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ON THE SIMULTANEOUS MULTI-POINT COLLISION OF THE RIGID SOLID

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Abstract. In this paper the problem of the collision of the rigid solids is generalized for the simultaneous multi-point ones. We considered the case of the collision between the rigid solid and many impulses and we proved that the problem has a determined solution only if the directions of the impulses satisfy certain conditions. We also discuss the case of the collision of a constrained rigid solid, as well the collision between two rigid solids. The obtained formulae are given in matrix form and they can be easily implemented in different calculation algorithms. The approach is based on screw theory. Some examples highlight the theory.

Key words*:* simultaneous multi-point collision, coefficient of restitution, impulses, constraints.

1. INTRODUCTION

The problem of the collision of rigid solids is discussed in many papers. The authors usually consider the collision of a rigid solid with one impulse and the collision of two rigid solids at only one point. The accepted working hypotheses are those stated in [2, 3, 9, 11, 12, 16, 17]: the values of the impact forces are high enough so that one may neglect all other forces acting on the rigid bodies; the collision has two phases: the compression and the expansion; the positions of the rigid solids remain constant during the impact; the tangential stiffness is infinite. Another problem is the definition of the coefficient of restitution. Three main definitions are considered now: Newton, Poisson and energetic. For the collision without friction the equality of the three coefficients is proved [4, 11, 12, 16, 17, 22]. The problem of the definition of the coefficient of restitution in the case of collision with friction is considered in [3, 4, 11, 12, 19, 22, 23], the authors proving that the consideration of the Newton or Poisson variants may lead to inconsistent results. Batlle [1], Brogliato [4], Glocker [11, 12], Pandrea and Stănescu [17], Pfeiffer [19] studied the collision with friction of two rigid solids for which the values of the three coefficients are not equal. Pandrea and Stănescu [17] proved the possibility that one coefficient of restitution may be greater than one in the case of collision with friction. Aspects concerning the collision of kinematic chains or multi-body systems are discussed in [1, 9, 10, 13, 18, 24, 25] for simple examples highlighting the main phenomena that may appear during the collision process. Most authors deal with the classic theorems in collision: the theorem of momentum and the theorem of moment of momentum obtaining systems of equations, which imply calculation difficulties. Brogliato [4] discusses the collisions considering a combination between the classical and the screw approaches. Pandrea [14, 15], Pandrea and Stănescu [16, 17], and Stănescu *et al*. [20] considered the screw coordinates for the problems of collision with and without friction, obtaining matrix equation which are easily implemented in various algorithms of calculations. Stronge [21] discusses the planar case of collision with friction, establishes some relations between different coefficients of restitution and conditions for their equality. Chatterjee *et al.* [5] studied the planar cases with friction at two points using a global energetic coefficient of restitution similar to Stronge's one. Djerassi [8] analyses the five types of planar collision with friction, establishes the coherence conditions for each type and proves that Stronge's hypothesis leads to the existence and uniqueness of a coherent and energy-consistent solution. Djerassi [6, 7] studies the five possible cases of planar collisions and proves that using the Newton coefficient of restitution the mechanical energy may increase in the situations of sticking or reverse sliding. Moreover, Poisson's coefficient of restitution leads to unique solution which is coherent and energy-consistent; there is concordance between the Poisson and Stronge coefficients of restitution and the Poisson coefficient of restitution is preferable.

The problems of simultaneous multi-point collisions are not generally discussed yet. In this paper we consider some aspects of this problem and we prove that a solution of it may be obtained only in some cases when the directions of the reaction impulses satisfy certain conditions. The main hypothesis we add to the previous ones consists in the simultaneous canceling of the normal velocities at the collision points and no jamb phenomenon [2] appears. We use the screw approach in order to obtain the general calculation formulae, which are given in matrix form for an easy implementation in calculation algorithms. The paper is divided as follows: in section 2 we present the notations used, in section 3 we discuss the case of the simultaneous multi-point collision of a free rigid solid, in section 4 is treated the case of the constrained rigid, while in section 5 we deal with the case of the collision of two free rigid solids. The last part of the paper is dedicated to conclusions.

