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HOMOLOGICAL PROPERTIES OF BANACH MODULES RELATED TO LOCALLY COMPACT GROUPS

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Abstract. In this paper, we investigate some homological properties of $M(G)_0^*$ and its dual $(M(G)_0^*)^*$ as Banach left $L^1(G)$-module and characterize homological properties such as projectivity, injectivity and flatness for them in terms of G.

Key words: locally compact group, Banach module, projectivity, injectivity, flatness.

1. INTRODUCTION

Throughout the paper, G denotes a locally compact group with a fixed left Haar measure m on G. Let $L^1(G)$ denote the group algebra of G as defined in [6] endowed with the convolution product "$*$" and the norm $\|.\|_1$. Let also $L^\infty(G)$ denote the usual Lebesgue space as defined in [6] equipped with the essential supremum norm $\|.\|_\infty$. We denote by $L_0^\infty(G)$ the subspace of $L^\infty(G)$ consisting of all functions $f \in L^\infty(G)$ that for every positive number ε, there is a compact subset K of G for which $\|f\chi_{G\setminus K}\|_\infty < \varepsilon$, where $\chi_{G\setminus K}$ denotes the characteristic function of $G \setminus K$ on G. For an extensive study of $L_0^\infty(G)$ see [7].

Let $M(G)$ be the measure algebra of G as defined in [6]. It is well-know that $M(G)$ is the dual space of $C_0(G)$, the space of all continuous functions on G vanishing at infinity. The space $M(G)$ equipped with the convolution product "$*$" and the total norm $\|.\|$ is a Banach algebra with the identity element δ_e, the Dirac measure at the identity element e of G. Let $M_a(G)$, $M_d(G)$ and $M_s(G)$ be the space of absolutely continuous, purely discontinuous and singular measures in $M(G)$, respectively. Then $M(G)$ is the direct sum of them. We denote by $M(G)_0^*$ the subspace of $M(G)^*$ consisting of all functionals $\lambda \in M(G)^*$ with the property that for every positive number ε, there exists a compact subset K of G for which $|\langle\lambda,\mu\rangle| < \varepsilon$, where $\mu \in M(G)$, $|\mu|(K) = 0$ and $\|\mu\| = 1$. Similar, we can define the spaces $M_d(G)_0^*$ and $M_s(G)_0^*$. A linear functional $F \in M(G)^{**}$ is said to have *compact carrier* if there exists a compact subset $K \subseteq G$ satisfying

$$\langle F,\lambda\rangle = \langle F,\chi_K\lambda\rangle$$

for all $\lambda \in M(G)^*$. The closure of all linear functionals with compact carrier is denoted by M_G. One can prove that the restriction map is an isometric isomorphism from M_G onto $(M(G)_0^*)^*$ and for every $\mu \in M(G)$

$$\mu = \lim_{K\in\mathscr{K},K\nearrow G} \mu_K,$$

where $\mathscr{K}$ denotes the set of all compact subsets of G ordered by upward inclusion and

$$\mu_K(E) = \mu(K \cap E)$$

for all Borel subsets of G. This shows that $M(G)$ can be isometrically embedded into $(M(G)_0^*)^*$. Similarly, the concept of compact carrier for $\lambda \in (M(G)_0^*)^*$ is defined. Then functionals in $(M(G)_0^*)^*$ with compact carrier are dense in $(M(G)_0^*)^*$; see [8].

For every $\phi \in L^1(G)$ and $\lambda \in M(G)_0^*$, the functions $\phi \cdot \lambda$ and $\lambda \cdot \phi$ in $M(G)_0^*$ are defined by

$$\langle \lambda \cdot \phi, \mu \rangle = \langle \lambda, \phi * \mu \rangle \quad \text{and} \quad \langle \phi \cdot \lambda, \mu \rangle = \langle \lambda, \mu * \phi \rangle$$

