THE PUBLISHING HOUSE
OF THE ROMANIAN ACADEMY

PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A
Volume 21, Number 4/2020, pp. 303–309

THE EFFECT OF A DISCONTINUOUS WEIGHT FOR A CRITICAL SOBOLEV PROBLEM

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Abstract. Let Ω be a smooth bounded domain of $\mathbb{R}^N, 2^* = \frac{2N}{N-2}; N \geqslant 3$; the critical exponent for the Sobolev embedding and p be a positive discontinuous function. We study the minimizing problem

$$\inf\left\{\int_\Omega p(x)|\nabla u|^2 \mathrm{d}x,\, u \in H_0^1(\Omega),\, \|u\|_{L^{2^*}(\Omega)} = 1\right\}.$$

We prove the existence of a minimizer under a geometrical condition on the domain.

Key words: critical Sobolev exponent, lack of compactness, best Sobolev constant, Pohozaev identity.

1. INTRODUCTION

Let Ω be a smooth bounded domain of $\mathbb{R}^N$, $N \geqslant 3$ and $2^* = \frac{2N}{N-2}$ the critical exponent for the Sobolev embedding. Define Ω_1 and Ω_2 two disjoint domains such that $\Omega = \Omega_1 \cup \Omega_2$ and the set $V(\Omega) = \left\{u \in H_0^1(\Omega),\ \int_\Omega |u|^{2^*} \mathrm{d}x = 1\right\}$. Denote by $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$, which is not empty, and define the barycenter function

$$\begin{array}{rcl} \beta : V(\Omega) & \longrightarrow & \mathbb{R}^N \\ u & \longmapsto & \displaystyle\int_\Omega x|u|^{2^*} \mathrm{d}x. \end{array} \tag{1}$$

We consider the minimizing problem

$$S(p) = \inf_{u \in V(\Omega),\, \beta(u) \in \Gamma} \int_\Omega p(x)|\nabla u|^2 \mathrm{d}x, \tag{2}$$

where p is a discontinuous function defined as follows:

$$p(x) = \begin{cases} p_1(x), & \text{if } x \in \Omega_1, \\ p_2(x), & \text{if } x \in \overline{\Omega}_2 \cap \Omega, \end{cases} \tag{3}$$

and p_i, $i = 1,2$ are some positive functions which satisfy the following assumptions.

1. The functions p_i are smooth on $\bar{\Omega}_i$ for $i = 1,2$.
2. For $i = 1,2$, $\alpha_i := \min\limits_{x \in \Omega_i} p_i(x)$ are strictly positive constants such that $\alpha_1 < \alpha_2$.

The study of this problem has many interesting properties [8, 16] and arising in a geometric problem, namely, Yamabe problem and the prescribe scalar curvature problem [1]. The invariance of the problem under dilation causes a lack of compactness. Besides to the failure of Palais-Smale condition has been the subject of several study of this type of problem. In fact, Bahri et al. in [5] gave positive answer to the Euler equation associated to this problem, when some homology group of the domain with coefficients in $\mathbb{Z}/2\mathbb{Z}$ is non trivial. In [7], Brezis el al. studied the following problem

$$\begin{cases} -\mathrm{div}(p(x)\nabla u) &= u^{2^*-1}+\lambda u \quad \text{in } \Omega, \\ u &> 0 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{4}$$

where Ω a smooth bounded domain of $\mathbb{R}^N$. Let λ_1 be the first eigenvalue of $-\Delta$ on Ω with zero boundary condition and λ^* denote a positive constant. The authors proved, in the case when p is constant, the existence of a solution of (4); if $n \geqslant 4$, for $\lambda \in]0,\lambda_1[$ and for $\lambda \in]\lambda^*,\lambda_1[$, if $n=3$. Further on, Hadiji et al. in [11] extended the previous result to the general case when p is a smooth positive function i.e. $p \in H^1(\Omega) \cup C(\overline{\Omega})$. The authors proved that the existence of the solution depends on the parameter λ, on the behavior of p near its minima, and on the geometry of the domain Ω.

