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CAPACITANCE OF A PARALLEL-PLATE CAPACITOR WITH ROUGH PLATES

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Abstract: The change of the capacitance of a parallel-plate capacitor as a result of roughness of the plates is perturbatively calculated. It is seen that the roughening of the plates increases the capacitance, as expected. However, the result is different from the one used in the literature. The inconsistency is explained.

Key words: capacitance, rough plates, perturbation

1. INTRODUCTION

Surface roughness affects the physical properties of micro/nanoelectromechanical system, for example [1–9]. Among the properties affected by roughness is the capacitance of a parallel-plate capacitor. In [10], this problem has been studied and a perturbative expression has been found for the capacitance of a parallel plate capacitor, when one of the plates is rough. Here it is argued that the result found in [10] is not correct. The correct result is given, and the inconsistency is explained. One of the places where having a correct theoretical expression for the capacitance of a parallel plate capacitor could be important, is measuring the dielectric constant of a material using capacitance measurements: an incorrect theoretical expression for the capacitance leads to a wrong value of the dielectric constant obtained from the measured value of the capacitance. As the capacitance is roughly (exactly, if the plates ar smooth) proportional to the inverse of the thickness, for example a surface of roughness about 10% of the thickness of the capacitor, say $(10\,\mathrm{nm})$ roughness compared to $(100\,\mathrm{nm})$ thickness, could change the capacitance by about the same 10%, and neglecting the roughness in the theoretical expression for the capacitor could lead to an error of the same $(10\,\mathrm{nm})$ magnitude in measuring the dielectric constant.

The scheme of the paper is the following. In section 2 the geometry and conventions are introduced. In section 3 the perturbative form of the electric potential inside the capacitor is found. In section 4 a perturbative expression for the capacitance is found, using two methods: the energy method and the charge method. Finally, section 5 is devoted to the discussion, specifically the reason for the discrepancy between the results found here and in [10].

2. THE GEOMETRY AND THE CONVENTIONS

The position vector is $(\boldsymbol{\rho}+\hat{\mathbf{z}}\,z)$, where $\boldsymbol{\rho}$ is perpendicular to $\hat{\mathbf{z}}$. $\boldsymbol{\nabla}$ and D denote differentiations with respect to $\boldsymbol{\rho}$ and z, respectively. The (two-dimensional) Fourier transformation of $\mathfrak{X}$ is denoted by $\tilde{\mathfrak{X}}$:

$$\tilde{\mathfrak{X}}(\boldsymbol{k}) = \int \mathrm{d}^2\rho\,\exp(-\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{\rho})\,\mathfrak{X}(\boldsymbol{\rho}). \tag{1}$$

The electrodes of the capacitor are two plates: the first the plane $z=0$, and the second a perturbed plane with

$$z = Z + h(\boldsymbol{\rho}). \tag{2}$$

Z is the unperturbed thickness of the capacitor, and h is the perturbation of the thickness, as indicated in figure (1).

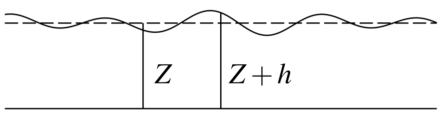

Fig. 1. The parallel plate capacitor with a rough plate

The surface area of the planar plate is S. It is assumed that

$$|h| \ll Z \ll L, \tag{3}$$

where L is the typical length size of the plates. The potential of first plate is 0 and that of the second is $(-E_0\,Z)$.

3. THE PERTURBATIVE SOLUTION FOR THE POTENTIAL

The potential ϕ satisfies the Laplace equation (in the region between the plates), and the boundary conditions

$$\phi(\boldsymbol{\rho},0)=0. \tag{4}$$

$$\phi[\boldsymbol{\rho},Z+h(\boldsymbol{\rho})]=-E_0\,Z. \tag{5}$$

For the corresponding field $\boldsymbol{E}$,

$$\boldsymbol{E}=-\boldsymbol{\nabla}\,\phi-\hat{\boldsymbol{z}}\,\mathrm{D}\,\phi, \tag{6}$$

A perturbative expansion for ϕ reads

$$\phi=\phi_0+\phi_1+\phi_2+\cdots, \tag{7}$$

where ϕ_j is of the order j in h. The boundary conditions become

$$\phi_j(\boldsymbol{\rho},0)=0. \tag{8}$$

$$\phi_1(\boldsymbol{\rho},Z)=-[(\mathrm{D}\,\phi_0)(\boldsymbol{\rho},Z)]\,h(\boldsymbol{\rho}). \tag{9}$$

$$\phi_2(\boldsymbol{\rho},Z)=-[(\mathrm{D}\,\phi_1)(\boldsymbol{\rho},Z)]\,h(\boldsymbol{\rho})-\frac{(\mathrm{D}^2\,\phi_0)(\boldsymbol{\rho},Z)}{2}\,[h(\boldsymbol{\rho})]^2. \tag{10}$$

