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SPLINE WAVELET MULTIRESOLUTION OF SCATTER DATA

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Abstract. In many environmental sciences, the primary information is obtained by direct measurements of natural phenomena. Most often, the measurement points are irregularly distributed and the measured values of the observed physical quantities are affected by human or device errors. There are also situations when the number of measurements points is so large that one can not process them all at a time. In this paper, we introduce a method intended to interpolate a big set of $\mathbb{R}^2$-randomly distributed data. The method also offers the possibility to cure the presence of outliers and moderate noisy data.

Key words: thin plate spline, polynomial interpolation, outliers filtering, multilevel representation.

1. INTRODUCTION

Throughout the paper, by scatter data we mean a set of data

$$\mathscr{S} := \{\boldsymbol{x}_i, z_i\}_{i=\overline{1,N}}, \quad \text{with} \quad \boldsymbol{x}_i = (x_i, y_i) \in D \subset \mathbb{R}^2, \quad z_i \in \mathbb{R}. \tag{1}$$

We assume there is a function $f(x,y)$ defined everywhere on D, called *objective function*, such that

$$f : D \to \mathbb{R}, \quad f(x_i, y_i) = z_i. \tag{2}$$

The scatter data interpolation problem is to recover the function f from the scatter data $\mathscr{S}$. As known, this is an ill-posed problem since there is an infinity of functions that coincide on $\mathscr{S}$. To deal with a treatable problem, one searches for a *model function* $\mathscr{Q}f$ in a certain functional space that has some desired physical relevant or mathematical properties and that approximates the scatter data $\mathscr{S}$. In this paper, a set of data distinguished by

1. irregular distribution of the points,
2. high cardinality,
3. data $\{z_i\}_{i=\overline{1,N}}$ corrupted by outliers or noise,

will be called *problematic scatter data*.

Our goal of building a method that interpolates the problematic scatter data is accomplished in two steps. Using an interpolation algorithm, we first obtain an everywhere defined function (*the raw data interpolation step*), and then we use the cubic spline wavelet basis functions to approximate it (*the post-processing step*).

Formally, the approximation scheme reads as

$$\mathscr{S} \xrightarrow{\mathscr{P}} \mathscr{B} \xrightarrow{\mathscr{Q}} \mathscr{W}, \tag{3}$$

where $\mathscr{B}$ is an intermediate space and $\mathscr{W}$ is the space generated by the spline wavelet basis functions.

Based on the hierarchical nested structure of the spline wavelet spaces, one can easily obtain a family of model functions as approximations of the objective function,

$$\mathscr{Q}^{(J,j)}f(x,y)=\sum_{k,l}a_{k,l}^{(J,j)}\varphi_{k,l}^{j}(x,y). \tag{4}$$

The accuracy of the approximation improves as level j increases. Thus, the information can be stored at the different levels of resolution, depending of the required accuracy.

The paper is organized as follows. In Section 2 we introduce three different interpolation methods as alternatives to obtain an everywhere defined function. Section 3 is devoted to the spline wavelet approximation of a everywhere defined function and to the multilevel representation of scatter data. In Section 4, we illustrate the performance of the scheme (3) by analyzing a theoretical case. Some final remarks and conclusions are given in Section 5.

2. RAW DATA INTERPOLATION STEP

The goal of the first stage in the scheme (3) is to obtain an everywhere defined function that interpolates or approximates the problematic scatter data:

$$\mathscr{P}f(\boldsymbol{x}_i)\simeq z_i,\quad i=\overline{1,N}. \tag{5}$$

Here and thereafter, by $\simeq$ we understand either the equality relation or a relation of approximation doubled by some invariant properties of $\mathscr{P}$ with respect to the polynomial functions. In this last case, one speaks about quasi interpolation.

From the existing methods, see [6] where 53 methods are comparatively studied, we select two that fit most our propose: thin plate interpolation method and polynomial natural neighbor.

2.1. Thin Plate Spline Interpolation (TPS) Method

There is a large literature devoted to the interpolation of scatter data by thin plate spline, see [7, 8, 12] to cite some works related to our problem.

