THE PUBLISHING HOUSE
OF THE ROMANIAN ACADEMY

PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A
Volume 22, Number 2/2021, pp. 119–125

CONGRUENCE IDENTITIES INVOLVING SUMS OF ODD DIVISORS FUNCTION

Mircea MERCA

University of Craiova, Department of Mathematics
A. I. Cuza 13, Craiova 200585, Romania
Corresponding author: Mircea MERCA, E-mail: mircea.merca@profinfo.edu.ro

Abstract. In this paper, inspired by a classical connections between partitions and divisors, we investigate some congruence identities involving sums of the odd divisor function $\sigma_{odd}(n)$ which is defined by $\sigma_{odd}(n) = \sum_{\substack{d|n \\ d \text{ odd}}} d$. In this context, we conjectured that the congruence

$$\sum_{k=-\infty}^{\infty} \sigma_{odd}\big(n-k(3k-1)/2\big) \equiv \begin{cases} n \pmod m, & \text{if } n = j(3j-1)/2,\ j \in \mathbb{Z}, \\ 0 \pmod m, & \text{otherwise.} \end{cases}$$

is valid for any positive integer n if and only if $m \in \{2,3,6\}$.

Key words: theta series, partitions, divisors.

1. INTRODUCTION

The object of our investigations is the divisor function $\sigma_{odd}(n)$ which is defined as the sum of the odd positive divisors of n, i.e.,

$$\sigma_{odd}(n) := \sum_{\substack{d|n \\ d \text{ odd}}} d.$$

Throughout this paper, we consider $\sigma_{odd}(n) = 0$ for $n \leqslant 0$. Recall that the function $\sigma_{odd}(n)$ is the coefficient of q^n in the following Lambert series expansion

$$\sigma_{odd}(n) = [q^n] \sum_{n=1}^{\infty} \frac{(2n-1)q^{2n-1}}{1-q^{2n-1}}, \qquad |q| < 1. \tag{1}$$

On the other hand, the function $\sigma_{odd}(n)$ appears naturally as the coefficients of a modular form. It is related to the eta η-Dedekind function and Eisenstein series $E_{2,2}$.

Recall [2, Chap. 3] that the Dedekind eta function $\eta(\tau)$ is given by

$$\eta(\tau) := q^{1/24} \prod_{n=1}^{\infty} (1-q^n), \tag{2}$$

where $q = e^{2\pi i \tau}$ and $\text{Im}(\tau) > 0$. It is well known that the η-function is a modular form of weight $1/2$ and level 1 for a certain character of order 24 of the metaplectic double cover of the modular group. The eta quotient $\eta(\tau)/\eta(2\tau)$ is equal to

$$\frac{\eta(\tau)}{\eta(2\tau)} = q^{-1/24} \prod_{n=1}^{\infty} (1-q^{2n-1}). \tag{3}$$

By the logarithmic derivative of this formula, we get the Eisenstein series

$$E_{2,2}(\tau) = -\frac{1}{24} + q\sum_{n=1}^{\infty} \frac{-(2n-1)q^{2n-2}}{q^{2n-1}} = -\frac{1}{24} - \sum_{n=1}^{\infty} \sigma_{odd}(n)q^n. \tag{4}$$

It is known that $E_{2,2}(\tau)$ is a modular form for the congruence subgroup $\Gamma_0(2)$ [5, pp. 18-19].

In [4, Chap. 3, Section 3.3], the odd divisor function $\sigma_{odd}(n)$ is related to the topic of *sums of four squares*. More details about arithmetic properties of $\sigma_{odd}(n)$ can be found in [3].

A partition of a positive integer n is a sequence of positive integers whose sum is n. The order of the summands is unimportant when writing the partitions of n, but for consistency, a partition of n will be written with the summands in a nonincreasing order [1]. The Euler partition function $p(n)$ gives the number of ways of writing the nonnegative integer n as a sum of positive integers, where the order of addends is not considered significant. For example, the partitions of 5 are 5, $4+1$, $3+2$, $3+1+1$, $2+2+1$, $2+1+1+1$ and $1+1+1+1+1$. Thus, $p(5) = 7$. The generating function of $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}$$

and the expansion starts as

$$\frac{1}{(q;q)_\infty} = 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + 11q^6 + 15q^7 + 22q^8 + 30q^9 + \cdots .$$

Here and throughout this paper, we use the following customary q-series notation:

$$(a;q)_n = \begin{cases} 1, & \text{for } n = 0, \\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & \text{for } n > 0; \end{cases}$$
$$(a;q)_\infty = \lim_{n\to\infty} (a;q)_n.$$

Because the infinite product $(a;q)_\infty$ diverges when $a \neq 0$ and $|q| \geqslant 1$, whenever $(a;q)_\infty$ appears in a formula, we shall assume $|q| < 1$.