2. NOTATIONS

We denote: O – the center of weight of the rigid solid; $Oxzy$ – the system of the principal central axes of inertia; m – the mass of the rigid; J_x, J_y, J_z – the moments of inertia; $\mathbf{v}^0$, $\mathbf{v}$ – the velocities of the point O before and after collision; $\boldsymbol{\omega}^0$, $\boldsymbol{\omega}$ – the angular velocities of the rigid solid before and after collision; A_i, $i=\overline{1,\ n}$ – the points at which the rigid solid is collided; P_i, $i=\overline{1,\ n}$ – the magnitudes of the impulses at the points A_i; $\mathbf{u}_i$, $i=\overline{1,\ n}$ – the unit vector of the impulse at the point A_i; $\mathbf{r}_i$, $i=\overline{1,\ n}$ – the vector $\mathbf{OA}_i$; a_i, b_i, c_i, and d_i, e_i, f_i – the projections of the vectors $\mathbf{u}_i$ and $\mathbf{r}_i\times\mathbf{u}_i$, respectively, on the axes Ox, Oy, Oz; $\{\mathbf{U}_i\}$ – the column matrix of the screw coordinates of the straight line of the impulse at the point A_i, $i=\overline{1,\ n}$; $\{\mathbf{P}\}$ – the column matrix of the impulses; $\{\mathbf{P}\}=[P_1 \quad \ldots \quad P_n]^{\mathrm{T}}$; v_{in}^0, v_{in}, $i=\overline{1,\ n}$ – the projections of the velocities of the points A_i on the directions of the impulses before and after collision; $[U]$ – the matrix given by $[\mathbf{U}]=[\{U_1\} \quad \ldots \quad \{U_n\}]$; k_i, $i=\overline{1,\ n}$ – the coefficients of restitution at the points A_i; $[\mathbf{K}]$ – the matrix given by $[\mathbf{K}]=\mathrm{diag}(k_i)$; $\{\mathbf{S}_i\}$, $i=\overline{1,\ n}$ – the matrix of the screw coordinates of the simple restrictions of the reaction impulses; $\{\mathbf{Q}_i\}$, $i=\overline{1,\ 6-n}$ – the matrix of the screw coordinates of the possible motions; $[\mathbf{S}]$ – the matrix $[\mathbf{S}]=[\{\mathbf{S}_1\} \quad \ldots \quad \{\mathbf{S}_n\}]$; $[\mathbf{Q}]$ – the matrix $[\mathbf{Q}]=[\{\mathbf{Q}_1\} \quad \ldots \quad \{\mathbf{Q}_{6-n}\}]$; ζ_i, $i=\overline{1,\ n}$ – the scalar values of impulses; ξ_i, $i=\overline{1,\ 6-n}$ – the scalar values of the velocities; $\{\zeta\}$ – the column matrix $\{\zeta\}=[\zeta_1 \quad \ldots \quad \zeta_n]^{\mathrm{T}}$; $\{\xi\}$ – the matrix column $\{\xi\}=\{\xi_1 \quad \ldots \quad \xi_{6-n}\}^{\mathrm{T}}$. The rest of notations may be found in [4, 14, 15, 16, 17].

3. THE SIMULTANEOUS COLLISION OF THE FREE RIGID AT MANY POINTS

The working schema is captured in Fig. 1. The theorems of momentum and moment of momentum lead to the matrix relation

$$\{\mathbf{V}\}-\{\mathbf{V}^0\}=[\mathbf{M}]^{-1}[\mathbf{U}]\{\mathbf{P}\}. \tag{1}$$

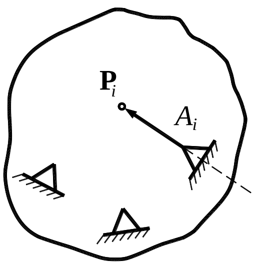

Fig. 1 – Working schema.

We assume that the rank of matrix $[\mathbf{U}]$ is equal to *n*. If this condition does not hold true, then [9] the support straight lines of the impulses must satisfy the following requirements: a) two straight lines do not coincide; b) three straight lines are not coplanar, concurrent or parallel; c) four straight lines are not: i) the generatrices of the same ruled quadric, ii) concurrent in space, iii) parallel in space, iv) situated in the same plane; c) five straight lines must: i) not intersect two given straight lines, ii) a part of them intersect a straight line, the rest of them being parallel to a given plane, iii) a part of them are situated in a plane, the rest being concurrent at a point of the plane, iv) some of them are situated in a plane, the rest of them being parallel to a straight line of the plane, v) some of them are parallel to a straight line, the rest of them being the straight line from infinite of the space; e) six straight lines must not: i) be normal to a family of helices of same step; ii) intersect the same straight line, iii) be parallel to a plane, iv) form two stars of concurrent straight lines, v) belong to two stars of parallel straight lines, vi) form a star of concurrent straight lines and a star of parallel straight lines.

