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PROGENY OF SUPERMASSIVE BLACK HOLES THROUGH THE COLLAPSE OF DARK MATTER BOSE-EINSTEIN CONDENSATES: A VARIATIONAL APPROACH

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Abstract. In this paper, we propose that the progeny of supermassive black holes occur due to the collapse of dark matter consisting of dark bosons at ultra-cold temperatures. For this purpose, supermassive black holes were modeled as attractive cosmological Bose-Einstein condensates and described by the Gross-Pitaevskii equation within the scope of the mean-field theory. The study developed in this paper was motivated by the possibility of observing the formation of supermassive black holes through the behavior of the chemical potential as a function of the number of dark bosons. The results demonstrated, through variational formalism, that the formation of supermassive black holes is possible. The results obtained in this paper can open the way for a better understanding about supermassive black holes and motivate new studies related to cosmological Bose-Einstein condensates such as supermassive Kerr black holes and quarks stars.

Key words: supermassive black holes, cosmological Bose-Einstein condensates, dark matter, variational method.

1. INTRODUCTION

The advent of the experimental realization of Bose-Einstein condensate (BEC) opened the way for a better understanding of ultracold atoms trapped [1–3]. Both dynamic and static properties of BECs consisting of dilute atomic gases in the ultracold temperature regime are well described by the mean-field equation known as the Gross-Pitaevskii equation (GPE) [4, 5].

In recent years, there has been an effort by the scientific community to understand the behavior of robust cosmic structures modeled as cosmological BECs [6, 7]. Among these cosmological BECs, we can mention the main constituents of the dark universe, i.e., dark matter and dark energy [8–13], supermassive black holes [14–16], neutron stars [17, 18], white dwarfs [19], primordial universe [20, 21], etc.

Nowadays, supermassive black holes (SMBH) have attracted interest from the scientific community. The Nobel Prize in Physics 2020 was divided, one half awarded to Roger Penrose "for the discovery that black hole formation is a robust prediction of the general theory of relativity", the other half jointly to Reinhard Genzel and Andrea Ghez "for the discovery of a supermassive compact object at the centre of our galaxy".

SMBHs can result from a variety of complex processes that occur at the center of galaxies [22]. Some current perspectives, however, propose that galaxies were formed around SMBHs which were the result of the gravitational collapse of dark matter (DM). In addition, dynamical evidence supports the existence of SMBHs in the centers of most nearby galaxies [23].

Recently, Gupta et al [15] studied collapse of ultra-light bosonic halo DM that is in a BEC phase to give rise to SMBHs on dynamical time scales, using the GPE in the framework of time-dependent variational method. Morikawa et al [16] explored the possibility that black holes form from the coherent waves of BEC which are supposed to form the DM for the Axion case with attractive interaction.

Thus, we propose in this paper a model that predicts the progeny of SMBHs through the gravitational collapse of DM, assuming that SMBHs can be described as attractive BECs. However, in order to corroborate the results obtained in Ref. [15], we investigated the collapse of the dark matter BEC from the behavior of the chemical potential as a function of the number of dark bosons in the stationary regime. For this purpose, the variational method formalism [24] is used to determine the wave function that describes the BECs from the GPE.

This paper is organised as follows. In Sec. 2, we propose a model described by the GPE, which is based on the mean-field theory. In Sec. 3, we present the variational formulation in order to solve the GPE using a Gaussian *ansatz*. In Sec. 4, we report the variational results regarding the collapse of cosmological BECs. Finally, in Sec. 5, we present the conclusions and final considerations.

2. THE MODEL

The scope of this paper is to investigate the possibility of the formation of SMBHs through the collapse of DM. For this, we propose that DM be constituted by bosonic particles called dark bosons, which have attractive interactions with each other. Thus, assuming ultra-low temperatures, it becomes possible to describe the behavior of SMBHs as cosmological BECs.

