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# A FAN-TYPE RESULT FOR THE EXISENCE OF RESTRICTED FRACTIONAL $(g,f)$-FACTORS

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**Abstract.** Let $h$ be a function defined on $E(G)$ with $h(e) \in [0,1]$ for all $e \in E(G)$. If $g(u) \le \sum_{e \ni u} h(e) \le f(u)$ for every $u \in V(G)$, then a graph $F_h$ with vertex set $V(G)$ and edge set $E_h$ is called a fractional $(g,f)$-factor of $G$ with indicator function $h$, where $E_h = \{e \in E(G) : h(e) > 0\}$. Let $M$ and $N$ be two sets of independent edges of $G$ such that $|M| = m$, $|N| = n$ and $M \cap N = \emptyset$. We say that $G$ admits a fractional $(g,f)$-factor with the property $E(m,n)$ if $G$ has a fractional $(g,f)$-factor $F_h$ satisfying $h(e) = 1$ for any $e \in M$ and $h(e) = 0$ for any $e \in N$. In this paper, we give a lower bound on Fan-type condition which guarantees graphs to admit fractional $(g,f)$-factors with the property $E(1,n)$, which is a generalization of Yu and Liu's previous result.

***Key words:*** graph; Fan-type condition; fractional $(g,f)$-factor; restricted fractional $(g,f)$-factors.

## 1. INTRODUCTION

All graphs considered in this paper are finite undirected graphs without loops nor multiple edges. Let $G = (V(G), E(G))$ be a graph, where $V(G)$ denotes the set of vertices of $G$ and $E(G)$ denotes the set of edges of $G$. For any $u \in V(G)$, we use $N_G(u)$ to denote the set of vertices adjacent to $u$ in $G$, and $d_G(u) = |N_G(u)|$ is the degree of $u$ in $G$. For any $X \subseteq V(G)$, $N_G(X) = \cup_{u \in X} N_G(u)$, we denote by $G[X]$ the subgraph of $G$ induced by $X$, and set $G - X = G[V(G) \setminus X]$. For a subset $E'$ of $E(G)$, we use $G - E'$ to denote the graph obtained from $G$ by deleting edges of $E'$. A subset $X$ of $V(G)$ is independent if $N_G(X) \cap X = \emptyset$. For two disjoint subsets $X$ and $Y$ of $V(G)$, we use $e_G(X,Y)$ to denote the number of edges joining $X$ to $Y$. We define the distance $d_G(u,v)$ between two vertices $u$ and $v$ as the minimum of the lengths of the $(u,v)$ paths of $G$. We use $\delta(G)$ to denote the minimum degree of $G$ and use $\Delta(G)$ to denote the maximum degree of $G$.

Let $g$ and $f$ be two integer-valued functions defined on $V(G)$ such that $0 \le g(u) \le f(u)$ for every $u \in V(G)$. A $(g,f)$-factor of $G$ is a spanning subgraph $F$ of $G$ such that $g(u) \le d_F(u) \le f(u)$ for all $u \in V(G)$. If $g(u) = a$ and $f(u) = b$ for every $u \in V(G)$, then a $(g,f)$-factor is an $[a,b]$-factor. A $[k,k]$-factor is simply called a $k$-factor.

Let $h$ be a function defined on $E(G)$ with $h(e) \in [0,1]$ for all $e \in E(G)$. If $g(u) \le \sum_{e \ni u} h(e) \le f(u)$ for every $u \in V(G)$, then a graph $F_h$ with vertex set $V(G)$ and edge set $E_h$ is called a fractional $(g,f)$-factor of $G$ with indicator function $h$, where $E_h = \{e \in E(G) : h(e) > 0\}$. A fractional $(g,f)$-factor is a fractional $[a,b]$-factor if $g(u) = a$ and $f(u) = b$ for all $u \in V(G)$. A fractional $[k,k]$-factor is simply called a fractional $k$-factor.

Let $M$ and $N$ be two sets of independent edges of $G$ such that $|M| = m$, $|N| = n$ and $M \cap N = \emptyset$. We say that $G$ admits a fractional $(g,f)$-factor with the property $E(m,n)$ if $G$ has a fractional $(g,f)$-factor $F_h$ satisfying $h(e) = 1$ for any $e \in M$ and $h(e) = 0$ for any $e \in N$.