The thin plate spline belongs to the family of radial basis functions (RBF). Let d be the dimension of the space $\mathbb{R}^d$ and q be an integer number. Let $\mathbb{P}_{d;q}:=\{p_a\}_{a=\overline{1,M}}$ be a basis set of polynomials $p_a:\mathbb{R}^d\to\mathbb{R}$ of degree less than q, where $M=\mathrm{C}^d_{q+d-1}$. The thin plate spline generating function of order q is defined by

$$\phi(r)=\begin{cases} r^{2q-d}, & d \text{ odd},\\ r^{2q-d}\log(r), & d \text{ even}.\end{cases} \tag{6}$$

Consider the family $\{\phi_k\}_{k=\overline{1,N}}$ of the translations of $\phi:\mathbb{R}_+\to\mathbb{R}$ defined by $\phi_k(\boldsymbol{x})=\phi(||\boldsymbol{x}-\boldsymbol{x}_k||)$. The interpolation of the scatter data $\mathscr{S}$ is sought as a linear combination of the functions $\{\phi_k\}_{k=\overline{1,N}}$ and $\{p_a\}_{a=\overline{1,M}}$

$$\mathscr{P}f(\boldsymbol{x})=\sum_{k=1}^{N}\alpha_k\phi_k(\boldsymbol{x})+\sum_{a=1}^{M}\beta_a p_a(\boldsymbol{x}). \tag{7}$$

The coefficients $\{\alpha_k\}_{k=\overline{1,N}}$ and $\{\beta_a\}_{a=\overline{1,M}}$ are determined by solving the linear equations

$$\sum_{k=1}^{N}\alpha_k\phi_k(\boldsymbol{x}_i)+\sum_{a=1}^{M}\beta_a p_a(\boldsymbol{x}_i)=z_i,\quad i=\overline{1,N},\quad \sum_{k=1}^{N}\alpha_k p_a(\boldsymbol{x}_k)=0,\quad a=\overline{1,M}. \tag{8}$$

The next algorithm is based on the singular value decomposition (SVD) of a matrix and can be used to find these coefficients as well as to filter noisy data. It involves a threshold value μ and it is similar with the ones introduced in [11], where the spectral decomposition is used instead of SVD.

Algorithm 1 SVD-Filter

1. Build the distance matrix $E_{ij} := \phi\left(||\boldsymbol{x}_i - \boldsymbol{x}_j||\right), i, j = \overline{1,N}$ and the matrix $T_{ia} := p_a(\boldsymbol{x}_i), i = \overline{1,N}, a = \overline{1,M}$.
2. Find the singular value decomposition $\quad \boldsymbol{E} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T$.
3. Find the level k such that $\sigma_k > \mu$ and build up the matrices

$$\boldsymbol{\Sigma}_k = \mathrm{diag}(\sigma_1, \cdots, \sigma_k), \qquad \boldsymbol{V}_k = \mathrm{col}\left(\boldsymbol{v}_1, \cdots, \boldsymbol{v}_k\right).$$

4. Find the QR decomposition $\quad \boldsymbol{V}_k^T \boldsymbol{T} = \boldsymbol{Q}\boldsymbol{R}$.
5. Define the matrix $\quad \boldsymbol{Z}_k := \mathrm{col}(\boldsymbol{q}_{M+1}, \cdots)$, and build the matrices $\quad \boldsymbol{E}_k = \boldsymbol{U}_k \boldsymbol{\Sigma}_k \boldsymbol{Z}_k, \quad \boldsymbol{A} = (\boldsymbol{E}_k | \boldsymbol{T})$.
6. Calculate

$$\begin{pmatrix} \boldsymbol{\delta}_k \\ \boldsymbol{\eta}_k \end{pmatrix} = \left(\boldsymbol{A}^T \boldsymbol{A}\right)^{-1} \boldsymbol{A}^T \boldsymbol{z}.$$

7. Define the coefficients of the development (7) by: $\quad \boldsymbol{\alpha} = \boldsymbol{U}_k \boldsymbol{Z}_k \boldsymbol{\delta}_k, \quad \boldsymbol{\beta} = \boldsymbol{\eta}_k$.

The TPS method is suitable for "small" number of data, $N \leq 1000$ by our experience. If the number of points is large, then the algorithm has no practical use, since it becomes time consuming or breaks down. To surmount such difficulty, we suggest to partition the scatter data and then apply the (SVD-Filter) algorithm on each element of the partition. Following this idea, we define the operator $\mathscr{P} : \mathscr{S} \longrightarrow \mathscr{B}$ in three steps:

Algorithm 2 Partitioned SVD-Filter

1. Build a partition $\{D_\alpha\}_{\alpha=1}^K$ of D and define $\{\mathscr{S}_\alpha\}_{\alpha=1}^K$ of the scatter data $\mathscr{S}$ as follows:

$$D = \bigcup_{\alpha=1}^{K} D_\alpha, \quad \mathscr{I}_\alpha = \{i | \boldsymbol{x}_i \in D_\alpha\}, \quad \mathscr{S}_\alpha = \{(\boldsymbol{x}_i, z_i) | i \in \mathscr{I}_\alpha\}. \tag{9}$$