BECs are well described by the GPE, in the context of the mean-field theory. The GPE is similar to the nonlinear Schrödinger equation (NLSE) [25]:

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi(\mathbf{r},t) = -\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r},t) + V_{ext}(\mathbf{r})\,\psi(\mathbf{r},t) + \Gamma\left|\psi(\mathbf{r},t)\right|^2\psi(\mathbf{r},t), \tag{1}$$

where $\mathrm{i} = \sqrt{-1}$ is the imaginary unit, $\hbar = h/2\pi$ is the reduced Planck constant, m is the mass of each dark boson, ∇^2 denotes the 3D Laplacian, $V_{ext}(\mathbf{r})$ is the external trapping potential and $\Gamma = 4\pi\hbar^2 a_s/m$ describes the strength of the coupling constant characterized by the s-wave scattering length a_s; it is positive for repulsive interactions and negative for attractive interactions[1]. The quantity $\psi(\mathbf{r},t)$ is the wave function, whose norm is equal to the number of particles:

$$\int_{-\infty}^{+\infty}\left|\psi(\mathbf{r},t)\right|^2 d\mathbf{r} = N, \tag{2}$$

where N is the total number of dark bosons.

The external potential $V_{ext}(\mathbf{r})$ responsible for the trapping of dark bosons is given by the gravitational potential energy due to a spherically symmetrical central compact remnant [15]:

$$V_{ext}(\mathbf{r}) = -\frac{GM_0 m}{r}, \tag{3}$$

where G is the universal gravitational constant and M_0 is the mass of the compact remnant.

3. VARIATIONAL FORMULATION

In general, Bose-Einstein condensation is a macroscopic occupation of the ground state of a bosonic quantum gas. In this context, it becomes reasonable to use a variational approach to deal with this quantum system. The variational method is based on choosing a trial wave function (*ansatz*) with some variational parameters. Such parameters are adjusted in order to minimize the energy of the system. Moreover, the fact that the universe is expanding creates a favorable environment for dark bosons to be in the BEC state [15]. Therefore, it is plausible that a considerable fraction of the DM is in the ground state.

In order to solve Eq. 1, we can formulate our model through variational formalism corresponding to minimizing the action:

$$S = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathscr{L}\,d\mathbf{r}\,dt, \tag{4}$$

[1] Due to attractive interactions, we describe the nonlinearity coefficient as $\Gamma \to -\Gamma = 4\pi\hbar^2\left|a_s\right|/m$.

where Lagrangian density $\mathscr{L}$ is given by:

$$\mathscr{L} = \frac{i\hbar}{2}(\psi\dot{\psi}^* - \psi^*\dot{\psi}) + \frac{\hbar^2}{2m}(\nabla\psi)^2 - \frac{GM_0m}{r}\psi^2 - \frac{\Gamma}{2}\psi^4. \tag{5}$$

The choice of the *ansatz* is very important. Regarding what is known about the shape of the BEC wave function, a Gaussian is probably a very reasonable *ansatz* for the case of BECs with attractive interactions ($a_s < 0$) [24]. Thus, variational solutions for Eq. 1 were obtained by assuming a nomalized Gaussian *ansatz*:

$$\psi(\mathbf{r},t) = \sqrt{\frac{N}{\pi^{3/2}\sigma^3}}\mathrm{e}^{-r^2/2\sigma^2}\mathrm{e}^{-\mathrm{i}\mu t/\hbar}, \tag{6}$$

where μ is the chemical potential and both norma N and width σ are variational parameters. The choice of this Gaussian ansatz is motivated by the interest in investigating the collapse of the BEC through the behavior of the chemical potential as a function of the number of dark bosons in the stationary regime.