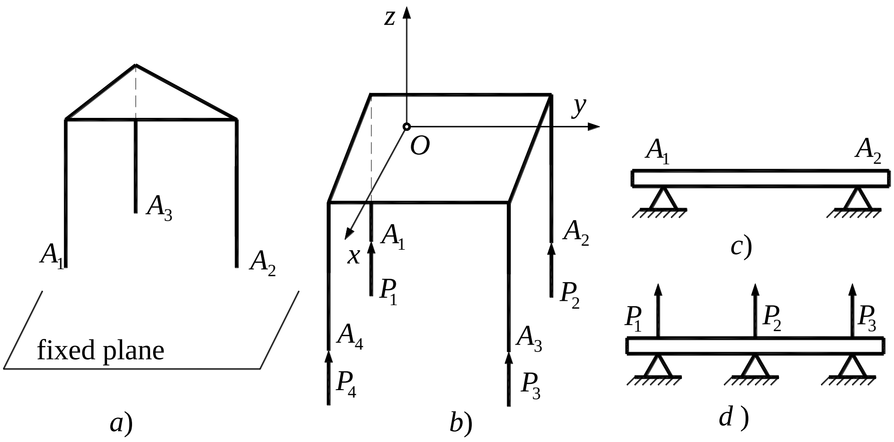

Fig. 2 – Examples of determined (a) and c)) and undetermined cases (b) and d)).

Based on these statements, it results that cases presented in Fig. 2a and Fig. 2c may be solved, while the cases presented in Fig. 2b and Fig. 2d are not determined ones. For instance, for the situation captured in Fig. 2d one may write the equation $[\mathbf{M}]\left\{\{\mathbf{V}\}-\{\mathbf{V}^0\}\right\}=P_1\{\mathbf{U}_1\}+P_2\{\mathbf{U}_2\}+P_3\{\mathbf{U}_3\}$. Since the directions of the impulses $\mathbf{P}_1$, $\mathbf{P}_2$ and $\mathbf{P}_3$ are parallel, one may write $\{\mathbf{U}_3\}=\lambda_1\{\mathbf{U}_1\}+\lambda_2\{\mathbf{U}_2\}$, and the previous equation leads us to $[\mathbf{M}]\left\{\{\mathbf{V}\}-\{\mathbf{V}^0\}\right\}=(P_1+\lambda_1P_3)\{\mathbf{U}_1\}+(P_2+\lambda_2P_3)\{\mathbf{U}_2\}$. It results that one can determine only the expressions $P_1+\lambda_1P_3$ and $P_2+\lambda_2P_3$, the impulses P_1, P_2 and P_3 remaining undetermined. In Fig. 2b we have the case of a table with four feet. The coordinates of the point A_i are x_i, y_i and z_i, $i=\overline{1,\ 4}$, while the unit vectors of the direction of impulses are $\mathbf{u}_i=\mathbf{k}$, $i=\overline{1,\ 4}$. It results $\mathbf{OA}_i\times\mathbf{u}_i=y_i\mathbf{i}-x_i\mathbf{j}$, $\{\mathbf{U}_i\}=[0\quad 0\quad 1\quad y_i\quad -x_i\quad 0]^{\mathrm{T}}$. On the other hand, the point A_4 belongs to the plane determined by the points A_1, A_2 and A_3; consequently A_4 may be written as an affine combination of the points A_1, A_2 and A_3, that is, there exist the constants λ_1, λ_2 and λ_3, $\lambda_1^2+\lambda_2^2+\lambda_3^2\neq 0$, such that $P_4=\lambda_1P_1+\lambda_2P_2+\lambda_3P_3$; it results the matrix relation $\{\mathbf{U}_4\}=\lambda_1\{\mathbf{U}_1\}+\lambda_2\{\mathbf{U}_2\}+\lambda_3\{\mathbf{U}_3\}$ and the linear system of three equations with four unknowns (λ_1, λ_2, λ_3, and λ_4) $\lambda_4=\lambda_1+\lambda_2+\lambda_3$, $\lambda_4x_4=\lambda_1x_1+\lambda_2x_2+\lambda_3x_3$, $\lambda_4y_4=\lambda_1y_1+\lambda_2y_2+\lambda_3y_3$. Since the points A_1, A_2 and A_3 are not collinear, one deduces that the determinant of the system is not equal to zero; hence the system is compatible determined. This situation corresponds to the case in which the directions of the impulses are four parallel straight lines in space.