We first introduce a well-known result on a Hamiltonian cycle (or 2-factor) of graph depending on Fan-type condition.

THEOREM 1 ([3]). *Let $G$ be a 2-connected graph of order $p \geq 3$. If*

$$\max\{d_G(u), d_G(v)\} \geq \frac{p}{2}$$

*for any two vertices $u$ and $v$ of $G$ with $d_G(u,v) = 2$, then $G$ admits a Hamiltonian cycle (or 2-factor).*

Niessen [11] generalized Theorem 1 to $k$-factors, which is shown in the following.

THEOREM 2 ([11]). *Let $k$ be an integer with $k \geq 1$ and $G$ a connected graph of order $p$ with $p \geq 8k^2 + 12k + 6$, $kp$ is even. If $\delta(G) \geq k$ and*

$$\max\{d_G(u), d_G(v)\} \geq \frac{p}{2}$$

*for any two vertices $u$ and $v$ of $G$ with $d_G(u,v) = 2$, then $G$ admits a $k$-factor.*

Yu and Liu [15] put forward a Fan-type condition for the existence of fractional $k$-factors in graphs.

THEOREM 3 ([15]). *Let $G$ a connected graph of order $p$ with $p \geq 8k^2 + 12k + 6$, where $k$ is a positive integer. If $\delta(G) \geq k$ and*

$$\max\{d_G(u), d_G(v)\} \geq \frac{p}{2}$$

*for any two vertices $u$ and $v$ of $G$ with $d_G(u,v) = 2$, then $G$ admits a fractional $k$-factor.*

For other results on graph factors see [1, 2, 4–6, 8–10, 12–14, 16–29]. In this paper, we investigate the existence of restricted fractional $(g,f)$-factors in graphs, and obtain a Fan-type condition for graphs having restricted fractional $(g,f)$-factors, which is shown in Section 2.

## 2. MAIN RESULTS

Motivated by Theorems 1–3, we verify the following theorem.

THEOREM 4. *Let $a, b, \lambda$ and $n$ be nonnegative integers with $2 \leq a \leq b - \lambda$, let $G$ be a graph of order $p$ with $p \geq \frac{(a+b)((a+b)(b-\lambda+1)+2n-1)}{a+\lambda-1} + \frac{(a+b)(b-\lambda+1)-2}{b-\lambda} + \frac{a+b}{(a+\lambda)(b-\lambda)}$, and let $g, f : V(G) \to Z$ be two functions such that $a \leq g(x) \leq f(x) - \lambda \leq b - \lambda$ for all $x \in V(G)$. If $\delta(G) \geq \frac{(b-\lambda)(b+2)}{a+\lambda-1} + 1$ and*

$$\max\{d_G(u), d_G(v)\} \geq \frac{(b-\lambda)p+2}{a+b}$$

*for any two vertices $u$ and $v$ of $G$ with $d_G(u,v) = 2$, then $G$ has a fractional $(g,f)$-factor with the property $E(1,n)$.*

*Remark.* The condition $\max\{d_G(u), d_G(v)\} \geq \frac{(b-\lambda)p+2}{a+b}$ in Theorem 4 is sharp, i.e., we cannot replace $\frac{(b-\lambda)p+2}{a+b}$ by $\frac{(b-\lambda)p+2}{a+b} - 1$.

Let $a, b, \lambda$ and $n$ be nonnegative integers with $2 \leq a = b - \lambda$ and $\beta$ be a sufficiently large integer with $\beta > 0$ and $n < (b-\lambda)\beta$. Set $G = K_{(b-\lambda)\beta} \vee (a+\lambda)\beta K_1$. Then we have $p = (b-\lambda)\beta + (a+\lambda)\beta = (a+b)\beta$ and

$$\frac{(b-\lambda)p+2}{a+b} - 1 < \max\{d_G(u), d_G(v)\} = (b-\lambda)\beta = \frac{(b-\lambda)p}{a+b} < \frac{(b-\lambda)p+2}{a+b}$$