for all $\mu \in M(G)$. It is easy to see that

$$M(G)_0^* = M_d(G)_0^* \oplus_\infty L_0^\infty(G) \oplus_\infty M_s(G)_0^*.$$

Hence for every $\lambda \in M(G)_0^*$, we have

$$\lambda = \lambda_d + \lambda_0 + \lambda_s$$

for some $\lambda_d \in M_d(G)_0^*$, $\lambda_0 \in L_0^\infty(G)$ and $\lambda_s \in M_s(G)_0^*$. Note that if $\mu \in M(G)$ and $\phi \in L^1(G)$, then $\mu * \phi \in L^1(G)$ and so

$$\langle \phi \cdot (\lambda_d + \lambda_s), \mu \rangle = \langle \lambda_d + \lambda_s, \mu * \phi \rangle = 0.$$

Thus $\phi \cdot (\lambda_d + \lambda_s) = 0$ which implies that

$$\phi \cdot \lambda = \phi \cdot \lambda_0 = \lambda_0 * \tilde{\phi} \in C_0(G),$$

where $\tilde{\phi}(x) = \phi(x^{-1})$ for all $x \in G$, see [7]. Similarly,

$$\lambda \cdot \phi = \lambda_0 \cdot \phi = \frac{1}{\Delta} \tilde{\phi} * \lambda_0 \in C_0(G),$$

where Δ denotes the modular function of G. So we can regard $M(G)_0^*$ as a Banach $L^1(G)$-bimodule with the module actions defined by "·". We also can prove that $(M(G)_0^*)^*$ is the dual bimodule of the Banach $L^1(G)$-bimodule $M(G)_0^*$ with the module operations defined by

$$\langle F \cdot \phi, \lambda \rangle = \langle F, \phi \cdot \lambda \rangle \quad \text{and} \quad \langle \phi \cdot F, \lambda \rangle = \langle F, \lambda \cdot \phi \rangle$$

for all $F \in (M(G)_0^*)^*$, $\lambda \in M(G)_0^*$ and $\phi \in L^1(G)$.

For every $H \in (M(G)_0^*)^*$ and $\lambda \in M(G)_0^*$, we define the bounded linear functional $H\lambda \in M(G)^*$ by

$$\langle H\lambda, \mu \rangle = \langle H, \lambda\mu \rangle,$$

in which

$$\langle \lambda\mu, \nu \rangle = \langle \lambda, \mu * \nu \rangle$$

foe all $\mu, \nu \in M(G)$. It is well-known from [8] that $H\lambda \in M(G)_0^*$. That is, $M(G)_0^*$ is a left introverted subspace of $M(G)^*$. So the dual space $(M(G)_0^*)^*$ of $M(G)_0^*$ is a Banach algebra with the first Arens product "$\diamond$" defined by

$$\langle F \diamond H, \lambda \rangle = \langle F, H\lambda \rangle,$$

for all $F, H \in (M(G)_0^*)^*$ and $\lambda \in M(G)_0^*$. Note that

$$F \cdot \phi = F \diamond \phi \quad \text{and} \quad \phi \cdot F = \phi \diamond F$$

for all $F \in (M(G)_0^*)^*$ and $\phi \in L^1(G)$. For the details, we refer the readers to [8]. For every $F \in (M(G)_0^*)^*$, there exist $F_d \in (M_d(G)_0^*)^*$, $F_0 \in L_0^\infty(G)^*$ and $F_s \in (M_s(G)_0^*)^*$ such that

$$F = F_d + F_0 + F_s.$$

Then for every $\phi \in L^1(G)$ we have $F \diamond \phi = F_0 \diamond \phi$ and $\phi \diamond F = \phi \diamond F_0$. Since $L^1(G)$ is an ideal in $L_0^\infty(G)^*$, it follows that $L^1(G)$ is an ideal in $(M(G)_0^*)^*$.