After choosing *ansatz*, our goal is to determine the Euler-Lagrange equations for the variational parameters. To this aim, we calculate an effective Lagrangian L by integrating the Lagrangian density over the space coordinates:

$$L = \langle\mathscr{L}\rangle = \int_{-\infty}^{+\infty}\mathscr{L}d\mathbf{r} = \int_{-\infty}^{+\infty}\left[\frac{i\hbar}{2}(\psi\dot{\psi}^* - \psi^*\dot{\psi}) + \frac{\hbar^2}{2m}(\nabla\psi)^2 - \frac{GM_0m}{r}\psi^2 - \frac{\Gamma}{2}\psi^4\right]d\mathbf{r}. \tag{7}$$

So, the substitution of *ansatz* (Eq. 6) in Eq. 7 yields:

$$L = N\left[\frac{3\hbar^2}{4m\sigma^2} - \frac{2GM_0mN}{\sqrt{\pi}\sigma} - \frac{\Gamma N}{4\sqrt{2}\pi^{3/2}\sigma^3} - \mu\right]. \tag{8}$$

Euler-Lagrange equations for variational parameters can be obtained by minimizing the effective Lagrangian:

$$\frac{\partial L}{\partial q} = 0, \tag{9}$$

where q are the generalized coordinates $q \equiv \{\sigma, N\}$. Thus, $\partial L/\partial\sigma = 0$ and $\partial L/\partial N = 0$ provide, respectively, the following variational equations:

$$0 = \frac{3\hbar^2}{2m\sigma^3} - \frac{2GM_0m}{\sqrt{\pi}\sigma^2} - \frac{3\Gamma N}{4\sqrt{2}\pi^{3/2}\sigma^4}, \tag{10}$$

$$\mu = \frac{3\hbar^2}{4m\sigma^2} - \frac{2GM_0m}{\sqrt{\pi}\sigma} - \frac{\Gamma N}{2\sqrt{2}\pi^{3/2}\sigma^3}. \tag{11}$$

4. RESULTS

4.1. Critical number of particles

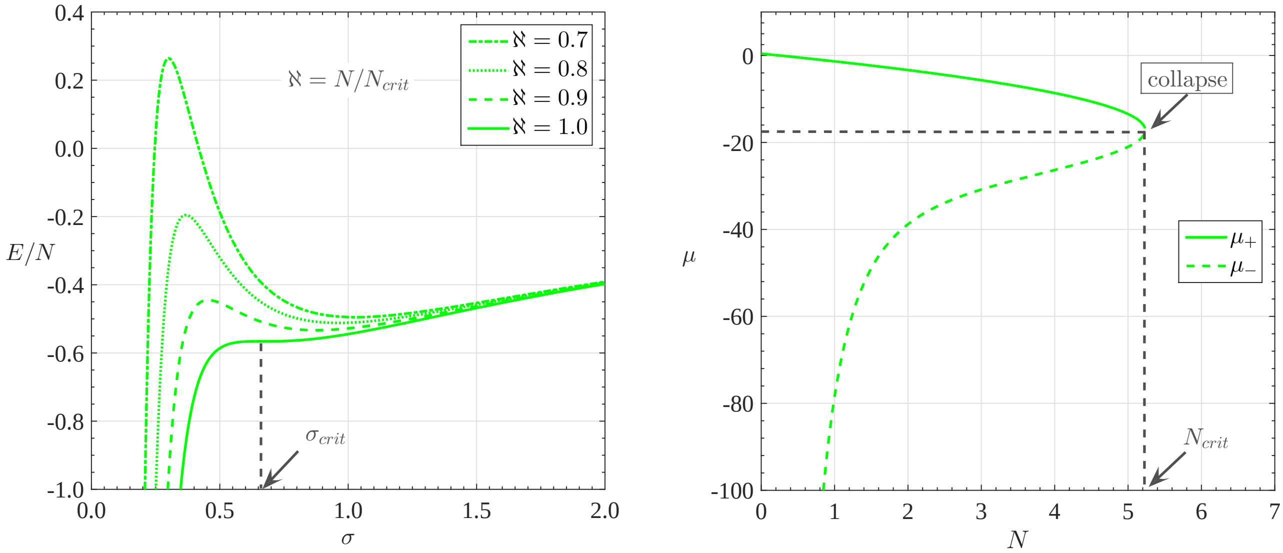
The equation for energy can be obtained through the thermodynamic definition of the chemical potential:

$$\mu = \frac{\partial E}{\partial N}. \tag{12}$$

Substituting Eq. 11 in Eq. 12, the following expression is obtained:

$$E = \frac{3\hbar^2N}{4m\sigma^2} - \frac{2GM_0mN}{\sqrt{\pi}\sigma} - \frac{\Gamma N^2}{4\sqrt{2}\pi^{3/2}\sigma^3}. \tag{13}$$

