THE PUBLISHING HOUSE
OF THE ROMANIAN ACADEMY

PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A
Volume 21, Number 3/2020, pp. 221–230

APPELL POLYNOMILAS ASSOCIATED WITH LÉVY PROCESSES AND APPLICATIONS

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Abstract. In this paper we investigate Appell polynomials associated with Lévy processes and their properties. The general mean value property and the moment representations of Appell polynomials are presented. Furthermore, we study Appell polynomials associated with killed Lévy processes at rate $r > 0$ and their running maximum, minimum. Some applications to optimal stopping problems with power reward functions are also reviewed and discussed.

MSC 2010: 60G51, 60E07, 60G40, 11B68.
Key words: Appell polynomials, Lévy processes, optimal stopping.

1. INTRODUCTION

A system of polynomials $\{Q_n(x)\}_{n=0}^{\infty}$ satisfying the recursive differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x}Q_n(x) = nQ_{n-1}(x), \quad n \geq 1,$$

is called Appell system. These polynomials were first investigated by Paul Appell [1]. Appell polynomials have been widely applied in a variety of mathematical areas, e.g., number theory, numerical analysis and recently in probability theory (see, e.g., [4, 7, 14, 20, 22, 24, 26]). Some typical examples of Appell polynomials are, e.g., the Bernoulli, Euler and Hermite polynomials. In particular, the Hermite polynomials are the unique Appell polynomials that are orthogonal.

There are several approaches to study Appell polynomials, e.g., Sheffer [23] introduced the method of generating functions where the system of Appell polynomials is associated with some analytic function. Costabile et al. [4] used the determinantal approach to investigate these polynomials. Recently, Ta [27] studied these polynomials via the probabilistic approach in which every random variable having finite moments or some exponential moments is associated with a system of Appell polynomials. The probabilistic approach offers us powerful tools to derive many useful properties of Appell polynomials and provides interesting new proofs of some classical results for Bernoulli, Euler and Hermite polynomials.

As remarked by Novikov and Shiryaev [14], Appell polynomials play an important role in studying optimal stopping problems for random walks. Our main interest in this paper is inspired by the works in [14, 27] and recent developments due to Kyprianou and Surya [11], Salminen [18] and Christensen et al. [3], that Appell polynomials are used to characterize the value function of optimal stopping problems with power reward functions under Lévy processes. Furthermore, Appell polynomials have a wide range in constructing time-space martingales for Lévy processes (see, e.g., [20, 21, 25]). Picard and Lefèvre [16] utilize such polynomials to find the probability distribution of the ruin time in insurance models. We also refer to [8, 12, 17] for further applications in insurance mathematics.

The paper is organized as follows. In the next section we give definitions of Appell polynomials in cases Lévy processes admit moment generating functions and have moments of all orders. Some general properties are investigated and scrutinized. In particular, under certain conditions Appell polynomials have moment representations. In section 3 we study properties of Appell polynomials associated with killed Lévy processes and running maximum at rate $r > 0$. In section 4 we discuss some applications of Appell polynomials to optimal stopping problems.

2. GENERAL PROPERTIES

We consider a real-valued Lévy process $X = (X_t)_{t\geq 0}$ starting from 0. Roughly speaking, Lévy processes are, in fact, continuous time Markov processes having independent and stationary increments. These processes have rich mathematical theory and are used to model many random phenomena, especially, in finance and industry. Familiar Lévy processes are, e.g., Brownian motion, Gamma processes and compound Poisson processes. We refer to [2,10,19] for detailed and rigorous materials on the theory of Lévy processes. Assume that X has some exponential moments, i.e.,

$$\mathbb{E}(e^{\lambda|X_t|}) < \infty \quad \text{for some} \quad \lambda > 0.$$

Definition 2.1. The family of polynomials $\{Q_n^{(X)}(x,t), n \geq 0, t \geq 0\}$ satisfying

$$\sum_{n=0}^{\infty} \frac{u^n}{n!} Q_n^{(X)}(x,t) = \frac{e^{xu}}{\mathbb{E}(e^{uX_t})} \quad \text{for all} \quad x \in R, \tag{1}$$

where $Q_n^{(X)}(x,t)$ is of order n, is called the Appell polynomials associated with the Lévy process X.

In case the process X has moments of all orders, Appell polynomials associated with X can be defined as follows.

