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A GAP CONDITION FOR THE ZEROS OF A CERTAIN CLASS OF FINITE PRODUCTS

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Abstract: We carry out complete membership to Kaplan classes of certain class of finite products with all zeros on unit circle. In this way we extend Sheil-Small's, Jahangiri's and our previous results. An interpretation of the obtained gap condition in terms of mass and density is given.

Key words: Kaplan classes, univalence, close-to-convex functions, critical points

1. INTRODUCTION

Let $\mathbb{C}$ be the set of complex numbers and let $\mathscr{A}$ denote the space of functions analytic in $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ given the usual topology of local uniform convergence. Let $\mathscr{H} \subset \mathscr{A}$ be the class of all functions f normalized by $f(0) = f'(0) - 1 = 0$ and such that $f' \neq 0$ in $\mathbb{D}$. Also let $\mathscr{S} \subset \mathscr{H}$ be the class of all functions univalent in $\mathbb{D}$.

The functions of the form $\mathbb{D} \ni z \mapsto 1 - z\mathrm{e}^{-\mathrm{i}t}$ for $t \in [0; 2\pi)$ play a central role in the univalent functions theory. Due to the result of Royster [11] they are used for example as an extremal functions in many articles (see [2], [3], [10]). Moreover, consider finite products of the form

$$\mathbb{D} \ni z \mapsto F_n(z; T_n; P_n) := \zeta \cdot \prod_{k=1}^{n} (1 - z\mathrm{e}^{-\mathrm{i}t_k})^{p_k} \ , \tag{1}$$

where $\zeta \in \mathbb{C} \setminus \{0\}$, $n \in \mathbb{N}$, $T_n := (t_1, t_2, \ldots, t_n)$ is an increasing sequence of values from $[0; 2\pi)$ such that $t_1 := 0$ and $P_n := (p_1, p_2, \ldots, p_n)$ is a sequence of real numbers. We note that all the zeros of the function $F_n(\cdot; T_n; P_n)$ lie on the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. Denote $s := \sum_{k=1}^{n} p_k$. Now suppose that $\lambda \in \mathbb{R}$, where $\mathbb{R}$ is the set of all real numbers. We define the class Π_λ of all $k \in \mathscr{A}$ such that $k \neq 0$ in $\mathbb{D}$ satisfying the following condition for every $z \in \mathbb{D}$,

$$\operatorname{Re}\left(\frac{zk'(z)}{k(z)}\right) \begin{cases} < \frac{\lambda}{2}, \text{ if } \lambda > 0 \\ > \frac{\lambda}{2}, \text{ if } \lambda < 0 \\ = 0, \text{ if } \lambda = 0 \ . \end{cases}$$

Finite products of the form (1), where $s = \lambda$ and p_k have the same sign (i.e. that of λ) are dense in Π_λ (see Sheil-Small [13]).

We define the class of analytic functions, namely $K(\alpha, \beta)$. Class $K(\alpha, \beta)$ together with two intertwined classes, $T(\alpha, \beta)$ and its dual, are the means used as universal tools to investigate many well-known subclasses of $\mathscr{S}$ (see Jahangiri [6–8], Ruscheweyh [12], Sheil-Small [13–16]). For $\alpha, \beta \geq 0$, Sheil-Small [13] defined the Kaplan class $K(\alpha, \beta)$ as the set of all functions $f \in \mathscr{A}$ that can be written in the form $f(z) = k(z)H(z)$ where $k \in \Pi_{\alpha-\beta}$ and $H \in \mathscr{A}$ is non-zero and satisfies the following condition for $z \in \mathbb{D}$,

$$|\arg H(z)| \leq \frac{\pi}{2} \min\{\alpha, \beta\} \ .$$

The class $K(\alpha, \beta)$ is called Kaplan class because using the Kaplan method [9], one can show that a function $f \in \mathscr{H}$ is close-to-convex of order $\alpha \geq 0$ if and only if $f' \in K(\alpha, \alpha + 2)$. The following characterization of Kalplan classes $K(\alpha, \beta)$ is due to Sheil-Small [13, Theorem 2.2].