In the case of attractive BECs, it is observed that the energy diverges as sigma tends to zero. This behavior occurs due to the dominance of the negative interaction energy. Consequently, a solution that minimizes energy is a wavepacket of zero width.



(a) energy per particle versus width

(b) chemical potential versus number of particles

Fig. 1 – Variational results for a spherically symmetrical cosmological BEC with attractive interactions.

Fig. 1a illustrates energy per particle as a function of the variational parameter. As $N \to N_{crit}$, the collapse of the BEC becomes inevitable. Fig. 1b shows the dependence of the chemical potential as a function of the number of particles. Stable (solid green line) and unstable (dotted green line) solutions were obtained using the variational method for a Gaussian *ansatz*. Here, the following parameters were considered to be dimensionless: $c = G = \hbar = \Gamma = m = M_0 = 1$.

However, if the number of particles is less than a certain critical number, the existence of a local minimum of the system energy is observed, which predicts the possibility of a stable BEC with a non-zero width, as illustrated Fig. 1a. In fact, stability occurs when gravitational trapping potential and attractive interactions between dark bosons are counterbalanced by repulsive quantum pressure (Heisenberg's uncertainty principle). As $N \to N_{crit}$, the minimum local energy tends to disappear, giving rise to a narrower BEC. In other words, the equilibrium condition ceases to exist because quantum pressure cannot counterbalance the potential gravitational trapping potential and attractive self-interactions between particles. In this limit, stable solutions do not exist and the collapse of the BEC becomes inevitable.

The N_{crit} satisfies the condition that both the first and second derivative of E with respect to the variational parameter σ is equal to zero [26]:

$$\frac{\partial E}{\partial \sigma} = 0 \to \frac{3\Gamma N^2}{4\sqrt{2}\pi^{3/2}\sigma^4} + \frac{2GM_0 mN}{\sqrt{\pi}\sigma^2} - \frac{3N\hbar^2}{2m\sigma^3} = 0, \tag{14}$$

$$\frac{\partial^2 E}{\partial \sigma^2} = 0 \to \frac{3\Gamma N^2}{\sqrt{2}\pi^{3/2}\sigma^5} + \frac{4GM_0 mN}{\sqrt{\pi}\sigma^3} - \frac{9N\hbar^2}{2m\sigma^4} = 0. \tag{15}$$

When the equations to the first (Eq. 14) and the second (Eq. 15) derivate are equal to each other, the critical variational parameter is given by:

$$\sigma_{crit} = \frac{3\sqrt{\pi}\hbar^2}{8GM_0 m^2}. \tag{16}$$

Using this critical parameter in Eq. 14, the critical value of particle number is given by:

$$N_{crit} = \frac{3\sqrt{2}\pi^2\hbar^4}{8\Gamma GM_0 m^3}. \tag{17}$$

4.2. Chemical potential

With respect to variational equations, Eq. 11 can be rewritten as the following quadratic equation:

$$\sigma^2 - \frac{3\sqrt{\pi}\hbar^2}{4GM_0m^2}\sigma + \frac{3\Gamma N}{8\sqrt{2}\pi GM_0 m} = 0, \tag{18}$$

whose analytical solution is trivial:

$$\sigma_j = \frac{3\pi^{3/2}\hbar^2 \pm \sqrt{9\pi^3\hbar^4 - 12\sqrt{2}\pi\Gamma GM_0m^3N}}{8\pi GM_0m^2}, \tag{19}$$