In [12] Hadiji et al. studied the existence and the multiplicity of the solution to the problem (4), first, when the set of minimizers for the weight p has a multiple connected component then, when the case when this set has one connected component and has a complex topology.

Recently, in [4] Baraket et al. gave positive answer to the problem (2), in the case when the functions $p_i, i \in \{1,2\}$, are positive constants. The authors proved the existence of a minimizers under some assumptions. Our result extends the previous one in the case when $p_i, i \in \{1,2\}$, are positive functions.

Remark that without the condition $\beta(u) \in \Gamma$ we have $S(p) = \alpha_1 S$, as one can verify concentrating an extremal function for the best Sobolev constant S near a point in the interior of the region Ω_1. In this case the infimum $S(p)$ is not attained.

2. STATEMENTS AND PROOFS OF RESULTS

We need to recall some results of Baraket et al. in [4], let

$$S_{\alpha_1,\alpha_2} = \inf\left\{\alpha_1 \int_{\mathbb{R}^N_+} |\nabla u|^2 \mathrm{d}x + \alpha_2 \int_{\mathbb{R}^N_-} |\nabla u|^2 \mathrm{d}x,\, u \in H^1(\mathbb{R}^N),\, u \neq 0 \text{ in } \mathbb{R}^N_\pm, \|u\|_{L^{2^*}(\mathbb{R}^N)} = 1\right\},$$

where $\mathbb{R}^N_+ = \{(x',x_N) \in \mathbb{R}^{N-1} \times [0,\infty[\}$ and $\mathbb{R}^N_- = \{(x',x_N) \in \mathbb{R}^{N-1} \times]-\infty,0]\}$. Set

$$S^+ = \inf\left\{\int_{\mathbb{R}^N_+} |\nabla u|^2 \mathrm{d}x, \quad u \in H^1(\mathbb{R}^N_+),\ u \neq 0 \text{ in } \mathbb{R}^N_+,\ \|u\|_{L^{2^*}(\mathbb{R}^N_+)} = 1\right\}$$

and

$$S^- = \inf\left\{\int_{\mathbb{R}^N_-} |\nabla u|^2 \mathrm{d}x, \quad u \in H^1(\mathbb{R}^N_-),\ u \neq 0 \text{ in } \mathbb{R}^N_-,\ \|u\|_{L^{2^*}(\mathbb{R}^N_-)} = 1\right\}.$$

It is easy to verify that (see for example [9]) $S^+ = S^- = \dfrac{S}{2^{\frac{2}{N}}}$, where S is the best constant of the Sobolev embedding defined by $S = \inf\limits_{u \in H^1(\mathbb{R}^N)\setminus\{0\}} \dfrac{\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x}{\left(\int_{\mathbb{R}^N} |u|^{2^*} \mathrm{d}x\right)^{\frac{2}{2^*}}}$. We need also to recall the following result from [4]

THEOREM 1. *The following equality holds*

$$S_{\alpha_1,\alpha_2} = \left(\frac{\alpha_1^{\frac{N}{2}} + \alpha_2^{\frac{N}{2}}}{2} \right)^{\frac{2}{N}} S.$$

We state now our main result

THEOREM 2. *Let Ω, Ω_1, Ω_2, p be as defined in the Introduction and let $x_0 \in \Gamma$. Assume that the following geometrical condition (g.c.) on Γ holds: in a neighborhood of x_0 , Ω_2 lies on one side of the tangent plane at x_0 and the mean curvature with respect to the unit inner normal of Ω_2 at x_0 is positive.*
Then $S(p)$ is attained by some $u \in H_0^1(\Omega)$.

The following proposition presents a strict lower bound for the minimizing problem

PROPOSITION 3. *The following inequality holds*

$$\alpha_1 S < S(p).$$

Proof. We have $S(p) \geqslant \min_{x\in\Omega} p(x) S$, so $S(p) \geqslant \alpha_1 S$. Arguing by contradiction, suppose that $S(p) = \alpha_1 S$. Assume that the equality holds and consider a minimizing sequence $(u_n)_{n\in\mathbb{N}}$, then for every $n \in \mathbb{N}$, $u_n \in V(\Omega)$, $\beta(u_n) \in \Gamma$ and $\lim_{n\to+\infty} \int_\Omega p(x)|\nabla u_n|^2 \mathrm{d}x = \alpha_1 S$.