Theorem A. *Let $f \in \mathscr{A}$ such that $f \neq 0$ in $\mathbb{D}$ and $\alpha, \beta \geq 0$. Then $f \in K(\alpha, \beta)$ if and only if, for $0 < r < 1$ and $\theta_1 < \theta_2 < \theta_1 + 2\pi$,*

$$\arg f(r\mathrm{e}^{\mathrm{i}\theta_2}) - \arg f(r\mathrm{e}^{\mathrm{i}\theta_1}) \geq -\alpha\pi - \frac{1}{2}(\alpha - \beta)(\theta_1 - \theta_2) \ ; \tag{2}$$

$$\arg f(r\mathrm{e}^{\mathrm{i}\theta_2}) - \arg f(r\mathrm{e}^{\mathrm{i}\theta_1}) \leq \beta\pi - \frac{1}{2}(\alpha - \beta)(\theta_1 - \theta_2) \ . \tag{3}$$

The two inequalities are equivalent, i.e. each implies the other.

As in the case of the class Π_λ, we assume further that numbers p_k in definition (1) have the same sign, namely positive and without loss of generality we assume a normalization $\zeta := 1$. We deduce from [4, Theorem 1.1] that $f_k \in K(1, 0)$ for any $k \in \mathbb{N}_n$. For the set of natural numbers $\mathbb{N}$ and for $N_m := \mathbb{N} \cap [1; m]$, the following theorem is a modified version of a result given by Sheil-Small [16, p. 248].

Theorem B (Sheil-Small)**.** *For any polynomial $Q \in \mathscr{H}_d$ of degree $n \in \mathbb{N} \setminus \{1\}$ with all zeros in $\mathbb{T}$, if λ is minimal arclength between two consecutives zeros of Q, then $Q \in K(1, 2\pi/\lambda - n + 1)$.*

Theorem B can also be deduced from [6], where Jahangiri obtained a certain gap condition for polynomials with all zeros in $\mathbb{T}$. In [4] we extended the Jahangiri's result for all $\alpha, \beta \geq 0$ and effectively determined complete membership to Kaplan classes of polynomials with all zeros in $\mathbb{T}$. In [5] we carried out complete membership to Kaplan classes of finite products of the form similar to (1), but with zeros simetrically situated in $\mathbb{T}$. In this article we determine a gap condition for zeros of function $F_n(\cdot; T_n; P_n)$ in Kaplan classes, that is with zeros arbitrarily situated on the circle and any positive powers. This aim was achieved in Theorem 1. Corollary 1 gives a description of the set Π containing all (α, β) such that $F_n(\cdot; T_n; P_n) \in K(\alpha, \beta)$ as a conjunction of linear inequalities. Example 1 shows the differences in membership to Kaplan classes between functions $F_n(\cdot; T_n; P_n) \in K(\alpha, \beta)$ depending on the sequences T_n and P_n. Moreover, we give an interpretation of the obtained gap condition in terms of mass and density.

2. MAIN THEOREMS

Assume that $t_{k+n} := t_k + 2\pi$ and $p_{k+n} := p_k$ for all $k, n \in \mathbb{N}$. Denote by $\tau_{a,b}$ the arclength of every arc of $\mathbb{T}$ that contains zeros $e^{it_{a+1}}, e^{it_{a+2}}, \ldots, e^{it_{a+b}}$ of function $F_n(\cdot; T_n; P_n)$ for any $a, b \in \{0\} \cup \mathbb{N}$. In particular for $b := 0$ the arc does not contain any zeros of $F_n(\cdot; T_n; P_n)$. Denote by τ_c the arclength of every arc of $\mathbb{T}$ that contains at least the mass $c > 0$, i.e. arc contains zeros of function $F_n(\cdot; T_n; P_n)$ such that the sum of theirs powers is grater or equal to c.