where $j \equiv \{-,+\}$. Here, σ_- and σ_+ represent, respectively, unstable and stable solutions. Substituting Eq. 19 in Eq. 11, we obtain the chemical potential $\mu_\pm$ whose dependence on the number of particles can be seen in Fig. 1b. The stability of solution σ_+ can be verified by the well-known Vakhitov-Kolokolov (VK) criterion. The VK criterion predicts the existence of regions of stability in BECs, provided the following condition is satisfied:

$$\frac{\partial\mu}{\partial N} < 0. \tag{20}$$

Fig. 1b illustrates the stability regions for solution σ_+. As the number of particles increases, the chemical potential tends to decrease, satisfying Eq. 20. However, when $N = N_{crit}$, the BEC inevitably collapses, making it unstable. At this limit, the VK criterion is not satisfied.

4.3. Collapse and formation of SMBHs

The formation of SMBHs can occur similarly to the collapse of attractive BECs. As mentioned in Sec. 4.1, the critical parameter determines the threshold between the stability and the collapse of the BEC. However, the critical parameter that determines the formation of SMBHs can be described by the Schwarzschild radius:

$$r_s \equiv \frac{2GM_{eff}}{c^2}, \tag{21}$$

where c is the speed of light and M_{eff} is the total effective mass.

It is important to note that we use a non-relativistic formalism in this paper. However, if the size of the BEC decreases beyond a certain limit, its density will be so great that the effects of General Relativity cannot be neglected.

Assuming spherical symmetry, we can consider the mass of the BEC enclosed within a sphere Ω of radius σ for the purpose of estimating the total effective mass of the BEC confined within a Gaussian of width σ [15]:

$$M_{eff} \equiv m\int_\Omega |\psi(\mathbf{r},t)|^2 d\mathbf{r} = \frac{4mN}{\sqrt{\pi}\sigma^3}\int_0^\sigma r^2 \mathrm{e}^{-r^2/\sigma^2} dr = \frac{4mN}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+3)\,n!} = \Lambda mN, \tag{22}$$

where $\Lambda \equiv \mathrm{erf}(1) - 2\left(\mathrm{e}\sqrt{\pi}\right)^{-1}$.

According to Morikawa et al [16], BECs of dark bosons can collapse to form SMBHs when the width of the BEC becomes smaller than the Schwarzschild radius. Therefore, it becomes reasonable to define the Schwarzschild radius (Eq. 21) as being equal to the critical parameter (Eq. 16):

$$\sigma_{crit} = r_s. \tag{23}$$

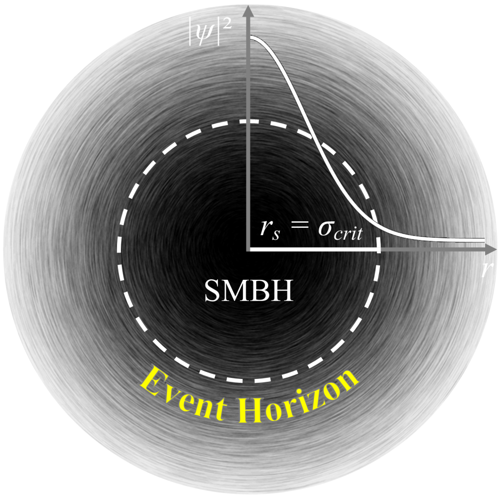


Fig. 2 – Illustration referring to the formation of SMBHs from the collapse of attractive BECs of dark matter. The Schwarzschild radius is the physical parameter that determines the extension of the event horizon, which delimits the boundaries of SMBHs. Similarly, the critical parameter determines the width of the attractive BECs on the verge of collapse.