Taking into account the relations $v_{in} = \{\mathbf{U}_i\}^{\mathrm{T}}[\boldsymbol{\eta}]\{\mathbf{V}\}$ and $v_{in}^0 = \{\mathbf{U}_i\}^{\mathrm{T}}[\boldsymbol{\eta}]\{\mathbf{V}^0\}$, where $[\boldsymbol{\eta}]$ is a square sixth order matrix given by $[\boldsymbol{\eta}] = \begin{bmatrix} [\mathbf{0}] & [\mathbf{I}] \\ [\mathbf{I}] & [\mathbf{0}] \end{bmatrix}$, one gets

$$\{\mathbf{V}_n\} - \{\mathbf{V}_n^0\} = [\mathbf{G}]\{\mathbf{P}\}\,. \tag{2}$$

Using the Newton model (for the simplicity of calculation), one obtains the equalities $v_{in} = -k_i v_{in}^0$, that is in matrix form,

$$\{\mathbf{V}_n\} - \{\mathbf{V}_n^0\} = -\big[[\mathbf{I}] + [\mathbf{K}]\big]\{\mathbf{V}_n^0\} \tag{3}$$

and from the expression (2), we deduce the matrix of impulses

$$\{\mathbf{P}\} = -[\mathbf{G}]^{-1}\big[[\mathbf{I}] + [\mathbf{K}]\big]\{\mathbf{V}_n^0\}\,. \tag{4}$$

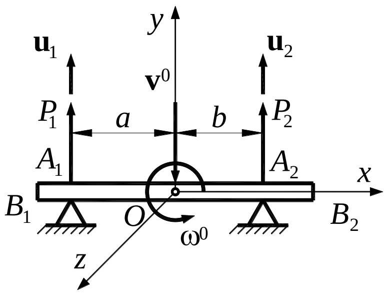

Fig. 3 – Planar collision of a bar.

As an example we consider the situation captured in Fig. 3, in which one knows: $a = l/2$, $b = l/2$, $B_1O = OB_2 = l$, the mass of the homogeneous bar m, v^0, ω^0, the coefficients of restitution k_1, k_2; numerically: $l = 1$ m, $m = 12$ kg, $k_1 = k_2 = k = 0.7$, $v^0 = -10$ m/s, $\omega^0 = 2$ rad/s. One successively obtains the values: $\mathbf{u}_1 = \mathbf{j}$, $\mathbf{u}_2 = \mathbf{j}$, $\mathbf{OA}_1 \times \mathbf{u}_1 = -a\mathbf{k} = -0.5\mathbf{k}$, $\mathbf{OA}_2 \times \mathbf{u}_2 = b\mathbf{k} = 0.5\mathbf{k}$, $J_z = ml^2/3 = 4$ kgm^2,

$$[\mathbf{U}] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & -a \\ 0 & 1 & 0 & 0 & 0 & b \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & -l/2 \\ 0 & 1 & 0 & 0 & 0 & l/2 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & -0.5 \\ 0 & 1 & 0 & 0 & 0 & 0.5 \end{bmatrix}^{\mathrm{T}},$$

$$[\mathbf{G}] = \begin{bmatrix} \dfrac{1}{m} + \dfrac{a^2}{J_z} & \dfrac{1}{m} - \dfrac{ab}{J_z} \\ \dfrac{1}{m} - \dfrac{ab}{J_z} & \dfrac{1}{m} + \dfrac{b^2}{J_z} \end{bmatrix} = \begin{bmatrix} \dfrac{7}{4m} & \dfrac{1}{4m} \\ \dfrac{1}{4m} & \dfrac{7}{4m} \end{bmatrix} = \frac{1}{48}\begin{bmatrix} 7 & 1 \\ 1 & 7 \end{bmatrix} \text{ kg}^{-1}, \quad v_{1n}^0 = v^0 + l\omega^0 = -8 \text{ m/s},$$

$$v_{2n}^0 = v^0 - l\omega^0 = -12 \text{ m/s}, \quad \{\mathbf{P}\} = -\frac{m}{12}\begin{bmatrix} 7(1+k_1)v_{1n}^0 - (1+k_2)v_{2n}^0 \\ -(1+k_1)v_{1n}^0 + 7(1+k_2)v_{2n}^0 \end{bmatrix} = \begin{bmatrix} 74.8 \\ 129.2 \end{bmatrix} \text{ Ns},$$

$$\{\mathbf{V}\} = \begin{bmatrix} 0 & 0 & \omega^0 - \dfrac{3}{8ml}(P_1 + P_2) & 0 & v^0 + \dfrac{P_1 + P_2}{m} & 0 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 0 & 0 & -4.375 & 0 & 7 & 0 \end{bmatrix}^{\mathrm{T}}, \quad \omega_x = 0 \text{ rad/s},$$

$$\omega_y = 0 \text{ rad/s}, \ \ \omega_z = -4.375 \text{ rad/s}, \ \ V_x = 0 \text{ m/s}, \ V_y = 7 \text{ m/s}, \ \ V_z = 0 \text{ m/s}.$$

After the collision, the center of weight of the bar has a vertical velocity equal to V_y, while the bar clockwise rotates with the angular velocity ω_z. One may also calculate the velocities of the points A_1 and A_2

before and after collision, obtaining $v_{A_1}^0=-11\ \mathrm{m/s}$, $v_{A_2}^0=-9\ \mathrm{m/s}$, $v_{A_1}=9.1875\ \mathrm{m/s}$, $v_{A_2}=4.8125\ \mathrm{m/s}$. The velocities satisfy the Euler relation, and cannot be obtained directly by $v_{A_i}=-k_i v_{A_i}^0$, $i=\overline{1,\ 2}$.