2. For each element of the partition, use the SVD-Filter algorithm to construct $\mathscr{P} f_\alpha$ associated to $\mathscr{S}_\alpha$:

$$\mathscr{P} f_\alpha(\boldsymbol{x}_i) \simeq z_i, \quad \forall i \in \mathscr{I}_\alpha.$$

3. Define the operator $\mathscr{P}$ by

$$\mathscr{P} f(\boldsymbol{x}) = \sum_{\alpha=1}^{K} \mathscr{P} f_\alpha(\boldsymbol{x}) \mathbf{1}_\alpha(\boldsymbol{x}), \quad \forall \boldsymbol{x} \in D. \tag{10}$$

Note that $\mathscr{P} f$ is a continuous function inside each D_α, but not on the entire D: it may have jumps on the boundary of D_α.

Another strategy to manage large sets of data is to use a nested sequence of sets of points $X_1 \subset X_2 \cdots \subset X_k$ and interpolate the residuals on each set X_k, see [2, 5].

PROPOSITION 1. *If each partition member $\mathscr{S}_\alpha$ of the scatter data set $\mathscr{S}$ contains a number of points greater than M, then the Partitioned SVD-Filter algorithm exactly reconstructs the polynomial function of order q.*

2.2. Natural Neighbor Polynomial

The TPS method is well-suited when dealing with clean data (no outliers or errors in data), otherwise it gives bad results because a wrong value influences all the coefficients. One way to cure such a problem is to use a local quasi interpolation method. The natural neighbor method and its different variants use the known data at the nearest neighbors to a query point and applies weights to them based on proportionate areas [13]. Such a method is local, uses only a subset of samples that surround a query point, and has the interpolated heights within the range of the samples used. It is also very fast for a relative small number of points, but can become very slow when the number of samples increases. Another drawback is that it can not omit an abnormal value in the neighborhood of the interpolating point.

Denote by $\mathscr{N}(\boldsymbol{x},k)$ the set of k-nearest sampling points $\boldsymbol{x}_i$ to a query point $\boldsymbol{x}$ and let

$$P(\boldsymbol{X};\mathscr{N}(\boldsymbol{x},k),m)=\sum_{0\le i+j\le m} a_{i,j}(X-x)^i(Y-y)^j \tag{11}$$

be the polynomial function that minimizes the L^1 distance

$$e(a)=\sum_{\boldsymbol{x}_j\in\mathscr{N}(\boldsymbol{x},k)}\left|z_j-P(\boldsymbol{x}_j;\mathscr{N}(\boldsymbol{x},k),m)\right|. \tag{12}$$

We define the operator $\mathscr{P}$ by

$$\mathscr{P}f(\boldsymbol{x})=P(\boldsymbol{x};\mathscr{N}(\boldsymbol{x},k),m). \tag{13}$$

Note that the method involves two parameters: the degree m of the polynomial base function and the number k of the samples in $\mathscr{N}(\cdot,k)$.

PROPOSITION 2. *If the number k of the nearest neighbors from $\mathscr{N}(\boldsymbol{x},k)$ and the degree m of the polynomial base function satisfy the inequality*

$$k>\frac{(m+1)(m+2)}{2},$$

then the polynomial natural neighbor method reconstructs the polynomial data of order m.

3. CUBIC SPLINE WAVELET POST-PROCESSING STEP

Recall that our method to solve the scatter data problem involves the operators $\mathscr{P}:\mathscr{S}\longrightarrow\mathscr{B}$ and $\mathscr{Q}:\mathscr{B}\longrightarrow\mathscr{W}$, and the solution of the problem is given by the composition $\mathscr{Q}\circ\mathscr{P}$. In the previous section devoted to raw data interpolation, we set up a method to define the operator $\mathscr{P}$.