Since $\int_\Omega p(x)|\nabla u_n|^2 \mathrm{d}x \geqslant \alpha_1 S$ then $\lim_{n\to+\infty} \int_\Omega |\nabla u_n|^2 \mathrm{d}x = S$. Therefore, there exists $x_0 \in \bar{\Omega}$ such that, for a subsequence, $|\nabla u_n|^2 \to S\,\delta_{x_0}$ and $|u_n|^{2^*} \to \delta_{x_0}$, where δ_{x_0} is the Dirac mass in x_0, see [14].

Since $\beta(u_n) \in \Gamma$ for every $n \in \mathbb{N}$, it follows that $x_0 \in \Gamma$ and $p(x_0) = \alpha_0 > \alpha_1$. Therefore $\lim_{n\to+\infty} \int_\Omega p(x)|\nabla u_n|^2 \mathrm{d}x = p(x_0) S > \alpha_1 S$, which gives a contradiction. □

If Γ is flat, that is, the mean curvature at any point of Γ is zero, then we have the following non-existence result

PROPOSITION 4. *Let $\Omega = B(0,R)$ and consider $\Gamma = \{x \in \Omega \,/\, x_N = 0\}$ which divides Ω into two subdomains Ω_1 and Ω_2. Then $S(p)$ is not attained.*

Proof. Indeed, if (2) is attained by u then $|u|$ is a minimization solution of (2) . Let us suppose that $S(p)$ is attained by some positive function $u \geqslant 0$. Then there exists a Lagrange multiplier $\lambda \in \mathbb{R}$ such that u satisfies

$$\begin{cases} -\mathrm{div}(p_1(x)\nabla u) = \lambda\, u^{2^*-1} & \text{in } \Omega_1, \\ -\mathrm{div}(p_2(x)\nabla u) = \lambda\, u^{2^*-1} & \text{in } \Omega_2, \\ p_1(x)\dfrac{\partial u}{\partial \nu_1} + p_2(x)\dfrac{\partial u}{\partial \nu_2} = 0 & \text{on } \Gamma, \\ u \neq 0 & \text{on } \Gamma \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5}$$

Let us suppose that $S(p)$ is attained by some positive function $u \geqslant 0$, On one hand, if we multiply (5) by $\nabla u \cdot x$ and we integrate on Ω_i we obtain

$$\begin{aligned} \int_{\Omega_i} -\mathrm{div}(p_i(x)\nabla u)\nabla u \cdot x \,\mathrm{d}x \;&=\; -\frac{n-2}{2}\int_{\Omega_i} p_i(x)|\nabla u|^2 \mathrm{d}x - \frac{1}{2}\int_{\Omega_i} \nabla p_i(x)\cdot x|\nabla u|^2 \mathrm{d}x \\ &-\; \frac{1}{2}\int_{\partial\Omega_i} p_i(x)\;(x\cdot \nu_i)|\frac{\partial u}{\partial \nu_i}|^2 \mathrm{d}s_x, \end{aligned} \tag{6}$$

where $i \in \{1,2\}$ and ν_i denote the outward normal to $\partial\Omega_i$. On the other hand, if we multiply

$$-\mathrm{div}(p_i(x)\nabla u) = u^{2^*-1}$$

by $\frac{n-2}{2}u$ and we integrate over Ω_i, we obtain,

$$\frac{n-2}{2}\int_{\Omega_i} p_i(x)|\nabla u|^2 \mathrm{d}x \quad = \quad \frac{n-2}{2}\int_{\Omega_i} |u(x)|^{2^*}\mathrm{d}x. \tag{7}$$

