graph if $R - \{v\}$ admits a perfect matching for every $v \in V(R)$. A graph $H$ is defined as a sun if $H = K_1$, $H = K_2$ or $H$ is the corona of a factor-critical graph $R$ with order at least three, i.e., $H$ is obtained from $R$ by adding a new vertex $w = w(v)$ together with a new edge $vw$ for any $v \in V(R)$. A big sun means a sun with order at least 6. We use $\text{sun}(G)$ to denote the number of sun components of $G$.


**Theorem 1** ( [2]). A graph $G$ has a $P_3$-factor if and only if
\[
\text{sun}(G - X) \leq 2|X| \quad \text{for all } X \subseteq V(G).
\]

**Theorem 2.** ( [4]). A connected graph $G$ is a $P_3$-factor covered graph if and only if
\[
\text{sun}(G - X) \leq 2|X| - \epsilon(X)
\]
for any vertex subset $X$ of $G$, where $\epsilon(X)$ is defined as follows:
\[
\epsilon(X) = \begin{cases} 
2, & \text{if } X \text{ is not an independent set;} \\
1, & \text{if } X \text{ is a nonempty independent set and } G - X \text{ admits a non-sun component;} \\
0, & \text{otherwise.}
\end{cases}
\]

We introduce a new parameter, i.e., sun toughness, which is denoted by $s(G)$. The sun toughness $s(G)$ of a graph $G$ was defined as follows:
\[
s(G) = \min \left\{ \frac{|X|}{\text{sun}(G - X)} : X \subseteq V(G), \text{sun}(G - X) \geq 2 \right\},
\]
if $G$ is not complete; otherwise, $s(G) = +\infty$.

A graph $G$ is defined as a $P_{2,k}$-factor uniform graph if $G$ admits a $P_{2,k}$-factor containing $e_1$ and excluding $e_2$ for any two distinct edges $e_1$ and $e_2$ of $G$, which is an extension of the concept of a $P_{2,k}$-factor covered graph. In this paper, we investigate the $P_{2,3}$-factor uniform graph and obtain a sun toughness condition for the existence of $P_{2,3}$-factor uniform graphs.

**Theorem 3.** Let $G$ be a 3-edge-connected graph. Then $G$ is a $P_{2,3}$-factor uniform graph if its sun toughness $s(G) > 1$.

### 2. THE PROOF OF THEOREM 3

**Proof of Theorem 3.** Since $G$ is 3-edge-connected, we have $|V(G)| \geq 4$. If $G$ is a complete graph, then it is easily seen that $G$ is a $P_{2,3}$-factor uniform graph by $|V(G)| \geq 4$. Next, we consider that $G$ is a non-complete graph.

Note that $G$ is 3-edge-connected. Thus, we know that $G' = G - e$ is connected for all $e = xy \in E(G)$. In order to justify Theorem 3, we only need to verify that $G'$ is $P_{2,3}$-factor covered. On the contrary, suppose that $G'$ is not $P_{2,3}$-factor covered. Then it follows from Theorem 2 that there exists some vertex subset $X$ of $G'$ such that
\[
\text{sun}(G' - X) \geq 2|X| - \epsilon(X) + 1.
\] (1)
Claim 1. \(|X| = 2\).

Proof. If \(|X| = 0\), then it follows from (1) that

\[ \text{sun}(G') \geq 1. \]  \hspace{1cm} (2)

Since \(G\) is 3-edge-connected and \(G' = G - e\), we have

\[ \text{sun}(G') \leq \omega(G') = 1. \]  \hspace{1cm} (3)

According to (2) and (3), we get

\[ \text{sun}(G') = \omega(G') = 1. \]

Note that \(|V(G')}| = |V(G)| \geq 4\). Therefore, \(G' \neq K_1\) and \(G' \neq K_2\). Thus, \(G'\) is a big sun. Obviously, there are at least three vertices with degree 1 in \(G'\), and so there is at least one vertex with degree 1 in \(G = G' + e\). This contradicts that \(G\) is 3-edge-connected.