Multiresolution Analysis of Scatter Data (MRA-SD)

In what follows, we consider $D:=[0,1]\times[0,1]\subset\mathbb{R}^2$. The coordinates of a point $\boldsymbol{x}_i$ will be denoted by (x_i,y_i). The cubic spline multiresolution analysis of the $\mathbb{L}^2([0,1])$ space introduced by Chui and Quack [1] consists in a set of closed finite dimensional subspaces $\mathbf{V}^j([0,1])$ and $\mathbf{W}^j([0,1])$, with $j\in\{j_0,j_0+1,\dots\}$ ($j_0\ge 3$) that exhibit the following properties:

1. $\mathbf{V}^j([0,1])\subset\mathbf{V}^{j+1}([0,1]),\quad \mathbf{V}^{j+1}([0,1])=\mathbf{V}^j([0,1])\oplus\mathbf{W}^j([0,1])$,
2. $\mathbf{V}^{j_0}([0,1])\bigoplus_{j=j_0}^{\infty}\mathbf{W}^j([0,1])=\mathbb{L}^2([0,1])$.

The scaling spaces $\mathbf{V}^j([0,1])$ and the wavelet spaces $\mathbf{W}^j([0,1])$ are generated by the spline functions denoted here by $\{\varphi_k^j(x)\}_k$ and $\{\psi_k^j(x)\}_k$, respectively. Using the tensor product, one constructs the space of the cubic spline functions defined on $[0,1]\times[0,1]$ at level j

$$\mathbf{VV}^j([0,1]^2)=\mathbf{V}^j([0,1])\otimes\mathbf{V}^j([0,1]).$$

A basis in this space is given by the functions $\varphi^j_{k,l}(x,y)$ defined by

$$\varphi^j_{k,l}(x,y) = \varphi^j_k(x)\varphi^j_l(y).$$

A function $f(x,y) : [0,1]\times[0,1] \to \mathbb{R}$ can be approximated by a function from the $\mathbf{VV}^N([0,1]^2)$ space using spline interpolation with knots (x_k, y_l), $x_k = k/2^N$, $y_l = l/2^N$, $k,l = \overline{0,2^N}$:

$$f^N_W(x,y) = \sum_{k,l} a^N_{k,l}(\boldsymbol{f})\varphi^N_{k,l}(x,y).$$

The coefficients $a^N_{k,l}(\cdot)$ can be calculated such that $f^N_W(x,y)$ interpolates the function f and reproduces the cubic polynomial

$$f^N_W(x_k,y_l) = f(x_k,y_l), \quad f^N_W(x,y) \equiv f(x,y), \quad \text{if} \quad f \in \pi_3,$$

with the cost of solving an algebraic system of linear equations. A simplified solution is to calculate the coefficients using the formula [4]

$$a^N_{k,l} = \sum_{m,n} T^N{}_{k,m} T^N{}_{l,n} f(x_m,y_n), \tag{14}$$

where the "projector" T^N is given by

$$T^N = \begin{pmatrix} 1 & 0 & \cdot & \cdot & 0 & \cdots & 0 \\ 7/18 & 18/18 & -9/18 & 2/18 & 0 & \cdots & 0 \\ -1/6 & 8/6 & -1/6 & 0 & 0 & \cdots & 0 \\ 0 & -1/6 & 8/6 & -1/6 & 0 & \cdots & 0 \\ & & & \ddots & & & \\ 0 & \cdots & 0 & -1/6 & 8/6 & -1/6 & 0 \\ 0 & \cdots & 0 & 0 & -1/6 & 8/6 & -1/6 \\ 0 & \cdots & 0 & 2/18 & -9/18 & 18/18 & 7/18 \\ 0 & \cdots & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{15}$$

For coefficients $a^N_{\cdot,\cdot}$ given by (14) and (15), the approximating function $f^N_W(x,y)$ loses the interpolating property, but it still reproduces the cubic polynomial.

By multiresolution of scatter data we mean to find a set of scaling and wavelet coefficients of an approximating function of the scatter data. The MRA-SD algorithm describes a procedure to obtain the multiresolution of scatter data and defines a hierarchical family of approximating functions. It uses the deconstruction and reconstruction algorithms for spline wavelets (extended for 2-D), firstly introduced in [9] for the 1-D case.

PROPOSITION 3. *If the interpolating function $\mathscr{P}f$ exactly reconstructs the polynomial function of the order three, then $\mathscr{Q}^{(J,j)}f$ given by MRA-SD algorithm reproduces the polynomial of the total degree three for any resolution level j.*

Propositions 1, 2, 3 show one can use the MRA-SD algorithm to set up a quasi-interpolation method of scatter data that exactly reproduces the polynomial function up to degree three.