Combining (6) and (7) we obtain

$$\begin{aligned}\frac{n-2}{2}\int_{\Omega_i} p_i(x)|\nabla u|^2\mathrm{d}x - \frac{1}{2}\int_{\Omega_i}\nabla p_i(x)\cdot x|\nabla u|^2\mathrm{d}x \quad &- \quad \frac{1}{2}\int_{\partial\Omega_i} p_i(x)(x\cdot\nu_i)\Big|\frac{\partial u}{\partial \nu_i}\Big|^2\mathrm{d}s_x \\ &= \quad \frac{n-2}{2}\int_{\Omega_i}|u(x)|^{2^*}\mathrm{d}x.\end{aligned} \tag{8}$$

So, $\int_{\Omega_i}\nabla p_i(x)\cdot x|\nabla u|^2\mathrm{d}x + \int_{\partial\Omega_i} p_i(x)(x\cdot\nu_i)\big|\frac{\partial u}{\partial\nu_i}\big|^2\mathrm{d}s_x = 0$. On the other hand, we have $\int_{\Omega}\nabla p(x)\cdot x|\nabla u|^2\mathrm{d}x + \int_{\partial\Omega} p(x)(x\cdot\nu)\big|\frac{\partial u}{\partial\nu}\big|^2\mathrm{d}s_x = 0$, where ν denote the outward normal to $\partial\Omega$, and $\int_{\Omega}\nabla p(x)\cdot x|\nabla u|^2\mathrm{d}x = \int_{\Omega_1}\nabla p_1(x)\cdot x|\nabla u|^2\mathrm{d}x + \int_{\Omega_2}\nabla p_2(x)\cdot x\big|\nabla u\big|^2\mathrm{d}x$. Then, by combining the above equations we obtain the Pohozaev identity

$$\int_{\Gamma}\Big[p_1(x)(x\cdot\nu)\Big|\frac{\partial u}{\partial\nu}\Big|^2 + p_2(x)(x\cdot\nu)\Big|\frac{\partial u}{\partial\nu}\Big|^2\Big]\mathrm{d}s_x = 0.$$

Since $B(0,R)$ is star-shaped about 0 then $x\cdot\nu > 0$ on $\partial\Omega$, which gives a contradiction. Therefore $S(p)$ is not attained. □

The proof of Theorem 2 follows from the following two Lemmas.

LEMMA 5. *Under the hypothesis of Theorem 2, we have: if $\alpha_1 S < S(p) < S_{\alpha_1,\alpha_2}$ then the infimum in (2) is attained.*

Proof. We follow the arguments of Baraket et al. from [4]. Let $(u_n) \subset H_0^1(\Omega)$ be a minimizing sequence for (2), that is,

$$\int_{\Omega} p(x)|\nabla u_n|^2\mathrm{d}x = S(p) + o(1), \tag{9}$$

$$\|u_n\|_{L^{2^*}} = 1, \tag{10}$$

and $\beta(u_n) \in \Gamma$. Easily we see that (u_n) is bounded in $H_0^1(\Omega)$, we may extract a subsequence still denoted by u_n, such that $u_n \rightharpoonup u$ weakly in$H_0^1(\Omega)$, $u_n \to u$ strongly in $L^2(\Omega)$, $u_n \to u$ a.e. on Ω, with $\|u\|_{L^{2^*}} \leqslant 1$. Set $v_n = u_n - u$, so that $v_n \rightharpoonup 0$ weakly in $H_0^1(\Omega)$, $v_n \to 0$ strongly in $L^2(\Omega)$, $v_n \to 0$ a.e. on Ω. Using (9) we write

$$\int_{\Omega} p(x)|\nabla u(x)|^2\mathrm{d}x + \int_{\Omega} p(x)|\nabla v_n(x)|^2\mathrm{d}x = S(p) + o(1), \tag{11}$$

since $v_n \rightharpoonup 0$ weakly in $H_0^1(\Omega)$. On the other hand, it follows from a result of Brezis-Lieb ([6], relation (1)) that $\|u+v_n\|_{L^{2^*}}^{2^*} = \|u\|_{L^{2^*}}^{2^*} + \|v_n\|_{L^{2^*}}^{2^*} + o(1)$, (which holds since v_n is bounded in L^{2^*} and $v_n \to 0$ a.e.). Thus, by (10), we have