for any two vertices $u, v \in V((a+\lambda)\beta K_1)$ with $d_G(u,v) = 2$. Let $g, f : V(G) \to Z$ be two functions with $g(u) = b - \lambda$ and $f(u) = a + \lambda$ for any $u \in V(G)$. Let $X = V(K_{(b-\lambda)\beta})$, $Y = V((a+\lambda)\beta K_1)$, $N = \{e_1, e_2, \cdots, e_n\}$ being a set of independent edges in $G$ and $H = G - N$. Then it follows that $|X| = (b-\lambda)\beta$, $|Y| = (a+\lambda)\beta$,

$d_{H-X}(Y)=0$ and $\varepsilon(X,Y)=2$. Hence, we obtain

$$\begin{aligned}\gamma_H(X,Y) &= f(X)+d_{H-X}(Y)-g(Y)\\ &= (a+\lambda)|X|-(b-\lambda)|Y|\\ &= (a+\lambda)(b-\lambda)\beta-(b-\lambda)(a+\lambda)\beta\\ &= 0<2=\varepsilon(X,Y).\end{aligned}$$

In light of Theorem 5, $H$ has no fractional $(g,f)$-factor with the property $E(1,0)$, that is, $G$ has no fractional $(g,f)$-factor with the property $E(1,n)$.

Let $n=0$ in Theorem 4. Then we get the following corollary.

COROLLARY 1. *Let $a,b,\lambda$ be nonnegative integers with $2\le a\le b-\lambda$, let $G$ be a graph of order $p$ with $p\ge \frac{(a+b)((a+b)(b-\lambda+1)-1)}{a+\lambda-1}+\frac{(a+b)(b-\lambda+1)-2}{b-\lambda}+\frac{a+b}{(a+\lambda)(b-\lambda)}$, and let $g,f:V(G)\to Z$ be two functions such that $a\le g(x)\le f(x)-\lambda\le b-\lambda$ for all $x\in V(G)$. If $\delta(G)\ge\frac{(b-\lambda)(b+2)}{a+\lambda-1}+1$ and*

$$\max\{d_G(u),d_G(v)\}\ge\frac{(b-\lambda)p+2}{a+b}$$

*for any two vertices $u$ and $v$ of $G$ with $d_G(u,v)=2$, then $G$ has a fractional $(g,f)$-factor with the property $E(1,0)$.*

Let $\lambda=0$ in Theorem 4. Then we obtain the following result.

COROLLARY 2. *Let $a,b$ and $n$ be nonnegative integers with $2\le a\le b$, let $G$ be a graph of order $p$ with $p\ge\frac{(a+b)((a+b)(b+1)+2n-1)}{a-1}+\frac{(a+b)(b+1)-2}{b}+\frac{a+b}{ab}$, and let $g,f:V(G)\to Z$ be two functions such that $a\le g(x)\le f(x)\le b$ for all $x\in V(G)$. If $\delta(G)\ge\frac{b(b+2)}{a-1}+1$ and*

$$\max\{d_G(u),d_G(v)\}\ge\frac{bp+2}{a+b}$$

*for any two vertices $u$ and $v$ of $G$ with $d_G(u,v)=2$, then $G$ has a fractional $(g,f)$-factor with the property $E(1,n)$.*

## 3. PROOF OF THEOREM 4

For any $X\subseteq V(G)$, let $\varphi(X)=\sum_{u\in X}\varphi(u)$, where $\varphi$ is a function defined on $V(G)$. Especially, $\varphi(\emptyset)=0$. Li, Yan and Zhang [7] put forward a characterization for graphs to have fractional $(g,f)$-factors with the property $E(1,0)$, which is used in the proof of Theorem 4.