4. THE COLLISION OF THE CONSTRAINED RIGID

The space of the possible motions is conjugate to the space of the reaction impulses; consequently [9], $[\mathbf{S}]^{\mathrm{T}}[\boldsymbol{\eta}][\mathbf{Q}]=[\mathbf{0}]$, $[\mathbf{Q}]^{\mathrm{T}}[\boldsymbol{\eta}][\mathbf{S}]=[\mathbf{0}]$. The matrices of the screw coordinates of the rigid solid's velocities and of the reaction impulses are $[\mathbf{Q}]\{\xi\}$ and $[\mathbf{S}]\{\zeta\}$, respectively. The general theorems read now

$$[\mathbf{M}][\mathbf{Q}]\left\{\{\xi\}-\{\xi^0\}\right\}=[\mathbf{U}]\{\mathbf{P}\}+[\mathbf{S}]\{\zeta\}, \tag{5}$$

where we assumed that the constraints are independent.

Multiplying the relation (5) by $[\mathbf{Q}]^{\mathrm{T}}[\boldsymbol{\eta}]$ and denoting $[\mathbf{M}_{red}]=[\mathbf{Q}]^{\mathrm{T}}[\boldsymbol{\eta}][\mathbf{M}][\mathbf{Q}]$, one gets

$$\{\mathbf{v}_n\}-\{\mathbf{v}^0\}=[\mathbf{Q}][\mathbf{M}_{red}]^{-1}[\mathbf{Q}]^{\mathrm{T}}[\boldsymbol{\eta}][\mathbf{U}]\{\mathbf{P}\}. \tag{6}$$

Multiplying the last expression by $[\mathbf{U}]^{\mathrm{T}}[\boldsymbol{\eta}]$ and taking into account the relation $\{\mathbf{v}_n\}-\{\mathbf{v}_n^0\}=-\left[[\mathbf{I}]+[\mathbf{K}][\mathbf{Q}]\right]\{\mathbf{v}_n^0\}$, it results the impulses at the collision points

$$\{\mathbf{P}\}=-\left[[\mathbf{U}]^{\mathrm{T}}[\boldsymbol{\eta}][\mathbf{Q}][\mathbf{M}_{red}]^{-1}[\mathbf{Q}]^{\mathrm{T}}[\boldsymbol{\eta}][\mathbf{U}]\right]^{-1}[\mathbf{U}]^{\mathrm{T}}[\boldsymbol{\eta}]\left[[\mathbf{I}]+[\mathbf{K}]\right]\{\mathbf{V}_n^0\}. \tag{7}$$

Now we calculate the velocities after collision, using equation (6).

Moreover, the expression (5) leads to

$$\{\zeta\}=-\left[[\mathbf{S}]^{\mathrm{T}}[\boldsymbol{\eta}][\mathbf{M}]^{-1}[\mathbf{S}]\right]^{-1}[\mathbf{S}]^{\mathrm{T}}[\boldsymbol{\eta}][\mathbf{M}]^{-1}[\mathbf{U}]\{\mathbf{P}\}, \tag{8}$$

while $[\mathbf{S}]\{\zeta\}$ gives us the screw coordinates of the impulses in the linkages.

If the matrices $[\mathbf{U}]$ and $[\mathbf{S}]$ are not independent, then $[\mathbf{Q}]^{\mathrm{T}}[\boldsymbol{\eta}]\cdot[\mathbf{U}]=[\mathbf{0}]$ and it results $\{\xi\}=\{\xi_0\}$ and $\{\mathbf{V}\}=\{\mathbf{V}^0\}$; hence, the motion remains unchanged.

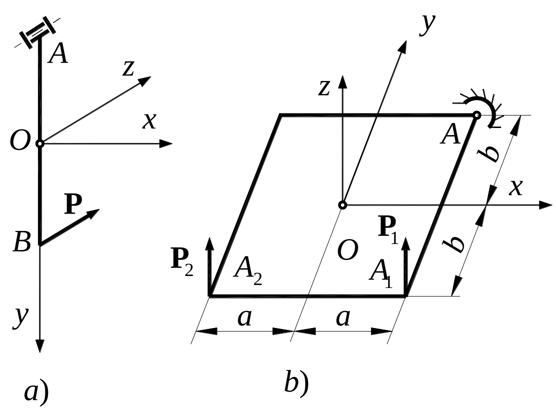

Fig. 4 – a) Spatial undetermined collision of a bar; b) spatial determined collision of a shell.