$$1 = \|u\|_{L^{2^*}}^{2^*} + \|v_n\|_{L^{2^*}}^{2^*} + o(1) \tag{12}$$

and therefore

$$1 \leqslant \|u\|_{L^{2^*}}^{2} + \|v_n\|_{L^{2^*}}^{2} + o(1). \tag{13}$$

Let $x_0=(x',x_{0N})$, denote by $\mathbb{R}^N_{+,x_0}=\left\{x=(x',x_N)\in\mathbb{R}^N \;/\; x'\in\mathbb{R}^{N-1},\ x_N>x_{0N}\right\}$ and $\mathbb{R}^N_{-,x_0}=\left\{x=(x',x_N)\in\mathbb{R}^N \;/\; x'\in\mathbb{R}^{N-1},\ x_N<x_{0N}\right\}$ and using the definition of S_{α_1,α_2}, extending v_j by 0 in $\mathbb{R}^N$ (still denoted by v_j) we obtain

$$\begin{aligned}
\|v_n\|^2_{L^{2^*}} &\leqslant \frac{1}{S_{\alpha_1,\alpha_2}}\left[\alpha_1\int_{\mathbb{R}^N_{+,x_0}}|\nabla v_n(x)|^2\mathrm{d}x+\alpha_2\int_{\mathbb{R}^N_{-,x_0}}|\nabla v_n(x)|^2\mathrm{d}x\right]\\
&\leqslant \frac{1}{S_{\alpha_1,\alpha_2}}\left[\int_{\Omega\cap\mathbb{R}^N_{+,x_0}}\alpha_1|\nabla v_n(x)|^2\mathrm{d}x+\int_{\Omega\cap\mathbb{R}^N_{-,x_0}}\alpha_2|\nabla v_n(x)|^2\mathrm{d}x\right]\\
&\leqslant \frac{1}{S_{\alpha_1,\alpha_2}}\left[\int_{\Omega\cap\mathbb{R}^N_{+,x_0}}p_1(x)|\nabla v_n(x)|^2\mathrm{d}x+\int_{\Omega\cap\mathbb{R}^N_{-,x_0}}p_2(x)|\nabla v_n(x)|^2\mathrm{d}x\right]\\
&\leqslant \frac{1}{S_{\alpha_1,\alpha_2}}\int_\Omega p(x)|\nabla v_n(x)|^2\mathrm{d}x. \qquad (14)
\end{aligned}$$

We claim that $u\not\equiv 0$. Indeed, suppose that $u\equiv 0$. From (11) we obtain $\int_\Omega p(x)|\nabla v_n|^2\mathrm{d}x=S(p)+o(1)$, then $\lim_{n\to+\infty}\int_\Omega p(x)|\nabla v_n|^2\mathrm{d}x=S(p)$. From (12) we see that $\lim_{n\to+\infty}\|v_n\|_{L^{2^*}}=1$. Or (14) gives that

$$\|v_n\|^2_{L^{2^*}}S_{\alpha_1,\alpha_2}\leqslant\int_\Omega p(x)|\nabla v_n|^2\mathrm{d}x.$$

Passing to the limit in the previous inequality we obtain $S_{\alpha_1,\alpha_2}\leqslant S(p)$. This contradicts the hypothesis $S(p)<S_{\alpha_1,\alpha_2}$. Consequently, $u\not\equiv 0$. Now, we deduce from (13) and (14) that

$$S(p)\leqslant S(p)\|u\|^2_{L^{2^*}}+\frac{S(p)}{S_{\alpha_1,\alpha_2}}\int_\Omega p(x)|\nabla v_n(x)|^2\mathrm{d}x+o(1). \qquad (15)$$