THEOREM 5 ([7]). *Let $G$ be a graph, and let $g,f:V(G)\to Z$ be two functions with $0\le g(x)\le f(x)$ for all $x\in V(G)$. Then $G$ has a fractional $(g,f)$-factor with the property $E(1,0)$ if and only if*

$$\gamma_G(X,Y)=f(X)+d_{G-X}(Y)-g(Y)\ge\varepsilon(X,Y)$$

*for any $X\subseteq V(G)$, where $Y=\{y:y\in V(G)\setminus X, d_{G-X}(y)\le g(y)\}$ and $\varepsilon(X,Y)$ is defined as follows:*

$$\varepsilon(X,Y)=\begin{cases}2, & if\ X\ is\ not\ independent,\\ 1, & if\ X\ is\ independent\ and\ there\ is\ an\ edge\ joining\ X\ and\ V(G)\setminus(X\cup Y),\ or\\ & there\ is\ an\ edge\ e=uv\ joining\ X\ and\ Y\ such\ that\ d_{G-X}(v)=g(v)\ for\ v\in Y,\\ 0, & otherwise.\end{cases}$$

*Proof of Theorem 4.* Assume that $G$ has no fractional $(g,f)$-factor with the property $E(1,n)$. Then there exist an edge $e$ and a set of independent edges $\{e_1,e_2,\cdots,e_n\}$ of $G$ such that $G$ has no fractional $(g,f)$-factor

$F_h$ with $h(e)=1$ and $h(e_i)=0$ for $1\le i\le n$. Set $N=\{e_1,e_2,\cdots,e_n\}$ and $H=G-N$. Then $H$ has no fractional $(g,f)$-factor with the property $E(1,0)$. In view of Theorem 5, there exists a subset $X$ of $V(H)$ such that

$$\gamma_H(X,Y)=f(X)+d_{H-X}(Y)-g(Y)\le\varepsilon(X,Y)-1, \tag{1}$$

where $Y=\{y:y\in V(H)\setminus X, d_{H-X}(y)\le g(y)\}$.

CLAIM 1. $Y\neq\emptyset$.

*Proof.* If $Y=\emptyset$, then by (1) we obtain

$$\varepsilon(X,Y)-1\ge\gamma_H(X,Y)=f(X)\ge(a+\lambda)|X|\ge 2|X|\ge|X|\ge\varepsilon(X,Y),$$

which is a contradiction. □

CLAIM 2. $d_{H-X}(Y)\ge d_{G-X}(Y)-\min\{2n,|Y|\}$.

*Proof.* Let $D=V(G)\setminus(X\cup Y)$ and $E_G(Y)=\{e:e=uv\in E(G),u,v\in Y\}$. Since $N=\{e_1,e_2,\cdots,e_n\}$ is a set of independent edges of $G$, we easily obtain

$$2|N\cap E_G(Y)|+|N\cap E_G(Y,D)|\le\min\{2n,|Y|\}. \tag{2}$$

It follows from (2) and $H=G-N$ that

$$\begin{aligned} d_{H-X}(Y) &= d_{G-N-X}(Y)\\ &= d_{G-X}(Y)-(2|N\cap E_G(Y)|+|N\cap E_G(Y,D)|)\\ &\ge d_{G-X}(Y)-\min\{2n,|Y|\}. \end{aligned}$$

The proof of Claim 2 is finished. □

CLAIM 3. $|Y|\ge b+3$.

*Proof.* Since $Y\neq\emptyset$ (by Claim 1), we may define

$$d=\min\{d_{G-X}(u):u\in Y\},$$

and choose $u_1\in Y$ with $d_{G-X}(u_1)=d$. Clearly, $0\le d\le b-\lambda+1$ by $H=G-N$ and the definition of $Y$. Moreover, we have

$$|X|+d=|X|+d_{G-X}(u_1)\ge d_G(u_1)\ge\delta(G),$$

that is,

$$|X|\ge\delta(G)-d\ge\frac{(b-\lambda)(b+2)}{a+\lambda-1}+1-d. \tag{3}$$

Let $|Y|\le b+2$. We shall consider two cases by the value of $d$.

• Case 1. $d=0$.