For instance, the case presented in Fig. 4a in which $OA=OB=l/2$ leads to the following values:

$[\mathbf{U}]=\{\mathbf{U}\}=[0\quad 0\quad 1\quad l/2\quad 0\quad 0]^{\mathrm{T}}$, $\{\mathbf{S}_1\}=[1\quad 0\quad 0\quad 0\quad 0\quad l/2]^{\mathrm{T}}$, $\{\mathbf{S}_2\}=[0\quad 1\quad 0\quad 0\quad 0\quad 0]^{\mathrm{T}}$,

$$\{\mathbf{S}_3\}=\begin{bmatrix}0 & 0 & 1 & -l/2 & 0 & 0\end{bmatrix}^{\mathrm{T}},\quad \{\mathbf{S}_4\}=\begin{bmatrix}0 & 0 & 0 & 1 & 0 & 0\end{bmatrix}^{\mathrm{T}},\quad \{\mathbf{S}_5\}=\begin{bmatrix}0 & 0 & 0 & 0 & 1 & 0\end{bmatrix}^{\mathrm{T}},$$

$$[\mathbf{Q}]=\{\mathbf{Q}\}=\begin{bmatrix}0 & 0 & 1 & -l/2 & 0 & 0\end{bmatrix}^{\mathrm{T}}.$$

One may easily observe that $\{\mathbf{U}\}=\{\mathbf{S}_3\}+l\{\mathbf{S}_4\}$, that is, the direction of the impulse $\mathbf{P}$ is situated in the space defined by the composed linkage at A; hence the collision at the point B along the axis Oz does not influence the motion.

Let us consider now the case presented in Fig. 4b in which a homogeneous shell spherically jointed at the point A is collided at the points A_1 and A_2. One knows: $a=b=0.5\ \mathrm{m}$, $m=120\ \mathrm{kg}$, $\omega_z^0=0\ \mathrm{rad/s}$, $v_x^0=0\ \mathrm{m/s}$, $v_y^0=0\ \mathrm{m/s}$, $\omega_x^0=8\ \mathrm{rad/s}$, $\omega_y^0=-2\ \mathrm{rad/s}$, $v_z^0=-5\ \mathrm{m/s}$, $k_1=k_2=0.7$. One successively obtains:

$$J_x=\frac{ma^2}{3},\quad J_y=\frac{ma^2}{3},\quad J_z=\frac{2ma^2}{3},\quad [\mathbf{M}_{red}]=\frac{ma^2}{3}\begin{bmatrix}4 & -3 & 0\\ -3 & 4 & 0\\ 0 & 0 & 2\end{bmatrix},\quad [\mathbf{S}]=[\mathbf{Q}]=\begin{bmatrix}1 & 0 & 0 & 0 & 0 & -a\\ 0 & 1 & 0 & 0 & 0 & a\\ 0 & 0 & 1 & a & -a & 0\end{bmatrix}^{\mathrm{T}},$$

$$[\mathbf{U}]=\begin{bmatrix}0 & 0 & 1 & -a & -a & 0\\ 0 & 0 & 1 & -a & a & 0\end{bmatrix}^{\mathrm{T}},\quad [\mathbf{S}]^{\mathrm{T}}[\boldsymbol{\eta}][\mathbf{Q}]=[\mathbf{0}],$$

$$\{\mathbf{P}\}=\begin{bmatrix}P_1\\ P_2\end{bmatrix}=\frac{7m}{12}\begin{bmatrix}a(1+k_2)(\omega_x^0-\omega_y^0)-(1+k_2)v_z^0\\ a(-3+k_1-4k_2)\omega_x^0+a(5+k_1+4k_2)\omega_y^0+(3-k_1+4k_2)v_z^0\end{bmatrix}=\begin{bmatrix}1190\\ -3808\end{bmatrix}\ \mathrm{Ns},$$

$$\{\zeta\}=-\frac{1}{56}\begin{bmatrix}0\\ 0\\ 8P_1-40P_2\end{bmatrix},\quad [\mathbf{S}]\{\zeta\}=-\frac{1}{56}\begin{bmatrix}0 & 0 & 8P_1-40P_2 & (8P_1-40P_2)a & -(8P_1-40P_2)a & 0\end{bmatrix}^{\mathrm{T}}=$$

$$=\begin{bmatrix}0 & 0 & -2890 & -1445 & 1445 & 0\end{bmatrix}^{\mathrm{T}}.$$

There is only one component of reaction impulse at the point A along the z-axis. One may easily verify the theorems of momentum and moment of momentum at the collision. Moreover, a convenient selection of the kinematic and dynamic parameters may lead to $[\mathbf{S}]\{\zeta\}=\{\mathbf{0}\}$, that is, point O may be considered a center of impulses (similar to the case of singular collision).