4. NUMERICAL APPLICATION

We now illustrate the performance of the proposed method by considering a theoretical test. The scatter data were generated by the polynomial function

$$f(x,y) = 1 + (x-0.5)^2 + (y-0.5)^2$$

Algorithm 3 Multiresolution Analysis of Scatter Data (MRA-SD)

1. Choose a resolution level J and define the knots: $\tilde{x}_i^J = \frac{i}{2^J}, \quad \tilde{y}_l^J = \frac{l}{2^J}, \quad i,l = \overline{0,2^J}$.
2. Define the function $\mathscr{P}f$ by using one of the methods (10) or (13).
3. Use the operator T^J given by (15) to evaluate

$$a_{k,l}^J = \sum_{m,n} P^J{}_{k,m} P^J{}_{l,n} \mathscr{P}f(\tilde{x}_m^J, \tilde{y}_n^J). \tag{16}$$

4. Use the deconstruction algorithm to find the scaling and wavelet coefficients

$$\{a_{k,l}^{j_0}\}_{k,l}; \quad \left\{ b_{k,l}^j, c_{k,l}^j, d_{k,l}^j \right\}_{k,l}^{j=j_0,J-1}. \tag{17}$$

5. Choose an intermediate level $j_0 \le j \le J$ and use the reconstruction algorithm to define the scaling coefficients $\{a_{k,l}^{(J,j)}\}_{k,l}$.
6. Define the quasi interpolation function of the scatter data $\mathscr{S}$ at the levels (J,j) by

$$\mathscr{Q}^{(J,j)} f(x,y) = \sum_{k,l} a_{k,l}^{(J,j)} \varphi_{k,l}^j(x,y). \tag{18}$$

on 5000 random uniformly distributed knots in $[0,1] \times [0,1]$. We use three kinds of data: clean data, noisy data and data with outliers. The noisy data were generated by considering

$$f_N(x_i,y_i) = f(x_i,y_i) + 0.2(U(0,1) - 0.5),$$

where $U(0,1)$ is a pseudo-random uniform function on the interval $(0,1)$. The outlier data were obtained by altering the values of 8 arbitrary distributed points (increasing their values by 20).

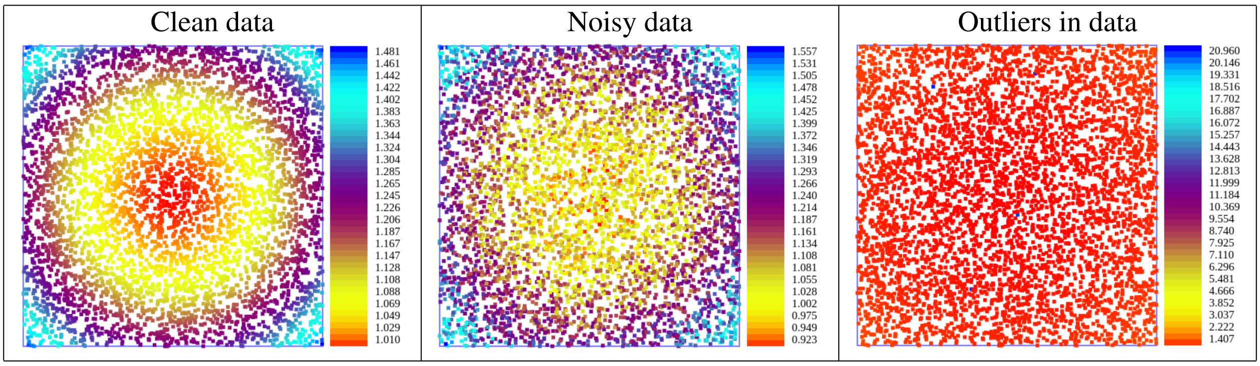

Fig. 1 – Scatter data. The distribution of the points and the colored codded value representation.

The errors in Table 1 were calculated using

$$\mathrm{RMSE} = \sqrt{\frac{1}{N}\sum_{i=1}^{N} \left| f(x_i,y_i) - \mathscr{Q}f^{(J,j)}(x_i,y_i) \right|^2}, \qquad ||\cdot||_\infty = \max_{i=1,N} \left| f(x_i,y_i) - \mathscr{Q}f^{(J,j)}(x_i,y_i) \right|.$$

The space partition of the scatter data used in Partitioned SVD-Filter has a hierarchical tree structure. A parent knot has four children and the brothers have approximately the same number of points.

We choose a polynomial function of degree two because all three methods exactly recover the polynomial data.