If \(|X| = 1\), then by (1) and \(\varepsilon(X) \leq 1\) we get \(\text{sun}(G' - X) \geq 2\). Let \(C\) be any sun component of \(G'\). If \(C = K_1\), then for \(x \in V(C)\) we have \(d_G(x) = 0\), and so \(d_G(x) \leq 2\) by \(|X| = 1\) and \(G = G' + e\). This contradicts that \(G\) is 3-edge-connected. If \(C = K_2\) or \(C\) is a big sun component of \(G'\), then there exist at least two vertices \(u\) and \(v\) with \(d_G(u) = d_G(v) = 1\). Combining this with \(|X| = 1\) and \(G = G' + e\), it is easily seen that \(d_G(u) \leq 2\) or \(d_G(v) \leq 2\). This contradicts that \(G\) is 3-edge-connected.

If \(|X| \geq 3\), then by (1) and \(\varepsilon(X) \leq 2\) we obtain \(\text{sun}(G' - X) \geq 2|X| - |\varepsilon(X)| + 1 \geq 2|X| - 1 \geq 5\). Combining this with \(\text{sun}(G' - X) \leq \text{sun}(G - X) + 2\), we have \(\text{sun}(G - X) \geq 3\). Using the definition of \(s(G)\), we obtain

\[ s(G) \leq \frac{|X|}{\text{sun}(G - X)} \leq \frac{|X|}{\text{sun}(G' - X) - 2} \]

\[ \leq \frac{|X|}{2|X| - 3} < \frac{3}{6 - 3} = 1, \]

which contradicts that \(s(G) > 1\). Therefore, \(|X| = 2\). Claim 1 is justified. \(\Box\)

In light of (1), \(\varepsilon(X) \leq |X|\) and Claim 1, we obtain

\[ \text{sun}(G' - X) \geq 2|X| - |\varepsilon(X)| + 1 \geq |X| + 1 = 3. \]  \hspace{1cm} (4)

It follows from (4) and \(G' = G - e\) that

\[ 3 \leq \text{sun}(G' - X) = \text{sun}(G - e - X) \leq \text{sun}(G - X) + 2, \]  \hspace{1cm} (5)

which implies

\[ \text{sun}(G - X) \geq 1. \]

Next, we consider two cases in light of the value of \(\text{sun}(G - X)\).

Case 1. \(\text{sun}(G - X) \geq 2\).

Using Claim 1, \(s(G) > 1\) and the concept of \(s(G)\), we have

\[ 1 < s(G) \leq \frac{|X|}{\text{sun}(G - X)} \]

\[ \leq \frac{|X|}{2} = 1, \]

a contradiction.

Case 2. \(\text{sun}(G - X) = 1\).

We denote by \(C_1\) the sun component of \(G - X\). From (5), we get that \(\text{sun}(G' - X) = 3\). Combining this with \(G' = G - e\), we know that \(C_1\) is also a sun component of \(G' - X\), and we denote by \(C_2\) and \(C_3\) the other two sun components of \(G' - X\). Thus, \(G' - X\) has at least two sun components other than \(C_1\), and so \(G - X\) has at least one vertex with degree 1. This contradicts that \(G\) is 3-edge-connected. \(\Box\)
components of $G' - X$. Obviously, one vertex of $e$ belongs to $V(C_2)$ and the other vertex of $e$ belongs to $V(C_3)$. Note that $e = xy$, and let $x \in V(C_2)$ and $y \in V(C_3)$.

**Subcase 2.1.** $C_2 \neq K_1$ or $C_3 \neq K_1$.

Without loss of generality, let $C_2 \neq K_1$. Then $C_2 = K_2$ or $C_2$ is a big sun.