Let $\mathfrak{A}$ be a Banach algebra. We denote the categories of Banach left $\mathfrak{A}$-modules, of Banach right $\mathfrak{A}$-modules and of Banach $\mathfrak{A}$-bimodules, by $\mathfrak{A}$-MOD, by MOD-$\mathfrak{A}$ and by $\mathfrak{A}$-MOD-$\mathfrak{A}$, respectively. For $E, F \in \mathfrak{A}$-MOD, we reserve the symbol ${}_{\mathfrak{A}}\mathscr{B}(E,F)$ for the set of all left $\mathfrak{A}$-module homomorphisms in $\mathscr{B}(E,F)$, the set of all bounded operators from E into F. Note that $\mathscr{B}(\mathfrak{A}, E) \in \mathfrak{A}$-MOD-$\mathfrak{A}$ with the module operations

$$(a \cdot T)(b) = T(ba) \quad \text{and} \quad (T \cdot a)(b) = T(ab)$$

for all $a, b \in \mathfrak{A}$ and $T \in \mathscr{B}(\mathfrak{A}, E)$. A left $\mathfrak{A}$-module homomorphism $T \in {}_{\mathfrak{A}}\mathscr{B}(E,F)$ is called *admissible* if $T(E)$ is closed and $\ker T$ and $T(E)$ are complemented subspaces of E and F, respectively. Also, T is called a *retraction* if it has a right inverse in ${}_{\mathfrak{A}}\mathscr{B}(F,E)$.

Let us recall that an element $P \in \mathfrak{A}$-MOD is called *projective*, if for every $E, F \in \mathfrak{A}$-MOD, each admissible epimorphism $\theta \in {}_{\mathfrak{A}}\mathscr{B}(E,F)$ and each $\sigma \in {}_{\mathfrak{A}}\mathscr{B}(P,F)$, there exists $\tau \in {}_{\mathfrak{A}}\mathscr{B}(P,E)$ such that

$$\theta \circ \tau = \sigma.$$

The set of all projective Banach left $\mathfrak{A}$-modules is denoted by $\mathfrak{A}$-PMOD. One can define the concept of projectivity for Banach $\mathfrak{A}$-bimodules. In the case where $\mathfrak{A}$ is a projective $\mathfrak{A}$-bimodule, it is called *biprojective*. Let also recall that an element $I \in \mathfrak{A}$-MOD is called *injective*, if for each $E, F \in \mathfrak{A}$-MOD, every admissible monomorphism $\theta \in {}_{\mathfrak{A}}\mathscr{B}(E,F)$ and every $\sigma \in {}_{\mathfrak{A}}\mathscr{B}(E,I)$, there exists $\tau \in {}_{\mathfrak{A}}\mathscr{B}(F,I)$ such that $\tau \circ \theta = \sigma$. The set of all injective Banach left $\mathfrak{A}$-modules is denoted by $\mathfrak{A}$-IMOD. Similarly, the set of all injective Banach right $\mathfrak{A}$-modules is denoted by IMOD-$\mathfrak{A}$. An element $E \in \mathfrak{A}$-MOD is called *flat*, if $E^* \in$ IMOD$-\mathfrak{A}$. The set of all flat Banach left $\mathfrak{A}$-modules is denoted by $\mathfrak{A}$-FMOD. A Banach algebra $\mathfrak{A}$ is called *biflat* if it is a flat Banach $\mathfrak{A}$-bimodule.

Homological properties of Banach modules have been studied by several authors [1, 3–5, 9]. For example, Dales and Polyakov [3] studied homological properties of modules over group algebras. They gave necessary and sufficient conditions for some Banach left $L^1(G)$-modules to have homological properties such as projectivity, injectivity and flatness.

In this paper, we continue these investigations for Banach left $L^1(G)$-modules $M(G)_0^*$ and $(M(G)_0^*)^*$ introduced in [8]. We characterize the locally compact groups such that these Banach modules are, respectively, (bi)projective, injective and (bi)flat.

2. PROJECTIVITY OF $M(G)_0^*$ AND $(M(G)_0^*)^*$

We commence this section with the following result.