Lemma 1. *For every $\rho > 0$ and $\alpha \geq 0$ such that $2\pi\rho - s + \alpha \geq 0$, the following equivalence holds*

$$\underset{m>0}{\forall} \tau_m \geq \frac{m-\alpha}{\rho} \iff \underset{l\in\mathbb{N}_n}{\forall} \underset{k\in\{0\}\cup\mathbb{N}_{n-1}}{\forall} \frac{\tau_{l,k}(s-\alpha) - 2\pi \sum\limits_{j=l+1}^{l+k} p_j}{2\pi - \tau_{l,k}} \leq 2\pi\rho - s + \alpha \ .$$

Proof. Fix $\rho > 0$ and $\alpha \geq 0$ such that $2\pi\rho - s + \alpha \geq 0$. First we prove

$$\underset{m>0}{\forall} \tau_m \geq \frac{m-\alpha}{\rho} \iff \underset{l\in\mathbb{N}_n}{\forall} \underset{k\in\mathbb{N}_n}{\forall} \tau_{l,k} \geq \frac{1}{\rho}\left(\sum_{j=l+1}^{l+k} p_j - \alpha\right) \ . \tag{4}$$

The implication (4) in direction $(\Rightarrow)$ follows from setting $m := \sum_{j=l+1}^{l+k} p_j$ and $\tau_m := \tau_{l,k}$. Now we prove implication (4) in direction $(\Leftarrow)$. Fix $m > 0$ and arc of length τ_m. The arc contains at least the mass m, it means that there exist $l, k \in \mathbb{N}_n$ such that $\tau_m = \tau_{l,k}$ and $\sum_{j=l+1}^{l+k} p_j \geq m$. Since $\rho > 0$, so

$$\tau_m = \tau_{l,k} \geq \frac{1}{\rho}\left(\sum_{j=l+1}^{l+k} p_j - \alpha\right) \geq \frac{m-\alpha}{\rho} \ .$$

Now taking arbitrary arclength $2\pi - \tau_{l,k}$ instead of any $\tau_{l,k}$, we get $s - \sum_{j=l+1}^{l+k} p_j$ instead of $\sum_{j=l+1}^{l+k} p_j$. Hence

$$\underset{l\in\mathbb{N}_n}{\forall} \underset{k\in\mathbb{N}_n}{\forall} \tau_{l,k} \geq \frac{1}{\rho}\left(\sum_{j=l+1}^{l+k} p_j - \alpha\right) \iff \underset{l\in\mathbb{N}_n}{\forall} \underset{k\in\{0\}\cup\mathbb{N}_{n-1}}{\forall} \frac{\tau_{l,k}(s-\alpha) - 2\pi \sum\limits_{j=l+1}^{l+k} p_j}{2\pi - \tau_{l,k}} \leq 2\pi\rho - s + \alpha \ .$$

□

Now we obtain the following gap condition for the zeros of $F_n(\cdot;T_n;P_n)$.

Theorem 1. *If $n\in\mathbb{N}\setminus\{1\}$, then for all $\alpha\geq 0$ and $\rho>0$ such that $2\pi\rho-s+\alpha\geq 0$,*

$$F_n(\cdot;T_n;P_n)\in K\left(\alpha,2\pi\rho-s+\alpha\right)$$

if and only if for every $m\in[0;s]$ the arclength τ_m of every arc of $\mathbb{T}$ has to satisfy

$$\tau_m\geq\frac{m-\alpha}{\rho}\ . \tag{5}$$

Proof. Fix $k\in\{0\}\cup\mathbb{N}_{n-1}$. For every $l\in\mathbb{N}_n$ let $\theta_1\in I_l$ and $\theta_2\in I_{l+k}$. By (3) for every $r\in[0;1)$ we obtain

$$\begin{aligned}&\arg F_n(r\mathrm{e}^{\mathrm{i}\theta_2};T_n;P_n)-\arg F_n(r\mathrm{e}^{\mathrm{i}\theta_1};T_n;P_n)=\\&=\sum_{j=1}^{n}p_j\left(\arctan\left(\frac{-r\sin(\theta_2-t_j)}{1-r\cos(\theta_2-t_j)}\right)-\arctan\left(\frac{-r\sin(\theta_1-t_j)}{1-r\cos(\theta_1-t_j)}\right)\right)\ .\end{aligned}$$