In light of (1), (3), $2\le a\le b-\lambda$ and $\varepsilon(X,Y)\le 2$, we obtain

$$\begin{aligned} \varepsilon(X,Y)-1 &\ge \gamma_H(X,Y)=f(X)+d_{H-X}(Y)-g(Y)\\ &\ge f(X)-g(Y)\ge(a+\lambda)|X|-(b-\lambda)|Y|\\ &\ge (a+\lambda)\Big(\frac{(b-\lambda)(b+2)}{a+\lambda-1}+1-d\Big)-(b-\lambda)(b+2)\\ &> a+\lambda\ge a\ge 2\ge\varepsilon(X,Y), \end{aligned}$$

which is a contradiction.

• Case 2. $1\le d\le b-\lambda+1$.

It follows from (3), Claim 2, $2 \le a \le b-\lambda$ and $1 \le d \le b-\lambda+1$ that

$$\begin{aligned}
\gamma_H(X,Y) &= f(X)+d_{H-X}(Y)-g(Y)\\
&\ge f(X)+d_{G-X}(Y)-\min\{2n,|Y|\}-g(Y)\\
&\ge f(X)+d_{G-X}(Y)-|Y|-g(Y)\\
&\ge (a+\lambda)|X|+d|Y|-|Y|-(b-\lambda)|Y|\\
&= (a+\lambda)|X|-(b-\lambda-d+1)|Y|\\
&\ge (a+\lambda)\Big(\frac{(b-\lambda)(b+2)}{a+\lambda-1}+1-d\Big)-(b-\lambda-d+1)(b+2)\\
&= (d-1)(b+2-a-\lambda)+\frac{(b-\lambda)(b+2)}{a+\lambda-1} \ge \frac{(b-\lambda)(b+2)}{a+\lambda-1}\\
&\ge \frac{a(a+\lambda+2)}{a+\lambda-1} > a \ge 2 \ge \varepsilon(X,Y),
\end{aligned}$$

which contradicts (1). Hence, we have $|Y| \ge b+3$. Claim 3 is proved. □

CLAIM 4. $d_{G-X}(u) \le b-\lambda+1 \le b+1$ *for any* $u \in Y$.

*Proof.* In view of the definitions of $Y$ and $N$, $H = G-N$, we have

$$d_{G-X}(u) = d_{H+N-X}(u) \le d_{H-X}(u)+1 \le g(u)+1 \le b-\lambda+1 \le b+1$$

for any $u \in Y$. □

CLAIM 5. $1 \le |X| \le \frac{(b-\lambda)p+1}{a+b}$.

*Proof.* If $X = \emptyset$, then it follows from (1), $2 \le a \le b-\lambda$ and Claims 2–3 that

$$\begin{aligned}
\varepsilon(X,Y)-1 &\ge \gamma_H(X,Y) = f(X)+d_{H-X}(Y)-g(Y)\\
&\ge f(X)+d_{G-X}(Y)-\min\{2n,|Y|\}-g(Y)\\
&\ge f(X)+d_{G-X}(Y)-|Y|-g(Y)\\
&= d_G(Y)-|Y|-g(Y) \ge \delta(G)|Y|-|Y|-(b-\lambda)|Y|\\
&= (\delta(G)-(b-\lambda+1))|Y| \ge \Big(\frac{(b-\lambda)(b+2)}{a+\lambda-1}+1-(b-\lambda+1)\Big)|Y|\\
&\ge \Big(\frac{(b-\lambda)(a+\lambda+2)}{a+\lambda-1}+1-(b-\lambda+1)\Big)|Y|\\
&= \frac{3(b-\lambda)}{a+\lambda-1}|Y| \ge \frac{3(b-\lambda)}{a+\lambda-1}(b+3)\\
&\ge \frac{3(b-\lambda)}{a+\lambda-1}(a+\lambda+3) > 3(b-\lambda) > 2 \ge \varepsilon(X,Y),
\end{aligned}$$

which is a contradiction. Therefore, $|X| \ge 1$.

On the other hand, by (1), $\varepsilon(X,Y) \le 2$ and $|X|+|Y| \le p$, we have

$$\begin{aligned}
1 &\ge \varepsilon(X,Y)-1 \ge \gamma_H(X,Y) = f(X)+d_{H-X}(Y)-g(Y)\\
&\ge f(X)-g(Y) \ge (a+\lambda)|X|-(b-\lambda)|Y|\\
&\ge (a+\lambda)|X|-(b-\lambda)(p-|X|)\\
&= (a+b)|X|-(b-\lambda)p,
\end{aligned}$$

which implies

$$|X| \le \frac{(b-\lambda)p+1}{a+b}.$$

Hence, we obtain that $1 \le |X| \le \frac{(b-\lambda)p+1}{a+b}$. □

CLAIM 6. $(b-\lambda)|Y| \ge (a+\lambda)|X|-1$.