If $C_2 = K_2$, then $\text{sun}(G - X \cup \{x\}) = \text{sun}(G' - X \cup \{x\}) = 3$. In view of $s(G) > 1$, Claim 1 and the concept of $s(G)$, we get

$$1 < s(G) \leq \frac{|X \cup \{x\}|}{\text{sun}(G - X \cup \{x\})} = \frac{|X| + 1}{3} = 1,$$

which is a contradiction.

If $C_2$ is a big sun. Then we write $R_0$ for the factor-critical graph in $C_2$. Thus, $d_C(u) = 1$ for any $u \in V(C_2) \setminus V(R_0)$ and $|V(R_0)| = \frac{|V(C_2)|}{2} \geq 3$. Note that $y \in V(C_3)$. If $x \in V(R_0)$, then we have

$$\text{sun}(G - X \cup \{x\}) = \text{sun}(G' - X \cup \{x\}) = 3.$$

In terms of Claim 1, $s(G) > 1$ and the concept of $s(G)$, we get

$$1 < s(G) \leq \frac{|X \cup \{x\}|}{\text{sun}(G - X \cup \{x\})} = \frac{1 + |X|}{3} = 1,$$

a contradiction. If $x \in V(C_2) \setminus V(R_0)$, then $\exists x_0 \in V(R_0)$ such that $xx_0 \in E(C_2)$. Thus, we obtain

$$\text{sun}(G - X \cup (V(R_0) \setminus \{x_0\}) \cup \{x\}) = \text{sun}(G' - X \cup (V(R_0) \setminus \{x_0\}) \cup \{x\}) = |V(R_0)| + 2.$$

Combining this with Claim 1 and the concept of $s(G)$, we get

$$s(G) \leq \frac{|X \cup (V(R_0) \setminus \{x_0\}) \cup \{x\}|}{\text{sun}(G - X \cup (V(R_0) \setminus \{x_0\}) \cup \{x\})} = \frac{|X| + |V(R')|}{|V(R_0)| + 2} = 1,$$

which contradicts that $s(G) > 1$.

**Subcase 2.2** $C_2 = K_1$ and $C_3 = K_1$.

Apparently, $C_2 \cup C_3 + e = K_2$, which is a sun component of $G - X$. Thus, $\text{sun}(G - X) = 2$. This contradicts that $\text{sun}(G - X) = 1$. Theorem 3 is testified.

### 3. REMARKS

**Remark 1.** We point out here that the sun toughness condition stated in Theorem 3 is sharp, that is, we cannot replace $s(G) > 1$ by $s(G) \geq 1$. Let $G = H \vee (K_2 \cup P_4)$, where $H = K_2$ and $P_4 = v_0v_1v_2v_3$. We easily calculate that $s(G) = \frac{|V(H) \cup \{v_1\}|}{\text{sun}(G - V(H) \cup \{v_1\})} = 1$ and $G$ is $3$-edge-connected. We write $e = v_1v_2$ and $G' = G - e$. Set $X = V(H) \subseteq V(G')$. Then $\epsilon(X) = 2$ and $\text{sun}(G' - X) = 3 > 2 = 2|X| - \epsilon(X)$. Using Theorem 2, $G'$ is not $P_{3,3}$-factor covered, and so $G$ is not $P_{3,3}$-factor uniform.

**Remark 2.** Now, we show that the edge-connectivity in Theorem 3 is sharp, that is, we cannot replace $3$-edge-connected by $2$-edge-connected. Let $G = K_1 \vee (K_2 \cup K_4)$. We easily see that $G$ is $2$-edge-connected and...
Let $G' = G - e$ for $e \in E(K_2)$. We choose $X = V(K_1)$, and so $\varepsilon(X) = 1$. Thus, we have $\sun(G' - X) = 2 > 1 = 2|X| - \varepsilon(X)$. In light of Theorem 2, $G'$ is not $P_{\geq 3}$-factor covered, and so $G$ is not $P_{\geq 3}$-factor uniform.

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