Consider the above equality with $r\to 1^-$, $\theta_1\neq t_l$ and $\theta_2\neq t_{l+k}$ for every $l\in\mathbb{N}_n$. Hence

$$\begin{aligned}&\lim_{r\to 1^-}\left(\arg F_n(r\mathrm{e}^{\mathrm{i}\theta_2};T_n;P_n)-\arg F_n(r\mathrm{e}^{\mathrm{i}\theta_1};T_n;P_n)\right)=\\&=\sum_{j=1}^{n}p_j\left(\arctan\left(\frac{-\sin(\theta_2-t_j)}{1-\cos(\theta_2-t_j)}\right)-\arctan\left(\frac{-\sin(\theta_1-t_j)}{1-\cos(\theta_1-t_j)}\right)\right)=\\&=\sum_{j=1}^{n}p_j\left(\arctan\left(\tan\left(\frac{\theta_2-t_j}{2}-\frac{\pi}{2}\right)\right)-\arctan\left(\tan\left(\frac{\theta_1-t_j}{2}-\frac{\pi}{2}\right)\right)\right)=\\&=\sum_{j=1}^{n}p_j\left(\frac{\theta_2-t_j-\pi}{2}-\pi\,\mathrm{Ent}\left(\frac{\theta_2-t_j}{2\pi}\right)-\frac{\theta_1-t_j-\pi}{2}+\pi\,\mathrm{Ent}\left(\frac{\theta_1-t_j}{2\pi}\right)\right)=\\&=\frac{\theta_2-\theta_1}{2}s-\pi\sum_{j=1}^{n}p_j\left(\mathrm{Ent}\left(\frac{\theta_2-t_j}{2\pi}\right)-\mathrm{Ent}\left(\frac{\theta_1-t_j}{2\pi}\right)\right)\ .\end{aligned}$$

For every $l\in\mathbb{N}_n$,

$$\begin{aligned}&\sum_{j=1}^{n}p_j\,\mathrm{Ent}\left(\frac{\theta_1-t_j}{2\pi}\right)=\sum_{j=1}^{l}p_j\,\mathrm{Ent}\left(\frac{\theta_1-t_j}{2\pi}\right)+\sum_{j=l+1}^{n}p_j\,\mathrm{Ent}\left(\frac{\theta_1-t_j}{2\pi}\right)=\\&=\sum_{j=1}^{l}0+\sum_{j=l+1}^{n}(-p_j)=-\sum_{j=l+1}^{n}p_j\ .\end{aligned}$$

Now we have two cases:

1. If $l+k\leq n$, then

$$\begin{aligned}&\sum_{j=1}^{n}p_j\,\mathrm{Ent}\left(\frac{\theta_2-t_j}{2\pi}\right)=\sum_{j=1}^{l+k}p_j\,\mathrm{Ent}\left(\frac{\theta_2-t_j}{2\pi}\right)+\sum_{j=l+k+1}^{n}p_j\,\mathrm{Ent}\left(\frac{\theta_2-t_j}{2\pi}\right)=\\&=\sum_{j=1}^{l+k}0+\sum_{j=l+k+1}^{n}(-p_j)=-\sum_{j=l+k+1}^{n}p_j\ ,\end{aligned}$$

and as a consequence

$$\sum_{j=1}^{n}p_j\left(\mathrm{Ent}\left(\frac{\theta_2-t_j}{2\pi}\right)-\mathrm{Ent}\left(\frac{\theta_1-t_j}{2\pi}\right)\right)=\sum_{j=l+1}^{l+k}p_j\ .$$

2. If $l+k>n$, then

$$\sum_{j=1}^{n} p_j \operatorname{Ent}\left(\frac{\theta_2 - t_j}{2\pi}\right) = \sum_{j=1}^{l+k-n} p_j \operatorname{Ent}\left(\frac{\theta_2 - t_j}{2\pi}\right) + \sum_{j=l+k-n+1}^{n} p_j \operatorname{Ent}\left(\frac{\theta_2 - t_j}{2\pi}\right) =$$
$$= \sum_{j=1}^{l+k-n} p_j + \sum_{j=l+k-n+1}^{n} 0 = \sum_{j=1}^{l+k-n} p_j = \sum_{j=n+1}^{l+k} p_j \ ,$$

and as a consequence

$$\sum_{j=1}^{n} p_j \left(\operatorname{Ent}\left(\frac{\theta_2 - t_j}{2\pi}\right) - \operatorname{Ent}\left(\frac{\theta_1 - t_j}{2\pi}\right)\right) = \sum_{j=l+1}^{l+k} p_j \ .$$