Combining (11) and (15) we obtain

$$\int_\Omega p(x)|\nabla u(x)|^2\mathrm{d}x+\int_\Omega p(x)|\nabla v_n(x)|^2\mathrm{d}x\leqslant S(p)\|u\|^2_{L^{2^*}}+\frac{S(p)}{S_{\alpha_1,\alpha_2}}\int_\Omega p(x)|\nabla v_n(x)|^2\mathrm{d}x+o(1).$$

Thus $\int_\Omega p(x)|\nabla u(x)|^2\mathrm{d}x\leqslant S(p)\|u\|^2_{L^{2^*}}+\left[\frac{S(p)}{S_{\alpha_1,\alpha_2}}-1\right]\int_\Omega p(x)|\nabla v_n(x)|^2\mathrm{d}x+o(1)$. Since $S(p)<S_{\alpha_1,\alpha_2}$, we deduce

$$\int_\Omega p(x)|\nabla u(x)|^2\mathrm{d}x\leqslant S(p)\|u\|^2_{L^{2^*}}, \qquad (16)$$

Therefore $\int_\Omega p(x)|\nabla u(x)|^2\mathrm{d}x=S(p)\|u\|^2_{L^{2^*}}$. It follows that $u_n\to u$ strongly in $L^{2^*}(\Omega)$ and $\beta(u)\in\Gamma$. This means that u is a minimum of $S(p)$. □

LEMMA 6. *Assume that there exists x_0 in the interior of Γ such that the condition (g.c.) holds. Then*

$$S(p)<S_{\alpha_1,\alpha_2}.$$

Proof. Let $\{\lambda_i(x_0)\}_{1\leqslant i\leqslant N-1}$, denote the principal curvatures and $H(x_0)=\frac{1}{N-1}\sum_{i=1}^{N-1}\lambda_i(x_0)$ the mean curvature at x_0 with respect to the unit normal. For simplicity, we suppose that $x_0=0$. Therefore we note $\{\lambda_i\}_{1\leqslant i\leqslant N-1}$

the principal curvatures at 0 and $H(0) = \frac{1}{N-1}\sum_{i=1}^{N-1}\lambda_i$. Let $R>0$, such that

$$B(R)\cap\Omega_1 = \left\{(x', x_N)\in B(R);\ x_N > \rho(x')\right\},$$

$$B(R)\cap\Omega_2 = \left\{(x', x_N)\in B(R);\ x_N < \rho(x')\right\},$$

$$B(R)\cap\Gamma = \left\{(x', x_N)\in B(R);\ x_N = \rho(x')\right\},$$

where $x' = (x_1, x_2, ..., x_{N-1})$ and $\rho(x')$ is defined by $\rho(x') = \sum_{i=1}^{N-1}\lambda_i x_i^2 + O(|x'|^3)$. We notice that the condition (g.c.) implies that $\rho(x')\geqslant 0$. Let us define, for $\varepsilon>0$ and for $t\in]0,1[$ the function

$$u_{x^k,\varepsilon,t}(x) = \begin{cases} \frac{\varphi(x)}{(\varepsilon+|\mathbf{x}'-(\mathbf{x}^k)'|^2+t^{-\frac{N-2}{2}}(\mathbf{x}_N-\mathbf{x}_N^k)^2)^{\frac{N-2}{2}}} & \text{if } x_N>\mathbf{0} \\ \frac{\varphi(x)}{(\varepsilon+|\mathbf{x}'-(\mathbf{x}^k)'|^2+(1-t)^{-\frac{N-2}{2}}(\mathbf{x}_N-\mathbf{x}_N^k)^2)^{\frac{\mathbf{N-2}}{2}}} & \text{if } x_N<\mathbf{0}, \end{cases}$$

where φ is a radial C^∞-function such that

$$\varphi(x) = \begin{cases} 1 & \text{if } |x-x^k|\leqslant \frac{R}{4} \\ 0 & \text{if } |x-x^k|\geqslant \frac{R}{2}, \end{cases}$$