The divisors of numbers have been studied from the point of view of partitions of integers for a long time. It is well know that Euler's partition function $p(n)$ and the sum of divisors function

$$\sigma(n) := \sum_{d|n} d$$

satisfy common recursive relations with only $p(0)$ different from $\sigma(0)$:

$$\sum_{k=-\infty}^{\infty} (-1)^k p\big(n - P_5(k)\big) = \delta_{0,n}, \quad \text{with} \quad p(0) = 1$$

and

$$\sum_{k=-\infty}^{\infty} (-1)^k \sigma\big(n - P_5(k)\big) = 0, \quad \text{with } \sigma(0) \text{ replaced by } n, \tag{5}$$

where $\delta_{i,j}$ is the Kronecker delta and

$$P_m(n) := \Big(\frac{m}{2} - 1\Big)n^2 - \Big(\frac{m}{2} - 2\Big)n$$

is the nth generalized m-gonal number. It is clear that the divisors functions $\sigma(n)$ and $\sigma_{odd}(n)$ have the same parity, i.e.,

$$\sigma(n) \equiv \sigma_{odd}(n) \pmod 2.$$

By identity (5), we easily deduce the following parity results.

THEOREM 1. *For $n \geqslant 0$,*

$$\sum_{k=-\infty}^{\infty} \sigma_{odd}\big(n-P_5(k)\big) \equiv 1 \pmod 2$$

if and only if n is an odd generalized pentagonal number.

It is well known that $\sigma_{odd}(n)$ is odd if and only if n is a square or a twice square. Thus we deduce the following parity result.

COROLLARY 1. *Let n be a positive integer. The number of representations of n as the sum of a generalized pentagonal number and a square or a twice square is odd if and only if n is an odd generalized pentagonal number.*

The first generalized pentagonal numbers are

$$0,1,2,5,7,12,15,22,26,35,40,51,\ldots$$

As we can see, 51 is an odd generalized pentagonal number that can be represented as a sum of a generalized pentagonal number and a square or twice square in five different ways:

$$51 = 1+2\cdot 5^2 = 2+7^2 = 15+6^2 = 26+5^2 = 35+4^2.$$

In this article, we investigate the positive integers m for which the following congruence identities are valid for any positive integer n:

$$\sum_{k=-\infty}^{\infty} \sigma_{odd}\big(n-P_m(k)\big) \equiv \begin{cases} n \pmod 2, & \text{if } n = P_m(j),\ j\in\mathbb{Z}, \\ 0 \pmod 2, & \text{otherwise,} \end{cases} \tag{6}$$

$$\sum_{k=-\infty}^{\infty} \sigma_{odd}\big(n-P_5(k)\big) \equiv \begin{cases} n \pmod m, & n = P_5(j),\ j\in\mathbb{Z}, \\ 0 \pmod m, & \text{otherwise,} \end{cases} \tag{7}$$

$$\sum_{k=-\infty}^{\infty} (-1)^{P_3(-k)}\sigma_{odd}\big(n-P_5(k)\big) \equiv \begin{cases} (-1)^{P_3(-j)}\cdot n \pmod m, & \text{if } n = P_5(j),\ j\in\mathbb{Z}, \\ 0 \pmod m, & \text{otherwise.} \end{cases} \tag{8}$$

It is clear that the case $m=5$ of (6) is the case $m=2$ of (7) and (8). Note that generalized 3-gonal numbers are triangular numbers and generalized 4-gonal numbers are squares of integers. It is known that generalized hexagonal numbers are identical with triangular numbers. We have the following equivalent form of the congruence (6): *The number of representation of n as the sum of a generalized m-gonal number and a square or a twice square is odd if and only if n is an odd generalized m-gonal number.*

There is a substantial amount of numerical evidence to conjecture the following assertions.