Hence

$$\lim_{r\to 1^-}\left(\arg F_n(r\mathrm{e}^{\mathrm{i}\theta_2};T_n;P_n) - \arg F_n(r\mathrm{e}^{\mathrm{i}\theta_1};T_n;P_n)\right) = \frac{\theta_2-\theta_1}{2}s - \pi\sum_{j=l+1}^{l+k} p_j \ .$$

Assume that

$$\Omega := \{(x,y,z)\in\mathbb{R}^3 : x\le y\le 2\pi + x \text{ and } z\in[0;1]\}$$

and

$$\Xi := \left\{(x,y,z)\in\mathbb{R}^3 : \underset{j\in\mathbb{N}}{\exists}(x=t_j \text{ or } y=t_j) \text{ and } z=1\right\} \ .$$

For all $\alpha,\beta\ge 0$ the function

$$\Omega\setminus\Xi \ni (\theta_1,\theta_2,r)\mapsto \varphi(\theta_1,\theta_2,r) := \arg F_n(r\mathrm{e}^{i\theta_2};T_n;P_n) - \arg F_n(r\mathrm{e}^{i\theta_1};T_n;P_n) + \frac{\alpha-\beta}{2}(\theta_1-\theta_2)$$

is harmonic on $\operatorname{int}(\Omega)$. Since

$$\liminf_{(\theta_1,r)\to(t_l,1^-)} \arctan\left(\frac{-r\sin(\theta_1-t_l)}{1-r\cos(\theta_1-t_l)}\right) = -\frac{\pi}{2} = \lim_{\theta_1\to t_l^+}\arctan\left(\frac{-\sin(\theta_1-t_l)}{1-\cos(\theta_1-t_l)}\right)$$

and

$$\limsup_{(\theta_2,r)\to(t_{l+n-k},1^-)} \arctan\left(\frac{-r\sin(\theta_2-t_{l+n-k})}{1-r\cos(\theta_2-t_{l+n-k})}\right) = \frac{\pi}{2} = \lim_{\theta_2\to t_{l+n-k}^-}\arctan\left(\frac{-\sin(\theta_2-t_{l+n-k})}{1-\cos(\theta_2-t_{l+n-k})}\right)$$

for $l\in\mathbb{N}_n$, so

$$\sup_{\zeta}\left(\limsup_{n\to+\infty}\varphi(\zeta_n)\right) = \sup_{(\theta_1,\theta_2,r)\in\operatorname{fr}(\Omega)\setminus\Xi}\varphi(\theta_1,\theta_2,r)\ , \tag{6}$$

where $\zeta:\mathbb{N}\to\operatorname{int}(\Omega)$ is a sequence such that $\lim_{n\to+\infty}\zeta_n\in\operatorname{fr}(\Omega)$. Therefore by [1, p. 8, Corollary 1.10] and (6) we obtain

$$\sup_{(\theta_1,\theta_2,r)\in\operatorname{int}(\Omega)}\varphi(\theta_1,\theta_2,r) \le \sup_{(\theta_1,\theta_2,r)\in\operatorname{fr}(\Omega)\setminus\Xi}\varphi(\theta_1,\theta_2,r)\ .$$

On the other hand by continouity of φ we get

$$\sup_{(\theta_1,\theta_2,r)\in\operatorname{int}(\Omega)}\varphi(\theta_1,\theta_2,r) = \sup_{(\theta_1,\theta_2,r)\in\Omega\setminus\Xi}\varphi(\theta_1,\theta_2,r) \ge \sup_{(\theta_1,\theta_2,r)\in\operatorname{fr}(\Omega)\setminus\Xi}\varphi(\theta_1,\theta_2,r)\ .$$