$k\in\{1,2\}$ and $p_k(x_k) = \min_{\Omega_k} p_k = \alpha_k$. There exists $t_0 = \frac{(\frac{\alpha_1}{\alpha_2})^{\frac{N}{2}}}{1+(\frac{\alpha_1}{\alpha_2})^{\frac{N}{2}}}$ such that

$$\sup_{t\in[0,1]}\frac{(\alpha_1 t^{\frac{2}{2^*}}+\alpha_2(1-t)^{\frac{2}{2^*}})S}{2^{\frac{2}{N}}} = \frac{(\alpha_1 t_0^{\frac{2}{2^*}}+\alpha_2(1-t_0)^{\frac{2}{2^*}})S}{2^{\frac{2}{N}}} = \left(\frac{\alpha_1^{\frac{N}{2}}+\alpha_2^{\frac{N}{2}}}{2}\right)^{\frac{2}{N}} S.$$

We note $Q(u) = Q_1(u)+Q_2(u)$ where $Q_i(u) = \frac{\int_{\Omega_i} p_i(x)|\nabla u|^2\mathrm{d}x}{\left(\int_\Omega |u|^{2^*}\mathrm{d}x\right)^{\frac{2}{2^*}}}$.

In order to obtain the result of Lemma 6, we use $u_{x^k,\varepsilon} =: (u_{x^k,\varepsilon,t_0})$ as a test function for $S(p)$.
From [2,4], direct computation gives

$$Q_1(u_{x^1,\varepsilon}) = \begin{cases} \frac{p_1(x^1)t_0^{\frac{2}{2^*}}S}{2^{\frac{2}{N}}} + p_1(x^1)SH(0)A(N)\varepsilon^{\frac{1}{2}}|\ln(\varepsilon)| + O(\varepsilon^{\frac{1}{2}}) & \text{if } N=3 \\ \frac{p_1(x^1)t_0^{\frac{2}{2^*}}S}{2^{\frac{2}{N}}} + p_1(x^1)SH(0)A(N)\varepsilon^{\frac{1}{2}} + O(\varepsilon|\ln(\varepsilon)|) & \text{if } N\geqslant 4 \end{cases} \tag{17}$$

and

$$Q_2(u_{x^2,\varepsilon}) = \begin{cases} \frac{p_2(x^2)(1-t_0)^{\frac{2}{2^*}}S}{2^{\frac{2}{N}}} - p_2(x^2)SH(0)A(N)\varepsilon^{\frac{1}{2}}|\ln(\varepsilon)| + O(\varepsilon^{\frac{1}{2}}) & \text{if } N=3 \\ \frac{p_2(x^2)(1-t_0)^{\frac{2}{2^*}}S}{2^{\frac{2}{N}}} - p_2(x^2)SH(0)A(N)\varepsilon^{\frac{1}{2}} + O(\varepsilon|\ln(\varepsilon)|) & \text{if } N\geqslant 4 \end{cases} \tag{18}$$

where $A(N)$ is a positive constant.

We denote $u_{0,\varepsilon}(x) = u_{0,\varepsilon,t_0}(x)$. Combining (17) and (18) we see that,

$$Q(u_{0,\varepsilon}) = \begin{cases} \frac{(\alpha_1 t_0^{\frac{2}{2^*}} + \alpha_2 (1-t_0)^{\frac{2}{2^*}})S}{2^{\frac{2}{N}}} - (\alpha_2 - \alpha_1) H(0) SA(N) \varepsilon^{\frac{1}{2}} |\ln(\varepsilon)| + O(\varepsilon^{\frac{1}{2}}) & \text{if } N = 3 \\ \frac{(\alpha_1 t_0^{\frac{2}{2^*}} + \alpha_2 (1-t_0)^{\frac{2}{2^*}})S}{2^{\frac{2}{N}}} - (\alpha_2 - \alpha_1) H(0) SA(N) \varepsilon^{\frac{1}{2}} + O(\varepsilon |\ln(\varepsilon)|) & \text{if } N \geqslant 4. \end{cases}$$