Table 1

The accuracy of the quasi-interpolation method described by the MRA-SD algorithm. The columns associated to RBF+SW and NN+SW contains the errors obtained when the Partitioned SVD-Filter and Natural Neighbor methods were used to define $\mathscr{P}f$, respectively. In the RBF+SW case, the multiresolution was generated using levels $(J,j)=(8,8)$, $\mu=0$ (No-F column), $(J,j)=(8,8)$, $\mu=1e-5$ (SVD-F column), and $(J,j)=(8,5)$, $\mu=1e-5$ (SVDW-F column). In the NN+SW case, the approximation was accomplished using levels $(J,j)=(6,5)$ for MRA-SD algorithm and parameters $k=20$ and $m=2$ for the Natural Neighbor method.

		RBF+SW			NN+SW
		No-F	SVD-F	SVDW-F	
Clean Data	RMSE	$2.4e-15$	$2.3e-15$	$2.0e-15$	$1.5e-15$
	$\|\|\cdot\|\|_\infty$	$1.6e-14$	$8.4e-15$	$1.5e-15$	$1.3e-14$
Outliers	RMSE	$8.5e-1$	$3.3e-1$	$3.0e-1$	$2.9e-5$
	$\|\|\cdot\|\|_\infty$	$2.4e+1$	$6.3e+0$	$4.9e+0$	$4.5e-4$
Noisy Data	RMSE	$6.0e-2$	$2.2e-2$	$1.9e-2$	$3.0e-2$
	$\|\|\cdot\|\|_\infty$	$1.0e+0$	$1.1e-1$	$8.3e-2$	$1.1e-1$

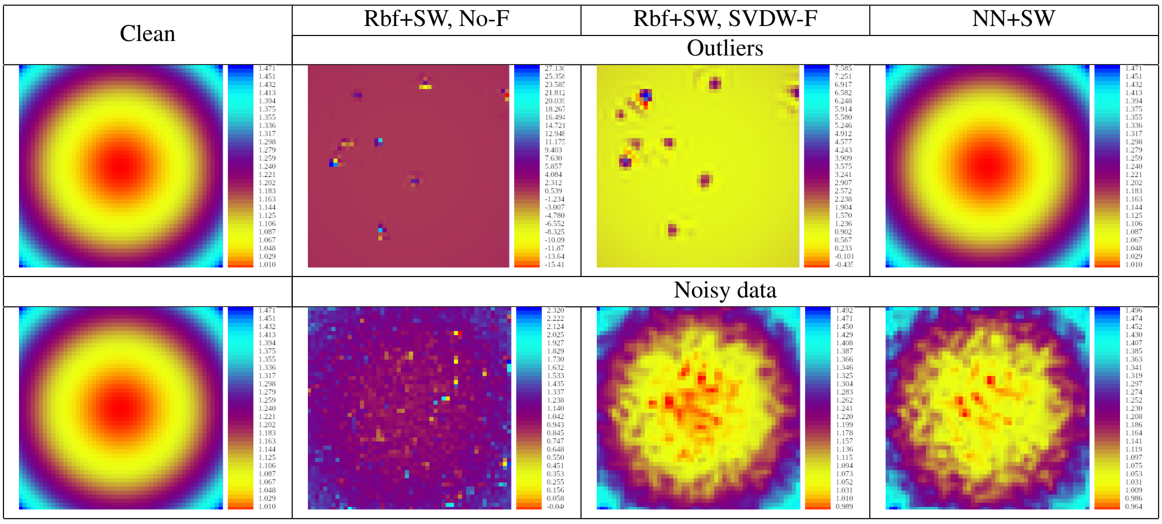

Fig. 2 – Gridded values of the scatter data. The domain was partitioned using a $40x40$ regular grid. The value in each cell was calculated using the integral average value of the function $\mathscr{Q}f^{(J,j)}(x,y)$, see Table 1 for details.

We note that:

(a) the Natural Neighbor interpolation polynomial has the capacity to remove the wrong influences of outliers,

(b) the SVD and spline wavelet filter can be used as denoising filter.

All the numerical results were obtained using the ASTERIX_IADS software developed by authors.

5. CONCLUSIONS

In this paper, we have constructed a scheme (scatter data interpolation) suited to problematic scatter data which are often encountered in environmental sciences. The novelty of the scheme consists of:

– a partition of the scatter data induced by a density criteria of point distribution, when the number of points is too large to perform a SVD-Filter algorithm;

– the polynomial built with the natural neighbor method to cure the data corrupted by outliers;

– the spline wavelet post-processing step.

The approximation using spline wavelets has many advantages, especially when one needs to evaluate the model function many times: a point evaluation of model function needs a very small number of coefficients that in addition can be very easily localized.

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