THEOREM 2.1. *Let G be a locally compact group. Then $M(G)_0^* \in L^1(G)$-PMOD if and only if G is finite.*

Proof. Let $E, F \in L^1(G)$-MOD, $\theta \in {}_{L^1(G)}\mathscr{B}(E,F)$ be an admissible epimorphism and $\sigma \in {}_{L^1(G)}\mathscr{B}(L_0^\infty(G), F)$. Let $\pi : M(G)_0^* \to L_0^\infty(G)$ be the canonical projection map. If $M(G)_0^* \in L^1(G)$-PMOD, then there exists $\tau \in {}_{L^1(G)}\mathscr{B}(M(G)_0^*, E)$ such that $\theta \circ \tau = \sigma \circ \pi$. It follows that

$$\theta \circ \tau \circ i = \sigma,$$

where $i : L_0^\infty(G) \to M(G)_0^*$ is the canonical injection map. Therefore, $L_0^\infty(G) \in L^1(G)$-PMOD. A similar argument as given in Theorem 3.1 in [3] shows that G is compact. Thus, $L^\infty(G) \in L^1(G)$-PMOD. Invoke Theorem 3.3 in [3] to conclude that G is finite. The proof will be complete if we only note that $M(G)_0^* = L^\infty(G)$ is a projective Banach left $L^1(G)$-module, when G is finite. □

We now investigate projectivity of $(M(G)_0^*)^*$ as Banach left $L^1(G)$-module.

THEOREM 2.2. *Let G be a locally compact group. Then the following assertions are equivalent:*
(a) $(M(G)_0^*)^* \in L^1(G)$-PMOD.
(b) $M(G) \in L^1(G)$-PMOD.
(c) *G is discrete.*

Proof. We define the map $T : M(G) \to (M(G)_0^*)^*$ by $\langle T(\mu), \lambda\rangle = \langle \lambda, \mu\rangle$ for all $\mu \in M(G)$ and $\lambda \in M(G)_0^*$. Then for every $f \in C_0(G)$ and $\mu \in M(G)$ we have

$$\langle R \circ T(\mu), f\rangle = \langle T(\mu), f\rangle = \langle f, \mu\rangle,$$

where $R : (M(G)_0^*)^* \to M(G)$ is the restriction map to $C_0(G)$. This shows that R is a retraction. So if $(M(G)_0^*)^* \in L^1(G)$-PMOD, then $M(G) \in L^1(G)$-PMOD. Hence (a) implies (b). Let $F \in (M(G)_0^*)^*$ and $\lambda \in M_d(G)_0^* \oplus_\infty M_s(G)_0^*$. Then for every $\phi \in L^1(G)$, we have $\lambda \cdot \phi = 0$ and so

$$\langle F \cdot \lambda, \phi\rangle = \langle F, \lambda \cdot \phi\rangle = 0.$$

If G is discrete, then $\delta_e \in L^1(G)$ and $L_0^\infty(G)^* = M(G)$. So

$$\langle F, \lambda\rangle = \langle \delta_e \cdot F, \lambda\rangle = \langle \delta_e, F \cdot \lambda\rangle = \langle F \cdot \lambda, \delta_e\rangle = 0.$$

It follows that $(M(G)_0^*)^* = M(G)$. Hence

$$(M(G)_0^*)^* = L^1(G) = \ell^1(G)$$

and $\ell^1(G)$ is a unital Banach algebra. Therefore, $(M(G)_0^*)^* \in L^1(G)$-PMOD. That is, (c) implies (a). By Theorem 2.6 in [3], (b) implies (c). □

3. INJECTIVITY OF $M(G)_0^*$ AND $(M(G)_0^*)^*$

An element $E \in \mathfrak{A}$-MOD is called *faithful* if $\mathfrak{A} \cdot x \neq 0$ for all nonzero elements $x \in E$. Let us remark from Proposition 1.7 in [3] that if $E \in \mathfrak{A}$-*mod* is faithfull, then $E \in \mathfrak{A}$-IMOD if and only if there exists a left $\mathfrak{A}$-module homomorphism $\rho : \mathscr{B}(\mathfrak{A}, E) \to E$ such that $\rho \circ \Pi_E = id_E$, where id_E is identity map on E and $\Pi_E : E \to \mathscr{B}(\mathfrak{A}, E)$ is defined by $\Pi_E(x)(a) = a \cdot x$ for all $x \in E$ and $a \in \mathfrak{A}$.