Of course ϕ_j's satisfy the Laplace equation for $0<z<Z$. These Laplace equations, and the above boundary conditions, are used to obtain the Fourier transformations of the perturbation terms:

$$\tilde{\phi}_j(\boldsymbol{k},z)=\frac{\sinh(kz)}{\sinh(kZ)}\,\tilde{\phi}_j(\boldsymbol{k},Z). \tag{11}$$

One has

$$\phi_0(\boldsymbol{\rho},z)=-E_0\,z. \tag{12}$$

So,

$$\tilde{\phi}_1(\boldsymbol{k},z)=E_0\,\frac{\sinh(kz)}{\sinh(kZ)}\,\tilde{h}(\boldsymbol{k}). \tag{13}$$

$(\mathrm{D}^2\,\phi_0)$ is zero. So,

$$\begin{aligned}\tilde{\phi}_2(\boldsymbol{k},Z)&=-\int\frac{\mathrm{d}^2\,k'}{(2\,\pi)^2}\,[\widetilde{(\mathrm{D}\,\phi_1)}(\boldsymbol{k}',Z)]\,\tilde{h}(\boldsymbol{k}-\boldsymbol{k}'),\\&=-E_0\int\frac{\mathrm{d}^2\,k'}{(2\,\pi)^2}\,[k'\,\coth(k'\,Z)]\,[\tilde{h}(\boldsymbol{k}')]\,\tilde{h}(\boldsymbol{k}-\boldsymbol{k}'),\end{aligned} \tag{14}$$

which results in

$$\tilde{\phi}_2(\boldsymbol{k},z)=-E_0\,\frac{\sinh(kz)}{\sinh(kZ)}\int\frac{\mathrm{d}^2\,k'}{(2\,\pi)^2}\,[k'\,\coth(k'\,Z)]\,[\tilde{h}(\boldsymbol{k}')]\,\tilde{h}(\boldsymbol{k}-\boldsymbol{k}'). \tag{15}$$

4. THE CAPACITANCE

The permeability of the medium between the electrodes is denoted by ε. One could calculate the capacitance C through the energy or the charge.

4.1. The energy method

One has

$$\begin{aligned}\frac{C\,(E_0 Z)^2}{\varepsilon} &= \int_{\mathbb{V}} \mathrm{d}V\,\boldsymbol{E}\cdot\boldsymbol{E},\\ &= 2I_1 + I_2 - I_3,\end{aligned} \tag{16}$$

where

$$I_1 = \int_{\mathbb{V}} \mathrm{d}V\,\boldsymbol{E}\cdot\boldsymbol{E}_0. \tag{17}$$

$$I_2 = \int_{\mathbb{V}} \mathrm{d}V\,(\boldsymbol{E}-\boldsymbol{E}_0)\cdot(\boldsymbol{E}-\boldsymbol{E}_0). \tag{18}$$

$$I_3 = \int_{\mathbb{V}} \mathrm{d}V\,\boldsymbol{E}_0\cdot\boldsymbol{E}_0. \tag{19}$$

One has

$$\begin{aligned}I_1 &= -\int_{\mathbb{V}} \mathrm{d}V\,(\boldsymbol{\nabla}\phi)\cdot\boldsymbol{E}_0,\\ &= \sum_a \psi_a \int_{\mathbb{S}_a} \mathrm{d}S\,\hat{\boldsymbol{n}}\cdot\boldsymbol{E}_0,\end{aligned} \tag{20}$$

where ψ_a is the value of the potential on the surface $\mathbb{S}_a$, $\hat{\boldsymbol{n}}$ is the unit vector normal to the boundary of $\mathbb{V}$ pointing inside $\mathbb{V}$, and use has been made of the fact that the $(\boldsymbol{\nabla}\cdot\boldsymbol{E}_0)$ vanishes. Using the same fact, one arrives at

$$\int_{\mathbb{S}_a} \mathrm{d}S\,\hat{\boldsymbol{n}}\cdot\boldsymbol{E}_0 = \int_{\mathbb{S}_{a0}} \mathrm{d}S\,\hat{\boldsymbol{n}}\cdot\boldsymbol{E}_0. \tag{21}$$