From Eq. 23, the following relation between the number of particles and the mass of each dark boson is obtained:

$$N = \frac{3\sqrt{\pi}}{16\Lambda}\frac{m_P^4}{M_0 m^3}, \tag{24}$$

where $m_P \equiv \sqrt{\hbar c/G}$ is the Planck mass. Thus, when the total number of particles tends to the critical number, it becomes possible to determine the relation between the mass of each dark boson and the the absolute value of scattering length:

$$N \to N_{crit} \dashrightarrow m = \sqrt{\frac{2}{\pi}\frac{G|a_s|m_P^4}{\Lambda\hbar^2}}. \tag{25}$$

4.4. Entropy

In addition to establishing the conditions for the formation of SMBHs, it becomes interesting to determine its entropy. Indeed, entropy can be quantified through the event horizon [27]. In the context of Schwarzschild's SMBHs, entropy is given by:

$$S_{\rm BH} = \frac{k_B c^3}{G\hbar}\frac{A}{4}, \tag{26}$$

where $A = 4\pi r_s^2$ is the area of the event horizon and k_B is the Boltzmann constant. Thus, replacing Eqs. 21, 24 and 25 in Eq. 26, we obtain:

$$S_{\rm BH} = \left(\frac{\sqrt{3}\pi\Lambda}{4}\frac{l_P}{a_s}\sqrt{\frac{m_P}{M_0}}\right)^4 k_B, \tag{27}$$

where $l_P \equiv \sqrt{\hbar G/c^3}$ is the Planck length. The results obtained predict that, in the event horizon, entropy can be determined in terms of the scattering length due to the attractive interactions between the dark bosons.

5. CONCLUSIONS

In this paper, we considered a scenario where the progeny of SMBHs occurs through the collapse of DM modeled as an attractive cosmological BEC. For this, the SMBH was described by the Gross-Pitaevskii mean-field equation, which was solved through variational formalism using a Gaussian trial wave function. The variational results demonstrated that it is possible to occur the collapse of DM for the formation of SMBHs as proposed in this paper. In addition, we demonstrated that, in the event horizon, the mass of each dark boson can be described in terms of the scattering length, which led us to determine the entropy of the SMBH. The results obtained in this paper can open the way for a better understanding about SMBHs and motivate new studies related to cosmological BECs.

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REFERENCES

1. M.H. ANDERSON, J.R. ENSHER, M.R. MATTHEWS, C.E. WIEMAN, E.A. CORNELL, *Observation of Bose-Einstein condensation in a dilute atomic vapor*, Science, **269**, *5221*, pp. 198–201, 1995.
2. K.B. DAVIS, M.O. MEWES, M.R. ANDREWS, N.J. VAN DRUTEN, D.S. DURFEE, D.M. KURN, W. KETTERLE, *Bose-Einstein condensation in a gas of sodium atoms*, Phys. Rev. Lett., **75**, pp. 3969–3973, 1995.
3. C.C. BRADLEY, C.A. SACKETT, J.J. TOLLETT, R.G. HULET, *Evidence of Bose-Einstein condensation in an atomic gas with attractive interactions*, Phys. Rev. Lett., **75**, pp. 1687–1690, 1995.
4. E.P. GROSS, *Structure of a quantized vortex in boson systems*, Il Nuovo Cimento (1955-1965), **20**, *3*, pp. 454–477, 1961.
5. L.P. PITAEVSKII, *Vortex lines in an imperfect Bose gas*, Sov. Phys. JETP, **13**, *2*, pp. 451–454, 1961.
6. V.T. TOTH, *Self-gravitating Bose-Einstein condensates and the Thomas-Fermi approximation*, Galaxies, **4**, *3*, p. 9, 2016.
7. C. GRUBER, A. PELSTER, *Bose-Einstein condensates in compact astrophysical objects*, In: *Selforganization in complex systems: The past, present, and future of Synergetics* (eds. G. Wunner, A. Pelster), Springer International Publishing, Cham, 2016, pp. 297–304.
8. V.A. POPOV, *Dark energy and dark matter unification via superfluid Chaplygin gas*, Physics Letters B, **686**, *4*, pp. 211–215, 2010.
9. J. FAN, *Ultralight repulsive dark matter and BEC*, Physics of the Dark Universe, **14**, pp. 84–94, 2016.
10. A.M. GAVRILIK, I.I. KACHURIK, M.V. KHELASHVILI, A.V. NAZARENKO, *Condensate of μ-Bose gas as a model of dark matter*, Physica A: Statistical Mechanics and its Applications, **506**, pp. 835–843, 2018.
11. E. KUN, Z. KERESZTES, S. DAS, L.Á. GERGELY, *Dark matter as a non-relativistic Bose-Einstein condensate with massive gravitons*, Symmetry, **10**, *10*, p. 520, 2018.
12. D. HARTLEY, C. KÄDING, R. HOWL, I. FUENTES, *Quantum simulation of dark energy candidates*, Phys. Rev. D, **99**, p. 105002, May 2019.
13. M. BASHKANOV, D.P. WATTS, *A new possibility for light-quark dark matter*, Journal of Physics G: Nuclear and Particle Physics, **47**, *3*, p. 03LT01, 2020.
14. A.A. HUJEIRAT, *On the viability of gravitational Bose-Einstein condensates as alternatives to supermassive black holes*, Monthly Notices of the Royal Astronomical Society, **423**, *3*, pp. 2893–2900, 2012.
15. P.D. GUPTA, E. THAREJA, *Supermassive black holes from collapsing dark matter Bose-Einstein condensates*, Classical and Quantum Gravity, **34**, *3*, p. 035006, 2017.
16. M. MORIKAWA, S, TAKAHASHI, *Supermassive black holes and dark halo from the Bose-condensed dark matter*, Proceedings (MDPI), **13**, *1*, p. 11, 2019.
17. C. GRUBER, A. PELSTER, *A theory of finite-temperature Bose-Einstein condensates in neutron stars*, The European Physical Journal D, **68**, *11*, p. 341, 2014.
18. C.J. PETHICK, T. SCHÄFER, A. SCHWENK, *Bose-Einstein condensates in neutron stars*, Cambridge University Press, 2017, pp. 573–592.
19. M.E. MOSQUERA, O. CIVITARESE, O.G. BENVENUTO, M.A. DE VITO, *Bose-Einstein condensation in helium white dwarf stars. I*, Physics Letters B, **683**, *2*, pp. 119–122, 2010.