Therefore

$$\sup_{(\theta_1,\theta_2,r)\in\operatorname{int}(\Omega)}\varphi(\theta_1,\theta_2,r) = \sup_{(\theta_1,\theta_2,r)\in\operatorname{fr}(\Omega)\setminus\Xi}\varphi(\theta_1,\theta_2,r)\ .$$

Consider the inequality (3) replacing $f := F_n(\cdot;T_n;P_n)$ for $\theta_1 < \theta_2 < 2\pi + \theta_1$ and $r \in [0;1)$,

$$\arg F_n(r\mathrm{e}^{\mathrm{i}\theta_2};T_n;P_n) - \arg F_n(r\mathrm{e}^{\mathrm{i}\theta_1};T_n;P_n) \le \beta\pi - \frac{\alpha-\beta}{2}(\theta_1-\theta_2)$$

or equivalently

$$\beta \ge \frac{2\arg F_n(r\mathrm{e}^{\mathrm{i}\theta_2};T_n;P_n) - 2\arg F_n(r\mathrm{e}^{\mathrm{i}\theta_1};T_n;P_n) - \alpha(\theta_2-\theta_1)}{2\pi - \theta_2 + \theta_1}\,. \tag{7}$$

Since $k \in \{0\} \cup \mathbb{N}_{n-1}$ is arbitrary chosen, so for all $\alpha \ge 0$, $\theta_1 \in I_l$ and $\theta_2 \in I_{l+k}$ there exists arc of arclength $\tau_{l,k}$ such that

$$\frac{(\theta_2-\theta_1)(s-\alpha) - 2\pi\sum_{j=l+1}^{l+k} p_j}{2\pi - (\theta_2-\theta_1)} = \frac{\tau_{l,k}(s-\alpha) - 2\pi\sum_{j=l+1}^{l+k} p_j}{2\pi - \tau_{l,k}}\,.$$

Therefore, for $\alpha,\beta \ge 0$, $F_n(\cdot;T_n;P_n) \in K(\alpha,\beta)$ if and only if

$$\underset{l\in\mathbb{N}_n}{\forall}\ \underset{k\in\{0\}\cup\mathbb{N}_{n-1}}{\forall}\ \beta \ge \frac{\tau_{l,k}(s-\alpha) - 2\pi\sum_{j=l+1}^{l+k} p_j}{2\pi - \tau_{l,k}}\,.$$

Setting $\beta := 2\pi\rho - s + \alpha$, by Lemma 1 we obtain the thesis of the theorem. □

Corollary 1. *If $n \in \mathbb{N}\setminus\{1\}$, then for all $\alpha \ge \max\{p_1,p_2,\dots,p_n\}$ and $\beta \ge 0$,*

$$F_n(\cdot;T_n;P_n) \in K(\alpha,\beta)$$

if and only if

$$(\alpha,\beta) \in \bigcap_{k=0}^{n-2}\left\{(x,y)\in\mathbb{R}^2 : y \ge \max_{l\in\mathbb{N}_n}\left(\frac{(t_{l+k+1}-t_l)(s-x) - 2\pi\sum_{j=l+1}^{l+k} p_j}{2\pi - t_{l+k+1} + t_l}\right)\right\} \tag{8}$$

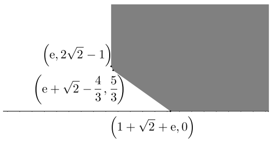
From Corollary 1 we see that the set of all classes $K(\alpha,\beta)$ for the function $F_n(\cdot;T_n;P_n)$ is an intersection of a finite number of closed half-planes. Formula (8) is convenient to determine the full membership to Kaplan classes of function $F_n(\cdot;T_n;P_n)$.

Remark 1. Let us notice that α occurring in Theorem 1 can be interpretated as a change in mass of arc, such that ρ is the minimal density of mass $m-\alpha$ on arc of arclength τ_m for all $m \in [0;s]$.