Definition 2.2. Let $X = (X_t)_{t\geq 0}$ be a Lévy process which has the moments up to order N, i.e.,

$$\mathbb{E}(|X_t|) < \infty, \quad n = 0, 1, \ldots, N.$$

The Appell polynomials $\{Q_n^{(X)}, n = 0, 1, 2, \ldots, N\}$ associated with X are defined via

$$Q_0^{(X)}(x,t) = 1 \quad \text{for all} \quad x, \tag{2}$$

$$Q_n^{(X)}(x,t) := \sum_{i=0}^{n} \binom{n}{i} Q_{n-i}^{(X)}(0,t) x^i, \quad n = 1, \ldots, N \tag{3}$$

where $Q_j^{(\xi)}(0), j = 1, 2, \ldots, n$ are generated by the recurrence formula

$$Q_j^{(X)}(0,t) = -\sum_{i=0}^{j-1} \binom{j}{i} Q_{j-i}^{(X)}(0,t) \mathbb{E}(X_t^j). \tag{4}$$

We now consider some properties of Appell-Lévy polynomials.

PROPOSITION 2.1. *It holds*

$$\begin{aligned} Q_n^{(X_{t+s})}(x, t+s) &= \sum_{k=0}^{n} \binom{n}{k} Q_k^{(X_s)}(0,s) Q_{n-k}^{(X_t)}(x,t) \\ &= \sum_{k=0}^{n} \binom{n}{k} Q_k^{(X_t)}(0,t) Q_{n-k}^{(X_s)}(x,s). \end{aligned}$$

Proof. Since Lévy processes have independent increments then

$$X_{t+s} = X_{t+s} - X_s + X_s \overset{(d)}{=} \hat{X}_t + X_s,$$

where $\hat{X}_t$ is identically distributed with X_t. So the claim follows from [27, Proposition 2.16]. □

Appell polynomials associated with some random variable ξ have the useful property (see [27]) which is called the mean value property, namely,

$$Q_n^{(\xi)}(x+\xi) = x^n, \quad n = 1,2,\ldots.$$

This property is utilized to derive many interesting new results and used to give simple proofs of some classical results. Next we will state a general mean value property for Appell-Lévy polynomials.

PROPOSITION 2.2 *(Mean value property). If X_t is independent of X_s, then*

$$\mathbb{E}(Q_n^{(X_s)}(X_t + X_s, s)) = \mathbb{E}(X_t^n), \quad n = 1,2\ldots. \tag{5}$$

In particular,

$$\mathbb{E}(Q_n^{(X_t)}(x + X_t, t)) = x^n.$$

Proof. We have

$$Q_n^{(X_s)}(X_t + X_s, s) = \sum_{k=0}^{n} \binom{n}{k} Q_k^{(X_s)}(X_s, s) X_t^{n-k}.$$

Hence,

$$\begin{aligned}\mathbb{E}(Q_n^{(X_s)}(X_t + X_s, s)) &= \sum_{k=0}^{n} \binom{n}{k} \mathbb{E}(Q_k^{(X_s)}(X_s, s) X_t^{n-k}) \\ &= \sum_{k=0}^{n} \binom{n}{k} \mathbb{E}(Q_k^{(X_s)}(X_s, s)) \mathbb{E}(X_t^{n-k}).\end{aligned}$$

Similarly, it can be proved that $\mathbb{E}(Q_n^{(X_s)}(X_s, s)) = 0$, for all $n \geq 1$ (see [27, p.274]). So we obtain (5). □

From (1), replacing x by X_t, the process $Y_t := \dfrac{e^{uX_t}}{\mathbb{E}(e^{uX_t})}$ is known as the Wald exponential martingale (see, e.g., [9]). Consequently, the polynomial process $Q_n^{(X)}(X_t, t)$ is a martingale (see, e.g., [20]). However, in case Lévy processes do not have exponential moments, a number of authors try to prove the martingale property of this polynomial process by using different tools, e.g., Solé and Utzet [25] overcome this difficulty by defining polynomials $Q_n^{(X_t)}(x,t)$ via Bell polynomials and exploit Itô formula to prove that the time-space process $Q_n^{(X_t)}(X_t, t)$ is a martingale. Nardo and Oliva [5] utilize tools in umbral calculus to build a new family of time-space polynomials. Here, we will show that this property can be proved easily by using the property of Appell polynomials associated with a sum of two independent variables.