5. THE SIMULTANEOUS MULTI-POINT COLLISION OF TWO FREE RIGIDS

The situation is presented in Fig. 5. We index by $i=1$ or $i=2$ the two rigid solids and use the previous formulae. In addition, we denote: $[\mathbf{G}]$ – the matrix $[\mathbf{G}]=[\mathbf{G}_1]+[\mathbf{G}_2]$; $\{\mathbf{V}_{12n}\}$ and $\{\mathbf{V}_{12n}^0\}$ – the column matrices given by $\{\mathbf{V}_{12n}\}=\{\mathbf{V}_{1n}\}-\{\mathbf{V}_{2n}\}$ and $\{\mathbf{V}_{12n}^0\}=\{\mathbf{V}_{1n}^0\}-\{\mathbf{V}_{2n}^0\}$.

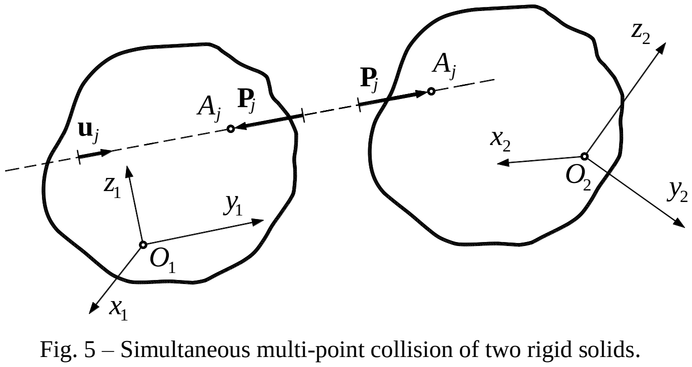

Fig. 5 – Simultaneous multi-point collision of two rigid solids.

For the two rigid solids one obtains the matrix equations

$$[\mathbf{M}_1]\left\{\{\mathbf{V}_1\}-\{\mathbf{V}_1^0\}\right\}=-[\mathbf{U}_1]\{\mathbf{P}\},\quad [\mathbf{M}_2]\left\{\{\mathbf{V}_2\}-\{\mathbf{V}_2^0\}\right\}=-[\mathbf{U}_2]\{\mathbf{P}\}, \tag{9}$$

wherefrom

$$\{\mathbf{V}_1\}-\{\mathbf{V}_1^0\}=-[\mathbf{M}_1]^{-1}[\mathbf{U}_1]\{\mathbf{P}\},\quad \{\mathbf{V}_{1n}\}-\{\mathbf{V}_{1n}^0\}=-[\mathbf{G}_1]\{\mathbf{P}\}, \tag{10}$$

$$\{\mathbf{V}_2\}-\{\mathbf{V}_2^0\}=[\mathbf{M}_2]^{-1}[\mathbf{U}_2]\{\mathbf{P}\},\quad \{\mathbf{V}_{2n}\}-\{\mathbf{V}_{2n}^0\}=[\mathbf{G}_2]\{\mathbf{P}\}. \tag{11}$$

It results

$$\{\mathbf{V}_{12n}\}-\{\mathbf{V}_{12n}^0\}=-[\mathbf{G}]\{\mathbf{P}\}. \tag{12}$$

Applying the Newton model, one gets

$$\{\mathbf{V}_{12n}\}=-[\mathbf{K}]\{\mathbf{V}_{12n}^0\}. \tag{13}$$

wherefrom

$$\{\mathbf{P}\}=[\mathbf{G}]^{-1}\left[[\mathbf{I}]+[\mathbf{K}]\right]\{\mathbf{V}_{12n}^0\}. \tag{14}$$

Replacing the column matrix $\{\mathbf{P}\}$ in the expression (10) and (11), one gets the velocities after collision.