20. R.C. FREITAS, G.A. MONERAT, G. OLIVEIRA-NETO, F.G. ALVARENGA, S.V.B. GONÇALVES, R. FRACALOSSI, E.V. CORRÊA SILVA, L.G. FERREIRA FILHO, *Primordial Universe with radiation and Bose-Einstein condensate*, Physics of the Dark Universe, **25**, p. 100325, 2019.
21. H. BERGERON, E. CZUCHRY, J.P. GAZEAU, P. MAŁKIEWICZ, *Quantum mixmaster as a model of the Primordial Universe*, Universe, **6**, *1*, p. 7, 2019.
22. M.J. REES, *Black hole models for active galactic nuclei*, Annual Review of Astronomy and Astrophysics, **22**, *1*, pp. 471–506, 1984.
23. G.E. ROMERO, G.S. VILA, *Black hole physics*, Springer Berlin Heidelberg, 2014, pp. 73–97.
24. V.M. PÉREZ-GARCÍA, H. MICHINEL, J.I. CIRAC, M. LEWENSTEIN, P. ZOLLER, *Dynamics of Bose-Einstein condensates: Variational solutions of the Gross-Pitaevskii equations*, Phys. Rev. A, **56**, pp. 1424–1432, 1997.
25. J. ROGEL-SALAZAR, *The Gross-Pitaevskii equation and Bose-Einstein condensates*, European Journal of Physics, **34**, *2*, pp. 247–257, 2013.
26. F. DALFOVO, S. GIORGINI, L.P. PITAEVSKII, S. STRINGARI, *Theory of Bose-Einstein condensation in trapped gases*, Rev. Mod. Phys., **71**, pp. 463–512, Apr. 1999.
27. C.A. EGAN, C.H. LINEWEAVER, *A larger estimate of the entropy of the universe*, The Astrophysical Journal, **710**, *2*, pp. 1825–1834, 2010.

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