THEOREM 3.1. *Let G be a locally compact group. Then* $M(G)_0^* \in L^1(G)$-IMOD *if and only if G is compact.*

Proof. Assume that G is compact. Then $L^1(G)$ is amenable. So $M(G)_0^* \in L^1(G)$-IMOD; see for example Proposition 1.11 in [3].

For the converse, suppose that G is not compact. Then there exist a subset S and an open, σ-compact and non-compact subgroup $\mathfrak{H}$ of G such that

$$G = \bigcup_{s \in S} s\mathfrak{H} = \bigcup_{s \in S} \mathfrak{H}s^{-1}.$$

Let $m_{\mathfrak{H}}$ be the restriction of m to the family of Borel subsets of $\mathfrak{H}$. Since $\mathfrak{H}$ is σ-compact and non-compact, we can choose sequences (K_i) and (C_i) of compact subsets of $\mathfrak{H}$ such that $K_i \subsetneqq \mathrm{int}K_{i+1}$, $m_{\mathfrak{H}}(C_i) > 0$ and $C_i \cap C_j = \emptyset$ whenever $i, j \in \mathbb{N}$ and $i \neq j$. We define $Q_1 : \ell^\infty \to M(\mathfrak{H})^*$ by

$$Q_1((\alpha_i)) = \sum_{i=1}^{\infty} \alpha_i \, \chi_{C_i}$$

and $Q_2 : M(\mathfrak{H})^* \to \mathscr{B}(L^1(\mathfrak{H}), C_0(\mathfrak{H}))$ by

$$Q_2(\lambda)(\phi) = \sum_{i=1}^{\infty} \Pi_{M(\mathfrak{H})_0^*}(\chi_{K_i}\lambda)(\chi_{L_i}\phi),$$

where $K_0 = \emptyset$, $L_i = K_i \setminus K_{i-1}$ and

$$\langle \chi_{K_i}\lambda, \mu \rangle = \langle \lambda, \mu_{K_i} \rangle$$

for all $\mu \in M(G)$. Let ι^* be the adjoint of the inclusion map $\iota : L^1(G) \to M(G)$. Take the nonzero element $\lambda \in M(G)_0^*$. Then for every $\phi \in L^1(G)$ we have

$$\langle \iota^*(\lambda), \phi \rangle = \langle \lambda, \iota(\phi) \rangle = \langle \lambda, \phi \rangle.$$

This together with the fact that $L^\infty(G) \in L^1(G)$-MOD is faithful shows that $M(G)_0^* \in L^1(G)$-MOD is faithful. So, if $M(G)_0^* \in L^1(G)$-IMOD, then there exists a left $L^1(G)$-module homomorphism

$$\rho : \mathscr{B}(L^1(G), M(G)_0^*) \to M(G)_0^*$$

such that $\rho \circ \Pi_{M(G)_0^*} = id_{M(G)_0^*}$. We now define the map $Q_3 : \mathscr{B}(L^1(G), C_0(G)) \to c_0$ by

$$Q_3(T) = \Big(\frac{1}{m_{\mathfrak{H}}(C_i)} \int_{C_i} R \circ \rho(T)\, dm_{\mathfrak{H}}\Big)_i,$$

where R is the natural map from $M(G)_0^*$ to $L_0^\infty(\mathfrak{H})$. Let

$$Q : \mathscr{B}(L^1(\mathfrak{H}), C_0(\mathfrak{H})) \to \mathscr{B}(L^1(G), C_0(G))$$

be the left $L^1(\mathfrak{H})$-module homomorphism used in the proof of Lemma 3.4 in [3]. We note that if K is a compact subset of $\mathfrak{H}$, then

$$Q_2(\chi_K) - \Pi_{M(\mathfrak{H})_0^*}(\chi_K)$$

has compact support K; i.e, for every $\phi \in L^1(G)$ with $\phi|_K = 0$, we have

$$(Q_2(\chi_K) - \Pi_{M(\mathfrak{H})_0^*}(\chi_K))(\phi) = 0.$$

By the argument used in the proof of Lemma 3.5 in [3], it can be shown that

$$R \circ \rho \circ Q(Q_2(\chi_K)) = R \circ \rho \circ Q(\Pi_{M(\mathfrak{H})_0^*}(\chi_K)).$$