*Proof.* In terms of (1) and $\varepsilon(X,Y) \le 2$, we get

$$\begin{aligned} 1 &\ge \varepsilon(X,Y)-1 \ge \gamma_H(X,Y) = f(X)+d_{H-X}(Y)-g(Y) \\ &\ge f(X)-g(Y) \ge (a+\lambda)|X|-(b-\lambda)|Y|, \end{aligned}$$

that is,

$$(b-\lambda)|Y| \ge (a+\lambda)|X|-1.$$

Claim 6 is verified. □

CLAIM 7. $|X| < \frac{(b-\lambda)p+2}{a+b} - (b-\lambda+1)$.

*Proof.* Assume that $|X| \ge \frac{(b-\lambda)p+2}{a+b} - (b-\lambda+1)$, that is, $(b-\lambda)p-(a+b)|X| \le (a+b)(b-\lambda+1)-2$. According to (1), $\varepsilon(X,Y) \le 2$, Claim 2 and $|X|+|Y| \le p$, we have

$$\begin{aligned} d_{G-X}(Y) &\le d_{H-X}(Y)+\min\{2n,|Y|\} \\ &\le g(Y)-f(X)+\varepsilon(X,Y)-1+2n \\ &\le (b-\lambda)|Y|-(a+\lambda)|X|+1+2n \\ &\le (b-\lambda)(p-|X|)-(a+\lambda)|X|+2n+1 \\ &= (b-\lambda)p-(a+b)|X|+2n+1 \\ &\le (a+b)(b-\lambda+1)+2n-1. \end{aligned}$$

Combining this with Claim 6 and $p \ge \frac{(a+b)((a+b)(b-\lambda+1)+2n-1)}{a+\lambda-1} + \frac{(a+b)(b-\lambda+1)-2}{b-\lambda} + \frac{a+b}{(a+\lambda)(b-\lambda)}$, we obtain

$$\begin{aligned} \frac{d_{G-X}(Y)}{(b-\lambda)|Y|} &\le \frac{(a+b)(b-\lambda+1)+2n-1}{(a+\lambda)|X|-1} \\ &\le \frac{(a+b)(b-\lambda+1)+2n-1}{(a+\lambda)\cdot\frac{(b-\lambda)p+2}{a+b}-(a+\lambda)(b-\lambda+1)-1} \\ &\le \frac{1}{b-\lambda}\Big(1-\frac{1}{a+\lambda}\Big), \end{aligned}$$

which implies

$$d_{G-X}(Y) \le \Big(1-\frac{1}{a+\lambda}\Big)|Y| = |Y| - \frac{1}{a+\lambda}|Y|. \tag{4}$$

It follows from (4), $2 \le a \le b-\lambda$ and Claim 3 that

$$d_{G-X}(Y) \le |Y| - \frac{1}{a+\lambda}|Y| \le |Y| - \frac{b+3}{a+\lambda} < |Y|-1. \tag{5}$$

Set $Y_0 = \{y \in Y : d_{G-X}(y) = 0\}$. It is easy to see that $|Y_0| \ge 2$ holds by (5). For any $y \in Y_0$, $d_G(y) \le |X| \le \frac{(b-\lambda)p+1}{a+b}$ by Claim 5. Note that $Y_0$ is an independent set of $G$. Combining this with the assumption of Theorem 4, the neighborhoods of the vertices in $Y_0$ are disjoint. Therefore, we obtain

$$|X| \ge |\cup_{y\in Y_0} N_G(y)| \ge \delta(G)|Y_0| \ge \Big(\frac{(b-\lambda)(b+2)}{a+\lambda-1}+1\Big)|Y_0|. \tag{6}$$