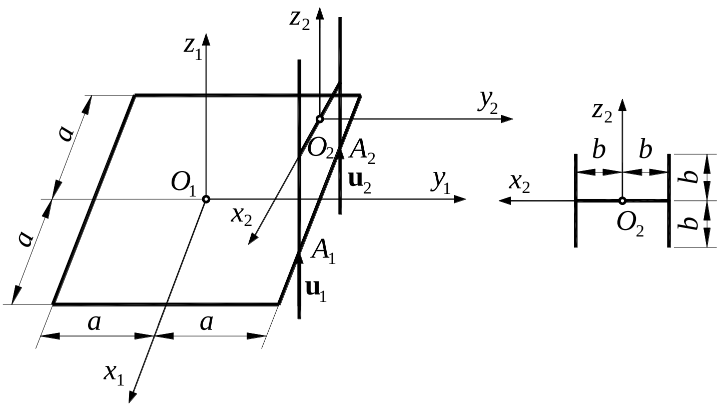

Fig. 6 – Numerical example.

As numerical example we consider the collision of two frames as in Fig. 6. One knows: $m_1=120\,\text{kg}$, $m_2=96\,\text{kg}$, $a=0.5\,\text{m}$, $b=0.25\,\text{m}$, $\omega_{1x}^0=2\,\text{rad/s}$, $\omega_{1y}^0=4\,\text{rad/s}$, $\omega_{1z}^0=0\,\text{rad/s}$, $v_{1x}^0=0\,\text{m/s}$, $v_{1y}^0=0\,\text{m/s}$, $v_{1z}^0=10\,\text{m/s}$, $\omega_{2x}^0=0\,\text{rad/s}$, $\omega_{2y}^0=4\,\text{rad/s}$, $\omega_{2z}^0=0\,\text{rad/s}$, $v_{2x}^0=0\,\text{m/s}$, $v_{2y}^0=0\,\text{m/s}$, $v_{2z}^0=-8\,\text{m/s}$, $k_1=0.7$, $k_2=0.7$. One obtains the following values: $J_{x_1}=80\ \text{kgm}^2$, $J_{y_1}=80\ \text{kgm}^2$, $J_{z_1}=40\ \text{kgm}^2$, $J_{x_2}=5.333\ \text{kgm}^2$, $J_{y_2}=4.666\ \text{kgm}^2$, $J_{z_2}=4.666\ \text{kgm}^2$, $\{\mathbf{V}_{1n}^0\}=[10\quad 12]^{\mathrm{T}}\,\text{m/s}$, $\{\mathbf{V}_{2n}^0\}=[-9\quad -7]^{\mathrm{T}}\,\text{m/s}$,

$$[\mathbf{U}_1]=\begin{bmatrix}0&0&1&0.5&-0.25&0\\0&0&1&0.5&-0.25&0\end{bmatrix}^{\mathrm{T}},\ [\mathbf{U}_2]=\begin{bmatrix}0&0&1&0&-0.25&0\\0&0&1&0&0.25&0\end{bmatrix}^{\mathrm{T}},\ [\mathbf{G}_1]=\begin{bmatrix}0.01224&0.01068\\0.01068&0.01224\end{bmatrix}\text{kg}^{-1},$$

$$[\mathbf{G}_2]=\begin{bmatrix}0.02381&-0.00298\\-0.00298&0.02381\end{bmatrix}\text{kg}^{-1},\quad \{\mathbf{P}\}=[738.286\quad 738.286]^{\mathrm{T}}\ \text{Ns},$$

$$\{\mathbf{V}_1\}=[0.7714\quad 4\quad 0\quad 0\quad 0\quad -2.3048]^{\mathrm{T}},\quad \{\mathbf{V}_2\}=[0\quad 4\quad 0\quad 0\quad 0\quad 7.3810]^{\mathrm{T}}.$$

For the second frame the motion remains unchanged, the value of the velocity along the z-axis being different, while for the first frame a new component appears for the rotation about the x-axis. Calculating the velocities at the points A_i for each frame, before and after collision, one may state that the simple formula $v_{A_i} = -k_i v_{A_i}^0$, $i = \overline{1,\ 2}$, is no longer valid.

6. CONCLUSIONS

In this paper we presented the theory for the simultaneous multi-point collisions of rigid solids. The impulses at the collision points, the reaction impulses and the distribution of velocities are deduced in all possible cases. Using the screw coordinates, the formulae were developed in matrix form and they could be easily implemented in various algorithms of calculations. The above theory is valid in the case when the collision is without friction, all normal velocities at the collision points vanish simultaneously and no jamb phenomenon is allowed. We also highlight the situations in which the problems are not determined. For each case examples are presented. The case of the simultaneous multi-point collisions with friction is more complicate due to the existence of the invariant directions of sliding at each collision point. The jamb phenomenon (existence of more than one phase of compression and restitution during the collision) was poorly studied in the literature. In future papers we will study the simultaneous multi-point collisions between more than two rigid bodies and will extend the theory to kinematic chains.

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