CONJECTURE 1. *The congruence* (6) *is valid for any positive integer n if and only if $m \in \{5,6\}$.*

CONJECTURE 2. *The congruence* (7) *is valid for any positive integer n if and only if $m \in \{2,3,6\}$.*

CONJECTURE 3. *The congruence* (8) *is valid for any positive integer n if and only if $m \in \{2,4\}$.*

In Section 2, we prove one implication of Conjecture 1, i.e., if $m \in \{5,6\}$, then (6) holds for any positive integer n. We remark that, the case $m=6$ of this implication reads as follows.

THEOREM 2. *For $n \geqslant 0$,*

$$\sum_{k=0}^{\infty} \sigma_{odd}\big(n-P_3(k)\big) \equiv 1 \pmod 2$$

if and only if n is an odd triangular number.

The other implication has been verified for all integers m with $m < 100000$. In Section 3, we prove one implication of Conjecture 2, i.e., if $m \in \{2,3,6\}$, then (7) holds for any positive integer n. In Section 4, we prove one implication of Conjecture 3, i.e., if $m \in \{2,4\}$, then (8) holds for any positive integer n.

2. PROOFS OF THEOREMS 1 AND 2

As usual, we denote by $Q(n)$ the number of integer partitions of n into odd parts. For example, $Q(7) = 5$ because the five partitions of 7 odd parts are 7, $5+1+1$, $3+3+1$, $3+1+1+1+1$, $1+1+1+1+1+1+1$. We remark that the generating function of $Q(n)$ is given by

$$\sum_{n=0}^{\infty} Q(n)q^n = \frac{1}{(q;q^2)_\infty} = (-q;q)_\infty.$$

The logarithmic differentiation of the generating function for $Q(n)$ gives:

$$\frac{d}{dq}\frac{1}{(q;q^2)_\infty} = -\frac{1}{(q;q^2)_\infty^2}\frac{d}{dq}(q;q^2)_\infty = \frac{1}{(q;q^2)_\infty}\sum_{n=1}^{\infty}\frac{(2n-1)q^{2n-2}}{1-q^{2n-1}}.$$

On the other hand, we have

$$\frac{d}{dq}\frac{1}{(q;q^2)_\infty} = \frac{d}{dq}\sum_{n=0}^{\infty} Q(n)q^n = \sum_{n=1}^{\infty} nQ(n)q^{n-1}.$$

Thus we deduce that

$$\sum_{n=1}^{\infty} nQ(n)q^n = (-q;q)_\infty \sum_{n=1}^{\infty}\frac{(2n-1)q^{2n-1}}{1-q^{2n-1}} = (-q;q)_\infty \sum_{n=1}^{\infty}\sigma_{odd}(n)q^n. \tag{9}$$

The following theta identity is often attributed to Gauss [1, p.23, eqs. (2.2.13)]:

$$1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2} = \frac{(q;q)_\infty}{(-q;q)_\infty}. \tag{10}$$

By (9) and (10), we obtain

$$\begin{aligned} nQ(n)+2\sum_{k=1}^{\infty}(-1)^k(n-k^2)Q(n-k^2) &= [q^n]\left(\frac{(q;q)_\infty}{(-q;q)_\infty}\cdot(-q;q)_\infty\sum_{n=1}^{\infty}\sigma_{odd}(n)q^n\right) = \\ &= \sum_{k=-\infty}^{\infty}(-1)^k\sigma_{odd}\big(n-k(3k-1)/2\big), \end{aligned}$$

where we have invoked Euler's pentagonal number theorem

$$(q;q)_\infty = \sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2}. \tag{11}$$

In this way, we deduce that

$$nQ(n) \qquad \text{and} \qquad \sum_{k=-\infty}^{\infty}(-1)^k\sigma_{odd}\big(n-k(3k-1)/2\big)$$

have the same parity. According to [6, Corollary 4.7], $Q(n)$ is odd if and only if n is a generalized pentagonal number. This concludes the proof of Theorem 1.

In order to prove Theorem 2, we consider another theta series identity of Gauss: [1, p.23, eqs. (2.2.13)]

$$\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \sum_{n=0}^{\infty} q^{n(n+1)/2}. \tag{12}$$

Considering this identity, we can write

$$\begin{aligned}\sum_{k=-\infty}^{\infty} (-1)^k Q\big(n-k(3k-1)\big) &= [q^n]\left((q^2;q^2)_\infty \cdot \frac{1}{(q;q^2)_\infty}\right) = \\ &= [q^n]\sum_{n=0}^{\infty} q^{n(n+1)/2} = \\ &= \begin{cases} 1, & \text{if } n \text{ is a triangular number,} \\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

Thus we deduce that

$$n\sum_{k=-\infty}^{\infty} Q\big(n-k(3k-1)\big) \equiv 1 \pmod 2 \tag{13}$$

if and only if n is an odd triangular number.