So there is no effect of the perturbation in I_1. That is,

$$I_1 = \int_{\mathbb{V}_0} \mathrm{d}V\,\boldsymbol{E}_0\cdot\boldsymbol{E}_0. \tag{22}$$

Or,

$$I_1 = \frac{C_0\,(E_0 Z)^2}{\varepsilon}. \tag{23}$$

So,

$$\frac{(C-C_0)\,(E_0 Z)^2}{\varepsilon} = I_1 + I_2 - I_3. \tag{24}$$

$$I_1 - I_3 = -E_0^2 \int \mathrm{d}^2\rho\, h(\rho). \tag{25}$$

Up to the second order in h,

$$I_2 = \int_{\mathbb{V}_0} \mathrm{d}V\, \boldsymbol{E}_1 \cdot \boldsymbol{E}_1 + \cdots . \tag{26}$$

So,

$$I_2 = \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \int_0^Z \mathrm{d}z\, |\tilde{\boldsymbol{E}}_1(\boldsymbol{k}, z)|^2. \tag{27}$$

Using

$$\tilde{\boldsymbol{E}}_1(\boldsymbol{k}, z) = -E_0\, \frac{\mathrm{i}\,\boldsymbol{k}\, \sinh(kz) + \hat{\boldsymbol{z}}\, k\, \cosh(kz)}{\sinh(kZ)}\, \tilde{h}(\boldsymbol{k}), \tag{28}$$

one arrives at

$$I_2 = E_0^2 \int \frac{\mathrm{d}^2 k}{(2\pi)^2}\, [k\, \coth(kZ)]\, |\tilde{h}(\boldsymbol{k})|^2 + \cdots . \tag{29}$$

So,

$$C - C_0 = \frac{C_0}{SZ} \left\{ \int \frac{\mathrm{d}^2 k}{(2\pi)^2}\, [k\, \coth(kZ)]\, |\tilde{h}(\boldsymbol{k})|^2 - \int \mathrm{d}^2 \rho\, h(\boldsymbol{\rho}) \right\} + \cdots , \tag{30}$$

Or,

$$C - C_0 = \frac{C_0}{Z} \left\{ \int \frac{\mathrm{d}^2 k}{(2\pi)^2}\, [k\, \coth(kZ)]\, \sigma(\boldsymbol{k}) - \frac{\tilde{h}(\boldsymbol{0})}{S} \right\} + \cdots , \tag{31}$$

where

$$\langle |\tilde{h}(\boldsymbol{k})|^2 \rangle =: S\, \sigma(\boldsymbol{k}). \tag{32}$$

4.2. The charge method

One has

$$\frac{C\, (E_0 Z)}{\varepsilon} = \int \mathrm{d}^2 \rho\, \hat{\boldsymbol{z}} \cdot \boldsymbol{E}(\boldsymbol{\rho}, 0). \tag{33}$$

So, up to the second order in h,

$$\begin{aligned} \frac{(C - C_0)\, (E_0 Z)}{\varepsilon} &= -\int \mathrm{d}^2 \rho\, [(\mathrm{D}\, \phi_1)(\boldsymbol{\rho}, 0) + (\mathrm{D}\, \phi_2)(\boldsymbol{\rho}, 0)] + \cdots , \\ &= -[\widetilde{(\mathrm{D}\, \phi_1)}(\boldsymbol{0}, 0) + \widetilde{(\mathrm{D}\, \phi_2)}(\boldsymbol{0}, 0)] + \cdots . \end{aligned} \tag{34}$$

$$\frac{C - C_0}{\varepsilon} = \frac{1}{Z^2} \left\{ -\tilde{h}(\boldsymbol{0}) + \int \frac{\mathrm{d}^2 k'}{(2\pi)^2}\, [k'\, \coth(k' Z)]\, [\tilde{h}(\boldsymbol{k}')]\, \tilde{h}(-\boldsymbol{k}') \right\}. \tag{35}$$

This is the same as (30) or (31), as expected.

5. DISCUSSION

Equation (30) is not the same as equation (33) in [10]. Apart from some constant coefficients, equation (33) in [10] reads

$$C - C_0 = \frac{C_0}{S}\,(K + L + M) + \cdots, \tag{36}$$

where

$$K = -\frac{1}{Z}\int \mathrm{d}^2\rho\, h(\boldsymbol{\rho}). \tag{37}$$

$$L = \int \frac{\mathrm{d}^2 k}{(2\pi)^2}\,[(k/Z)\,\coth(kZ)]\,|\tilde{h}(\boldsymbol{k})|^2. \tag{38}$$

$$M = \int \frac{\mathrm{d}^2 k}{(2\pi)^2}\,k^2\,|\tilde{h}(\boldsymbol{k})|^2. \tag{39}$$

The result of the present text is

$$C - C_0 = \frac{C_0}{S}\,(K + L) + \cdots \tag{40}$$

K is the first order term, while L and M are second order. In fact K is assumed to be zero in equation (33) in [10]. The difference between our result and equation (33) in [10] is the presence of M in equation (33) in [10].