On the other hand, it follows from (4) that

$$\Big(1-\frac{1}{a+\lambda}\Big)|Y| \ge d_{G-X}(Y) \ge |Y|-|Y_0|,$$

which implies

$$|Y_0| \geq \frac{1}{a+\lambda}|Y|. \tag{7}$$

In light of (6), (7), $2 \leq a \leq b-\lambda$ and Claim 1, we have

$$\begin{aligned}
(a+\lambda)|X| &\geq (a+\lambda)\Big(\frac{(b-\lambda)(b+2)}{a+\lambda-1}+1\Big)|Y_0| \\
&\geq \Big(\frac{(b-\lambda)(b+2)}{a+\lambda-1}+1\Big)|Y| \\
&> (b-\lambda)|Y|+|Y| \geq (b-\lambda)|Y|+1,
\end{aligned}$$

which contradicts Claim 6. Hence, Claim 7 holds. □

CLAIM 8. $e_G(X,Y) \leq (b-\lambda+2)|X|$.

*Proof.* Since $|Y| \geq b+3$ by Claim 3 and $d_{G-X}(u) \leq b-\lambda+1 \leq b+1$ for every $u \in Y$ by Claim 4, there exist at least two independent vertices $u,v \in Y$. Moreover, it follows from Claims 4 and 7 that

$$\begin{aligned}
\max\{d_G(u),d_G(v)\} &\leq \max\{d_{G-X}(u)+|X|,d_{G-X}(v)+|X|\} \\
&\leq (b-\lambda+1)+|X| < (b-\lambda+1)+\frac{(b-\lambda)p+2}{a+b}-(b-\lambda+1) \\
&= \frac{(b-\lambda)p+2}{a+b}
\end{aligned}$$

for any two vertices $u,v \in Y$. In terms of the above inequalities and the hypothesis of Theorem 4, $G[N_G(x)\cap Y]$ is complete for every $x \in X$. Note that $X \neq \emptyset$ by Claim 5. Combining this with Claim 4, we have $e_G(x,Y) \leq \Delta(G[Y])+1 \leq b-\lambda+2$. Therefore, $e_G(X,Y) \leq (b-\lambda+2)|X|$ holds. □

Note that $\varepsilon(X,Y) \leq |X|$. It follows from (1), Claims 2, 5, 6, 8 and $\delta(G) \geq \frac{(b-\lambda)(b+2)}{a+\lambda-1}+1$ that

$$\begin{aligned}
\varepsilon(X,Y)-1 &\geq \gamma_H(X,Y) = f(X)+d_{H-X}(Y)-g(Y) \\
&\geq f(X)+d_{G-X}(Y)-\min\{2n,|Y|\}-g(Y) \\
&\geq f(X)+d_{G-X}(Y)-|Y|-g(Y) \\
&\geq (a+\lambda)|X|+d_{G-X}(Y)-|Y|-(b-\lambda)|Y| \\
&= (a+\lambda)|X|+d_G(Y)-e_G(X,Y)-(b-\lambda+1)|Y| \\
&\geq (a+\lambda)|X|+\delta(G)|Y|-(b-\lambda+2)|X|-(b-\lambda+1)|Y| \\
&= (a-b+2\lambda-2)|X|+(\delta(G)-(b-\lambda+1))|Y| \\
&\geq (a-b+2\lambda-2)|X|+\Big(\frac{(b-\lambda)(b+2)}{a+\lambda-1}+1-(b-\lambda+1)\Big)|Y| \\
&= (a-b+2\lambda-2)|X|+\Big(\frac{b+2}{a+\lambda-1}-1\Big)(b-\lambda)|Y| \\
&\geq (a-b+2\lambda-2)|X|+\Big(\frac{b+2}{a+\lambda-1}-1\Big)((a+\lambda)|X|-1) \\
&\geq (a-b+2\lambda-2)|X|+\Big(\frac{b+2}{a+\lambda-1}-1\Big)(a+\lambda-1)|X| \\
&= (\lambda+1)|X| \geq |X| \geq \varepsilon(X,Y),
\end{aligned}$$

which is a contradiction. Theorem 4 is verified. □

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