On the other hand, taking into account (9), we can write

$$\begin{aligned}\sum_{k=-\infty}^{\infty} (-1)^k \big(n-k(3k-1)\big) Q\big(n-k(3k-1)\big) &= [q^n]\left((q^2;q^2)_\infty \sum_{n=1}^{\infty} nQ(n)q^n\right) = \\ &= [q^n]\left(\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\sum_{n=1}^{\infty} \sigma_{odd}(n)q^n\right) = \\ &= \sum_{k=0}^{\infty} \sigma_{odd}\big(n-k(k+1)/2\big).\end{aligned}$$

By this identity, taking into account that $k(3k-1)$ is even, we deduce that

$$\sum_{k=0}^{\infty} \sigma_{odd}\big(n-k(k+1)/2\big) \equiv n\sum_{k=-\infty}^{\infty} Q\big(n-k(3k-1)\big) \pmod 2.$$

The proof of Theorem 2 follows easily considering (13).

3. CONGRUENCES MODULO 2, 3 AND 6

The congruence provided by Theorem 1 motivates us to look for other similar results involving the divisor function σ_{odd} and generalized pentagonal numbers. We experimentally found that the coefficient of q^n in the series

$$\begin{aligned}&(-q;q)_\infty \frac{(q^3;q^3)_\infty}{(-q^3;q^3)_\infty}\sum_{n=1}^{\infty}\frac{(2n-1)q^{2n-1}}{1-q^{2n-1}} = \\ &= q+2q^2+6q^3+6q^4+11q^5+12q^6+19q^7+18q^8+24q^9+30q^{10}+36q^{11}+36q^{12} \\ &\quad +36q^{13}+48q^{14}+57q^{15}+60q^{16}+60q^{17}+66q^{18}+72q^{19}+84q^{20}+84q^{21}+106q^{22}+\cdots\end{aligned} \tag{14}$$

is congruent to 0 modulo 6 if and only if n is not a generalized pentagonal number or n is a generalized pentagonal number congruent to 0 modulo 6. For $0 < r < 6$ we notice that the coefficient of q^n in (14) is congruent to r modulo 6 if and only if n is a generalized pentagonal number congruent to r modulo 6. Considering the Jacobi triple product identity

$$(q;q)_\infty(z;q)_\infty(q/z;q)_\infty = \sum_{n=-\infty}^{\infty}(-1)^n z^n q^{n(n-1)/2}, \quad |q|<1,\ z\neq 0,$$

we deduce that

$$(-q;q)_\infty\frac{(q^3;q^3)_\infty}{(-q^3;q^3)_\infty} = (-q;q^3)_\infty(-q^2;q^3)_\infty(q^3;q^3)_\infty = \sum_{n=-\infty}^{\infty} q^{n(3n-1)/2}.$$

Thus we can state the following result.

THEOREM 3. *For $n>0$, $m\in\{2,3,6\}$,*

$$\sum_{k=-\infty}^{\infty}\sigma_{odd}\big(n-P_5(k)\big)\equiv\begin{cases} n \pmod m, & \text{if } n=P_5(j),\ j\in\mathbb{Z},\\ 0 \pmod m, & \text{otherwise.}\end{cases}$$

Proof. The proof of this theorem is quite similar to the proof of Theorem 2. We have

$$\begin{aligned}\sum_{k=-\infty}^{\infty}\sigma_{odd}\big(n-k(3k-1)/2\big) &= [q^n]\left(\Big(\sum_{n=-\infty}^{\infty}q^{n(3n-1)/2}\Big)\Big(\sum_{n=1}\sigma_{odd}(n)q^n\Big)\right)=\\ &= [q^n]\left((-q;q)_\infty\frac{(q^3;q^3)_\infty}{(-q^3;q^3)_\infty}\sum_{n=1}^{\infty}\frac{(2n-1)q^{2n-1}}{1-q^{2n-1}}\right)=\\ &= nQ(n)+2\sum_{k=1}^{\infty}(-1)^k(n-3k^2)Q(n-3k^2)=\\ &= n\left(Q(n)+2\sum_{k=1}^{\infty}(-1)^kQ(n-3k^2)\right)-6\sum_{k=1}^{\infty}(-1)^k k^2 Q(n-3k^2)\end{aligned}$$

and

$$\begin{aligned}Q(n)+2\sum_{k=1}^{\infty}(-1)^kQ(n-3k^2) &= [q^n]\left((-q;q)_\infty\frac{(q^3;q^3)_\infty}{(-q^3;q^3)_\infty}\right)=\\ &= [q^n]\sum_{n=-\infty}^{\infty}q^{n(3n-1)/2}=\\ &= \begin{cases}1, & \text{if } n \text{ is a generalized pentagonal number},\\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