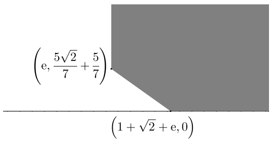
Example 1. Consider functions:

$$\begin{aligned}
f_1 &:= F_3\left(\cdot;(0,1/2\pi,7/6\pi);(1,\sqrt{2},\mathrm{e})\right)\,,\\
f_2 &:= F_3\left(\cdot;(0,1/2\pi,7/6\pi);(1,\mathrm{e},\sqrt{2})\right)\,,\\
f_3 &:= F_3\left(\cdot;(0,1/2\pi,7/6\pi);(\sqrt{2},1,\mathrm{e})\right)\,,\\
f_4 &:= F_3\left(\cdot;(0,1/2\pi,7/6\pi);(\mathrm{e},1,\sqrt{2})\right)\,,\\
f_5 &:= F_3\left(\cdot;(0,1/2\pi,7/6\pi);(\sqrt{2},\mathrm{e},1)\right)\,,\\
f_6 &:= F_3\left(\cdot;(0,1/2\pi,7/6\pi);(\mathrm{e},\sqrt{2},1)\right)\,.
\end{aligned}$$

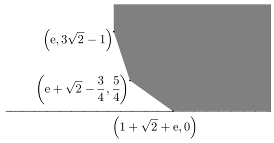
The following figures show complete membership to Kaplan classes of f_1, f_2, f_3, f_4, f_5 and f_6.



Kaplan classes of f_1 and f_2



Kaplan classes of f_3 and f_4



Kaplan classes of f_5 and f_6

REFERENCES

1. S. AXLER, P. BOURDON, W. RAMEY, *Harmonic Function Theory*, Springer-Verlag New York, 2001.

2. P. L. DUREN, *Univalent functions*, Grundlehren der Mathematischen Wissenschaften (259), Springer-Verlag, New York, 1983. ISBN: 0-387-90795-5

3. A. W. GOODMAN, *Univalent functions*, vol. II, Mariner Pub. Co., Inc., Tampa, Florida, 1983.

4. S. IGNACIUK, M. PAROL, *Zeros of complex polynomials and Kaplan classes*, Analysis Mathematica, 2020. (accepted for publication) https://doi.org/10.1007/s10476-020-0044-8

5. S. IGNACIUK, M. PAROL, *Kaplan classes of a certain family of functions*, Annales UMCS Sectio A Mathematica, 2020. (accepted for publication)

6. M. JAHANGIRI, *A gap condition for the zeroes of certain polynomials in Kaplan classes $K(\alpha,\beta)$*, Mathematika 34, pp. 53-63, 1987.

7. M. JAHANGIRI, *Weighted convolutions of certain polynomials*, Bull. Austral. Math. Soc. 40(3), pp. 397-405, 1989.

8. M. JAHANGIRI, *On the gap between two classes of analytic functions*, Proc. Indian Acad. Sci. Math. Sci. 99(2) pp. 123-126, 1989.

9. W. KAPLAN, *Close-to-convex schlicht functions*, Michigan Math. J. 1, pp. 169-185, 1952.

10. Y. J. KIM, E. P. MERKES, *On certain convex sets in the space of locally schlicht functions*, Trans. Amer. Math. Soc. 196, pp. 217–224, 1974.

11. W. C. ROYSTER, *On the univalence of a certain integral*, Michigan Math. J. 12, pp. 385-387, 1965.

12. S. RUSCHEWEYH, *Convolutions in Geometric Function Theory*, Seminaire de Math. Sup. 83, Les Presses de laUniversit‘e de Montr‘eal, 1982.

13. T. SHEIL-SMALL, *The Hadamard product and linear transformations of classes of analytic functions*, J. Analyse Math. 34, pp. 204-239, 1978.

14. T. SHEIL-SMALL, *Coefficients and integral means of some classes of analytic functions*, Proc. Amer. Math. Soc. 88(2), pp. 275-282, 1983.

15. T. SHEIL-SMALL, *Some remarks on Bazilevič functions*, J. Analyse Math. 43, pp. 1-11, 1983/84.

16. T. SHEIL-SMALL, *Complex Polynomials* Cambridge University Press 2002.

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