It follows that

$$\begin{aligned} R \circ \rho(Q(Q_2(\chi_K))) &= R \circ \rho \circ Q(\Pi_{M(\mathfrak{H})_0^*}(\chi_K)) \\ &= R \circ \rho \circ \Pi_{M(G)_0^*}(I(\chi_K)) \\ &= R \circ I(\chi_K) \\ &= \chi_K, \end{aligned}$$

where $I : L^\infty(\mathfrak{H}) \to L^\infty(G)$ is the natural embedding. Hence

$$Q_3(Q(Q_2(\chi_K))) = \Big(\frac{m_{\mathfrak{H}}(C_i \cap K)}{m_{\mathfrak{H}}(C_i)}\Big)_i.$$

Set

$$Q_4 := Q_3 \circ Q \circ Q_2 \circ Q_1.$$

For $(\alpha_i)_i \in c_0$, we have

$$Q_4((\alpha_i)) = \sum_{j=1}^{\infty} \alpha_j \, Q_3(Q(Q_2(\chi_{C_j}))) = (\alpha_i)_i.$$

Hence Q_4 is a projection from ℓ^∞ onto c_0, which contradicts Theorem 0.1.16 in [5]. Therefore, G is compact, as claimed. □

Let us recall that a locally compact group G is called *amenable* if there is a left invariant mean on $L^\infty(G)$.

THEOREM 3.2. *Let G be a locally compact group. Then the following statements hold:*
(i) $M(G)_0^* \in L^1(G)$-FMOD *if and only if G is amenable.*
(ii) $(M(G)_0^*)^* \in L^1(G)$-FMOD.

Proof. It is shown in [3] that $L^1(G)$ is always a flat Banach left $L^1(G)$-module and $C_0(G) \in L^1(G)$-FMOD if and only if G is amenable. Hence the theorem will be proved if we recall from [9] that $E \in L^1(G)$-FMOD if and only if $L^1(G) \cdot E \in L^1(G)$-FMOD. □

As an immediate consequence of Theorem 3.2 we present the following result.

COROLLARY 3.3. *Let G be a locally compact group. Then* $(M(G)_0^*)^* \in$ IMOD-$L^1(G)$ *if and only if G is amenable.*

In the following, let us remark from Theorem 2.9.65 in [2] that a Banach algebra $\mathfrak{A}$ is amenable if and only if $\mathfrak{A}$ has a bounded approximate identity and $\mathfrak{A}$ is biftat.

PROPOSITION 3.4. *Let G be a locally compact group. Then the following statements hold:*
(i) $M(G)_0^*$ *is biprojective if and only if G is compact.*
(ii) $M(G)_0^*$ *is always biflat.*
(iii) *If* $(M(G)_0^*)^*$ *is either biprojective or biflat, then G is amenable.*

Proof. First not that $M(G)^* = C_0(G)^{**}$ is a Banach algebra with respect to the first Arens product. For every $\lambda \in M(G)^*$, the involution λ is defined by

$$\langle \lambda^*, \nu \rangle = \overline{\langle \lambda, \bar{\nu} \rangle},$$

where $\bar{\nu}(E) = \overline{\nu(E)}$. It is easy to see that $M(G)_0^*$ is closed with respect to the norm-topology of $M(G)_0^*$ and the involution "$*$". Hence $M(G)_0^*$ is a commutative C^*-algebra.