For $m\in\{2,3,6\}$, we deduce that

$$\sum_{k=-\infty}^{\infty}\sigma_{odd}\big(n-k(3k-1)/2\big)\equiv\begin{cases} n \pmod m, & \text{if } n \text{ is a generalized pentagonal number},\\ 0 \pmod m, & \text{otherwise.}\end{cases}$$

This concludes the proof. □

4. CONGRUENCES MODULO 2 AND 4

In this section, we prove the following congruence identity.

THEOREM 4. *For $n > 0$, $m \in \{2,4\}$,*

$$\sum_{k=-\infty}^{\infty} (-1)^{P_3(-k)} \sigma_{odd}\big(n - P_5(k)\big) \equiv \begin{cases} (-1)^{P_3(-j)} \cdot n \pmod m, & \text{if } n = P_5(j),\ j \in \mathbb{Z}. \\ 0 \pmod m, & \text{otherwise.} \end{cases}$$

Proof. The proof of this theorem is quite similar to the proof of Theorem 3. Considering Euler's pentagonal number theorem (11), we deduce that

$$(-q;q)_\infty \frac{(q^2;q^2)_\infty}{(-q^2;q^2)_\infty} = (-q;q^2)_\infty (q^2;q^2)_\infty = \sum_{n=-\infty}^{\infty} (-1)^{n(n-1)/2} q^{n(3n-1)/2}.$$

We can write

$$Q(n) + 2\sum_{k=1}^{\infty} (-1)^k Q(n-2k^2) = [q^n] \left((-q;q)_\infty \frac{(q^2;q^2)_\infty}{(-q^2;q^2)_\infty} \right)$$
$$= \begin{cases} (-1)^{j(j-1)/2}, & \text{if } n = j(3j-1)/2,\ j \in \mathbb{Z}, \\ 0, & \text{otherwise} \end{cases}$$

and

$$\begin{aligned} \sum_{k=-\infty}^{\infty} (-1)^{k(k-1)/2} \sigma_{odd}\big(n - k(3k-1)/2\big) &= [q^n] \left(\Big(\sum_{n=-\infty}^{\infty} (-1)^{n(n-1)/2} q^{n(3n-1)/2} \Big) \Big(\sum_{n=1} \sigma_{odd}(n) q^n \Big) \right) \\ &= [q^n] \left((-q;q)_\infty \frac{(q^2;q^2)_\infty}{(-q^2;q^2)_\infty} \sum_{n=1}^{\infty} \frac{(2n-1)q^{2n-1}}{1-q^{2n-1}} \right) \\ &= nQ(n) + 2\sum_{k=1}^{\infty} (-1)^k (n-2k^2) Q(n-2k^2) \\ &= n \left(Q(n) + 2\sum_{k=1}^{\infty} (-1)^k Q(n-2k^2) \right) - 4\sum_{k=1}^{\infty} (-1)^k k^2 Q(n-2k^2). \end{aligned}$$

The proof follows easily. □

REFERENCES

1. G.E. ANDREWS, *The theory of partitions*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1998 (reprint of the 1976 original).
2. T.M. APOSTOL, *Modular functions and Dirichlet series in number theory, Second edition*, Graduate Texts in Mathematics, vol. 41, Springer-Verlag, New York, 1990.
3. C. BALLANTINE, M. MERCA, *Jacobi's four and eight squares theorems*, Mediterr. J. Math., **16**, p. 26, 2019.
4. B.C. BERNDT, *Number theory in the spirit of Ramanujan*, Student Mathematical Library, vol. 34, AMS, Providence, Rhode Island, 2006.
5. F. DIAMOND, J. SHURMAN, *A first course in modular forms*, Springer-Verlag, New York, 2005.
6. M. MERCA, *Combinatorial interpretations of a recent convolution for the number of divisors of a positive integer*, J. Number Theory, **160**, pp. 60–75, 2016.

Received March 16, 2021