The origin of M in [10] is the first integral on the right-hand side of (32a) in [10]. There equation (32b) has been used to calculate those terms. And the problem is the first equality in equation (32b) there. The correct equation is

$$\int \mathrm{d}^2\rho\, h\,\boldsymbol{\nabla}\cdot\boldsymbol{\nabla} h = -\int \mathrm{d}^2\rho\,(\boldsymbol{\nabla} h)\cdot(\boldsymbol{\nabla} h). \tag{41}$$

The minus sign on the right-hand side is missing in the first equality of (32b) in [10]. The correct equation ensures that the first integral on the right-hand side of (32a) in [10] vanishes. Hence M won't appear in the final result.

An example which shows M shouldn't be there is provided with

$$h(\boldsymbol{\rho}) = h_0\left(1 - \frac{\rho^2}{R^2}\right)^{\mu}\,\Theta(R - \rho), \tag{42}$$

where Θ is the Heaviside (step) function. This h describes a plate which is smooth except for a bump of the radius R and the height $|h_0|$. $\mu = (1/2)$ corresponds to a hemisphere, and increasing μ makes the bump more like a step. One arrives at

$$\tilde{h}(\boldsymbol{k}) = \pi\, h_0 R^2\,\Gamma(\mu + 1)\,(kR/2)^{-(\mu+1)}\,\mathrm{J}_{\mu+1}(kR), \tag{43}$$

So,

$$|\tilde{h}(\boldsymbol{k})|^2 \sim k^{-2\mu-3}, \quad (kR) \gg 1. \tag{44}$$

This shows that L is finite but M is infinite for

$$0 < \mu \leq \frac{1}{2}. \tag{45}$$

So, with the condition (45) our expression gives a finite value for $(C-C_0)$, while the expression in [10] produces an infinite value. But (45) corresponds to a finite and continuous bump on one of the plates, which should result in a finite change for the capacity. So M should not be there.

Alternatively,

$$M=\int \mathrm{d}^2\rho\,|\boldsymbol{\nabla} h|^2, \tag{46}$$

which is infinite if the derivative of h behaves like $s^{\mu-1}$ with

$$\mu\le(1/2). \tag{47}$$

Obviously, if (45) holds then (47) holds and M is infinite.

The relative importance of the terms L and M can be simply deduced for cases where the Fourier transform of h is narrowly peaked around some wave number k_m, that is, the perturbations are mainly of wavelength k_m. In that case,

$$\frac{M}{L}=\begin{cases}(k_\mathrm{m}Z)^2, & (k_\mathrm{m}Z)\ll 1\\ (k_\mathrm{m}Z), & (k_\mathrm{m}Z)\gg 1\end{cases} \tag{48}$$

As an example, consider

$$h(\boldsymbol{\rho})=h_0\exp\left(-\frac{\rho^2}{2\sigma^2}\right)\cos(\boldsymbol{k}_0\cdot\boldsymbol{\rho}), \tag{49}$$

where σ and $\boldsymbol{k}_0$ are constants. This is a sinusuidal roughness with an amplitude which decreases as the distance from the center of the plate increases. One arrives at

$$\tilde{h}(\boldsymbol{k})=2\pi h_0\sigma^2\exp\left[-\frac{\sigma^2}{2}(k^2+k_0^2)\right]\cosh(\sigma^2\boldsymbol{k}_0\cdot\boldsymbol{k}). \tag{50}$$

Defining the dimensionless parameters a and b through

$$a=\sigma k_0, \tag{51}$$

$$b=\frac{\sigma}{Z}, \tag{52}$$

and the dimensionless functions $\mathscr{L}$ and $\mathscr{M}$ through

$$L=h_0^2\mathscr{L}, \tag{53}$$

$$M=h_0^2\mathscr{M}, \tag{54}$$

it is seen that $\mathscr{L}$ depends on a and b while $\mathscr{M}$ depends on only a. Figure 2 shows the behavior of $\mathscr{L}$ and $\mathscr{M}$ for some parameter ranges.

These arguments show that the effect of the redundant term is negligible for small wave numbers, but increases as the wave number increases. So for small wave numbers the additional term M (which should not be there) doesn't affect the result. But for large wave numbers this term becomes important and completely changes the result (wrongly, of course).