It is well-known from [4] that a C^*-algebra of a locally compact group is biprojective if and only if G is compact. Hence (i) holds. For (ii), it suffices to note that $M(G)_0^*$ is amenable; see Example 2.3.4 in [10]. Finally, if $(M(G)_0^*)^*$ is biflat, then $(M(G)_0^*)^*$ is amenable. Since $L^1(G)$ is an ideal in $(M(G)_0^*)^*$, it follows that $L^1(G)$ is amenable and so G is amenable. To complete the proof, we recall that every biprojective module is biflat. □

Let $E \in L^1(G)$-MOD. A functional $\Lambda \in E^*$ is called *augmentation invariant* if every $x \in E$ and $\phi \in L^1(G)$, we have

$$\langle \Lambda, \phi \cdot x \rangle = \varphi_G(\phi) \langle \Lambda, x \rangle,$$

where $\varphi_G : M(G) \to \mathbb{C}$ is defined by $\varphi_G(\mu) = \mu(G)$. In the case where Λ is a non-zero augmentation invariant functional in E^*, then E is said to be *augmentation invariant*.

PROPOSITION 3.5. *Let G be a locally compact group. Then* $M(G)_0^*$ *is augmentation invariant if and only if G is compact.*

Proof. Let $M(G)_0^*$ be augmentation invariant. Then there exists a non-zero functional $\Lambda : M(G)_0^* \to \mathbb{C}$ such that

$$\langle \Lambda, \phi \cdot \lambda \rangle = \varphi_G(\phi)\langle \Lambda, \lambda \rangle$$

for every $\phi \in L^1(G)$ and $\lambda \in M(G)_0^*$. Choose an element $\lambda \in M(G)_0^*$ with $\langle \Lambda, \lambda \rangle \neq 0$ and a positive function with norm one in $L^1(G)$, say ϕ. Then $\phi \cdot \lambda \in C_0(G)$. If $\tilde{\Lambda}$ is the restriction map Λ to $C_0(G)$, then

$$\langle \tilde{\Lambda}, \phi \cdot \lambda \rangle = \langle \Lambda, \phi \cdot \lambda \rangle = \varphi_G(\phi)\langle \Lambda, \lambda \rangle = \langle \Lambda, \lambda \rangle.$$

Hence $\tilde{\Lambda}$ is non-zero on $C_0(G)$. This shows that $\tilde{\Lambda}$ is an augmentation invariant on $C_0(G)$. Form 17.19 (c) in [6] we see that G is compact. For the converse, we only need to note that if $\iota : L^1(G) \to M(G)$ is the inclusion map and $\Lambda \in L^\infty(G)^*$ is a augmentation invariant, then $\iota^{**}(\Lambda)$ is a augmentation invariant for $M(G)_0^*$. □

We finish the paper with the following result.

PROPOSITION 3.6. *The Banach left $L^1(G)$-module $(M(G)_0^*)^*$ is always augmentation invariant.*

Proof. Let $\mathscr{K}$ denote the set of all compact subsets of G ordered by upward inclusion. Then the bounded net (χ_{K_α}) has a weak* cluster point in $(M(G)_0^*)^{**}$, say Λ. Let F be a linear functional in $(M(G)_0^*)^*$ with compact carrier $K_{\alpha_0} \in \mathscr{K}$. Then for every $\phi \in L^1(G)$ we have

$$\begin{aligned} \langle \Lambda, \phi \cdot F \rangle &= \lim_\alpha \langle \chi_{K_\alpha}, \phi \cdot F \rangle = \lim \langle F, \chi_{K_\alpha} \cdot \phi \rangle = \lim_\alpha \langle F, (\chi_{K_\alpha} \cdot \phi)\chi_{K_{\alpha_0}} \rangle \\ &= \varphi_G(\phi) \lim_\alpha \langle F, \chi_{K_\alpha} \rangle = \varphi_G(\phi)\langle \Lambda, F \rangle. \end{aligned}$$

From this and Proposition 2.24 in [8] we see that Λ is an augmentation invariant functional for $(M(G)_0^*)^*$. □

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