Therefore, using the definition of t_0, we obtain

$$Q(u_{0,\varepsilon}) \leqslant \begin{cases} \left(\frac{\alpha_1^{\frac{N}{2}} + \alpha_2^{\frac{N}{2}}}{2} \right)^{\frac{2}{N}} S - (\alpha_2 - \alpha_1) H(0) SA(N) \varepsilon^{\frac{1}{2}} |\ln(\varepsilon)| + O(\varepsilon^{\frac{1}{2}}) & \text{if } N = 3 \\ \left(\frac{\alpha_1^{\frac{N}{2}} + \alpha_2^{\frac{N}{2}}}{2} \right)^{\frac{2}{N}} S - (\alpha_2 - \alpha_1) H(0) SA(N) \varepsilon^{\frac{1}{2}} + O(\varepsilon |\ln(\varepsilon)|) & \text{if } N \geqslant 4. \end{cases}$$

Finally, since $\alpha_1 < \alpha_2$ then we obtain the desired result. □

REFERENCES

1. T. AUBIN, *Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire*, J. Math. Pures Appl., **55**, pp. 269-296, 1976.
2. ADIMURTHI, G. MANCINI, *The Neumann problem for elliptic equation with critical non linearity*, 65th birthday of Prof. Prodi, Scuola Normale Superiore, Pisa, Ed. by Ambrosetti and Marino, 1991.
3. S.L. ADIMURTHI, S.L. YADAVA, *Positive solution for Neumann problem with critical non linearity on boundary*, Comm. In Partial Diff. Equations, **16**, *11*, pp. 1733-1760, 1991.
4. S. BARAKET, R. HADIJI, H. YAZIDI *The effect of a discontinuous weight for a critical Sobolev problem*, App. Anal., **97**, *14*, pp. 2544-2553, 2018.
5. A. BAHRI, J.M. CORON, *On a nonlinear elliptic equation involving the critical Sobolev exponent: the effect of the topology of the domain*, Comm. Pure Appl. Math., **41**, pp. 253-294, 1988.
6. H. BREZIS, E. LIEB, *A relation between pointwise convergence of functions and convergence of functionals*, Proc. A.M.S., **88**, *3*, pp. 486-490, 1983.
7. H. BREZIS, L. NIRENBERG, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math., **36**, *4*, pp. 437-477, 1983.
8. J.M. CORON, *Topologie et cas limite des injections de Sobolev*, C.R. Acad. Sci. Paris Sér. I Math., **299**, pp. 209-212, 1984.
9. G. CERAMI, D. PASSASEO, *Nonminimizing positive solutions for equations with critical exponents in the half-space*, SIAM J. Math. Anal., **28**, *4*, pp. 867-885, 1997.
10. A.V. DEMYANOV, A.I. NAZAROV, *On the existence of an extremal function in Sobolev embedding theorems with critical exponents*, Algebra & Analysis, **17**, *5*, pp. 105-140, 2005 (Russian); English transl.: St.Petersburg Math. J. **17**, *5*, pp. 108-142, 2006.
11. R. HADIJI, H. YAZIDI, *Problem with critical Sobolev exponent and with weight*, Chinese Annal. Math. ser. B, **28**, *3*, pp. 327-352, 2007.
12. R. HADIJI, R. MOLLE, D. PASSASEO, H. YAZIDI, *Localization of solutions for nonlinear elliptic problems with weight*, C. R. Acad. Sci. Paris, Séc. I Math., **343**, pp. 725-730, 2006.
13. M.F. FURTADO, B.N. SOUZA, *Positive and nodal solutions for an elliptic equation with critical growth*, Commun. Contemp. Math., **18**, *02*, 16 p., 2016.
14. P.L. LIONS, *The concentration-compactness principle in the calculus of variations, The limit case, part 1 and part 2*, Rev. Mat. Iberoamericana, **1**, *1*, pp. 145-201, 1985 and **1**, *2*, pp. 45-121, 1985.
15. M. STRUWE, *Global compactness result for elliptic boundary value problems involving limiting nonlinearities*, Math. Z., **187**, pp. 511-517, 1984.
16. G. TALENTI, *Best constants in Sobolev inequality*, Ann. Mat. Pura Appl., **110**, pp. 353-372, 1976.

Received March 5, 2020