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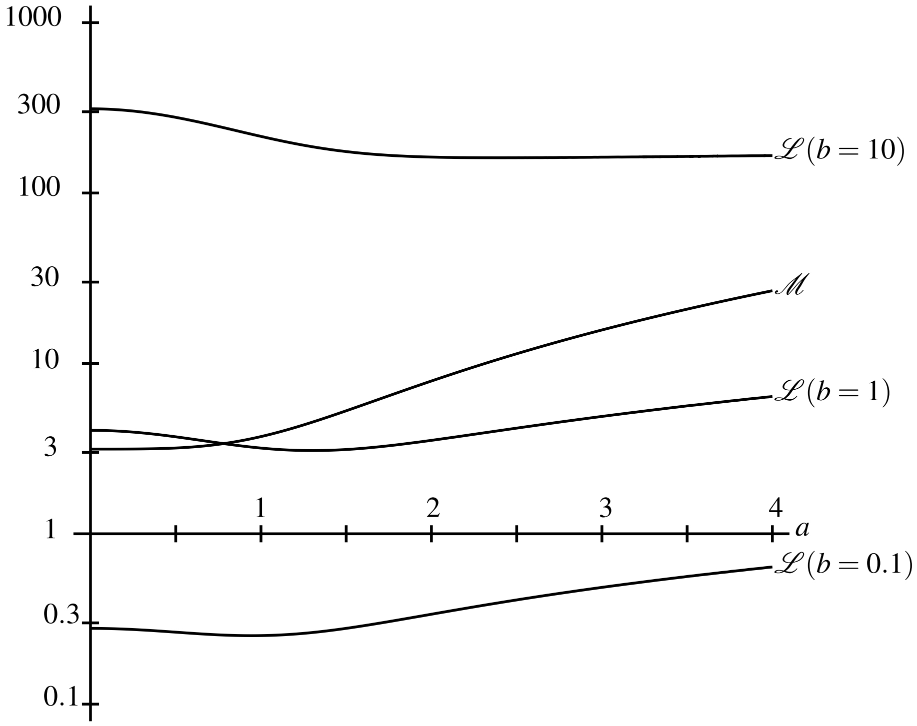

Fig. 2. The correct ($\mathscr{L}$) and redundant ($\mathscr{M}$) contributions to the second order perturbation

REFERENCES

1. G . PALASANTZAS, *Roughness and surface width of self-affine fractal surfaces via the K-correlation model*, Physical Review B, **48**, pp. 14472–14478, 1993.
2. K. KOMVOPOULOS, *Surface engineering and microtribology for microelectromechanical systems*, Wear, **200**, pp. 305–327, 1996.
3. G. PALASANTZAS, *Self-affine roughness influence on the pull-in voltage in capacitive electromechanical devices*, Journal of Applied Physics, **98**, 034505, 2005.
4. M. SAUSSE LHERNOULD, A. DELCHAMBRE, S. RÉGNIER, P. LAMBERT, *Electrostatic forces in micromanipulations: review of analytical models and simulations including roughness*, Applied Surface Science, **253**, pp. 6203–6210, 2007.
5. W. M. VAN SPENGEN, V. TURQ, J. W. FRENKEN, *The description of friction of silicon MEMS with surface roughness: virtues and limitations of a stochastic PrandtlTomlinson model and the simulation of vibration-induced friction reduction*, Beilstein Journal of Nanotechnology, **1**, pp. 163–171, 2010.
6. H. LÜTH, Solid surfaces, interfaces and thin films, 6th edition, Springer, 2015.
7. S. GOPALAKRISHNAN, A. DASGUPTA, D. R. NAIR, *Study of the effect of surface roughness on performance of rf mems capacitive switches through 3-d geometric modeling*, IEEE Journal of Electron Devices Society, **4**, 451–458, 2016.
8. S. TORABI, M. CHERRY, E. A. DUIJNSTEE, V. M. LE CORRE, Li QIU, J. C. HUMMELEN, G. PALASANTZAS, L. J. A. KOSTER, *rough electrode creates excess capacitance in thin-film capacitors*, Applied Materials and Interfaces, **9**, 27290–27297, 2017.
9. M. FESHANJERDI, A. MALEKAN, *contact electrification between randomly rough surfaces with identical materials*, Journal of Applied Physics, **125**, 165302, 2019.
10. Y.-P. Zhao, G-C. WANG, T.-M. LU, G. PALASANTZAS, J. T. M. DE HOSSON, *Surface roughness effect on capacitance and leakage current of an insulating film*, Physical Review B, **60**, pp. 9157–9164, 1999.

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