PROPOSITION 2.3. *The time-space process $Q_n^{(X)}(X_t, t)$ is a martingale.*

Proof. For all $s \leq t$, X_{t-s} and X_s are independent and $X_t \overset{(d)}{=} X_{t-s} + X_s$. Again from [27, Proposition 2.16] it follows

$$Q_n^{(X_t)}(x+y, t) = \sum_{k=0}^{n} \binom{n}{k} Q_k^{(X_{t-s})}(x, t-s) Q_{n-k}^{(X_s)}(y, s).$$

Now replacing x, y by $X_t - X_s, X_s$, respectively, and using the mean value property we have

$$\begin{aligned}\mathbb{E}\Big(Q_n^{(X_t)}(X_t,t)|\mathscr{F}_s\Big) &= \mathbb{E}\Big(\sum_{k=0}^{n}\binom{n}{k}Q_k^{(X_{t-s})}(X_t - X_s, t-s)Q_{n-k}^{(X_s)}(X_s,s)|\mathscr{F}_s\Big)\\ &= \sum_{k=0}^{n}\binom{n}{k}Q_{n-k}^{(X_s)}(X_s,s)\mathbb{E}\Big(Q_k^{(X_{t-s})}(X_{t-s},t-s)|\mathscr{F}_s\Big)\\ &= \sum_{k=0}^{n}\binom{n}{k}Q_{n-k}^{(X_s)}(X_s,s)\mathbb{E}\Big(Q_k^{(X_{t-s})}(X_{t-s},t-s)\Big) = Q_n^{(X_s)}(X_s,s).\end{aligned}$$

□

The following result shows that in some case the polynomial $Q_n^{(X)}(x,t)$ has the moment representation.

PROPOSITION 2.4. *Assume that the Lévy process X is symmetric. Then*

$$Q_n^{(X)}(x,t) = \mathbb{E}(x + iX_t)^n, \tag{6}$$

where i is the imaginary unit. In particular,

$$Q_{2n}^{(X)}(0,t) = (-1)^n \mathbb{E}(X_t)^{2n}.$$

Proof. Since X is symmetric, the Laplace exponent ψ of X_1 is a non-negative symmetric function satisfying

$$\mathbb{E}(e^{iuX_t}) = \mathbb{E}(\cos(uX_t)) = e^{-t\psi(u)}.$$

Consequently,

$$\mathbb{E}(e^{iuX_t}) = \frac{1}{\mathbb{E}(e^{uX_t})}.$$

From (1) we obtain

$$\sum_{n=0}^{\infty}\frac{u^n}{n!}Q_n^{(X)}(x,t) = \frac{e^{xu}}{\mathbb{E}(e^{uX_t})} = \mathbb{E}(e^{u(x+iX_t)}) = \sum_{n=0}^{\infty}\frac{u^n}{n!}\mathbb{E}(x+iX_t)^n,$$

and, hence,

$$Q_n^{(X)}(x,t) = \mathbb{E}(x + iX_t)^n.$$

□

Example 2.1. Let X be Brownian motion. Hence, X is a symmetric Lévy process. It is clear that Appell polynomial $Q_n^{(X)}(x,t)$ coincides with Hermite polynomial $H_n(x,t) = (\frac{t}{2})^{n/2}He_n(x/\sqrt{2t})$, where He_n is the classical Hermite polynomial (see, e.g., [25, p 6]). We have the moment representation of $H_n(x,t)$

$$Q_n^{(X)}(x,t) \equiv H_n(x,t) = \mathbb{E}(x + iX_t)^n,$$

and

$$Q_{2n}^{(X)}(0,t) = (-1)^n \frac{(2n)!t^n}{2^n n!}.$$

3. APPELL POLYNOMIALS ASSOCIATED WITH RUNNING MAXIMUM AND MINIMUM

In this section we study some properties of Appell polynomials associated with running maximum and mininum of Lévy processes up to exponential time. These polynomials have many useful properties and can be used to characterized optimal stopping problems driven by Lévy processes.

Let T be an exponential distributed random variable with parameter $r > 0$, independent of X and $M_T := \sup_{s \le T} X_s$ and $I_T := \inf_{s \le T} X_s$ denote by the maximum and minimum of Lévy process X up to time T, respectively. We now study properties of Appell polynomials associated with M_T and I_T. Recall that by the Wiener-Hopf factorization we have

$$X_T \overset{(d)}{=} M_T + \hat{I}_T,$$

where $\hat{I}_T$ is an independent copy of I_T, i.e., $\hat{I}_T \overset{(d)}{=} I_T$ and $\hat{I}_T$ is independent of M_T. Denote by Π the Lévy measure, it is defined on $\mathbb{R} \setminus \{0\}$ and has properties $\int (1 \wedge x^2)\Pi(dx) < \infty$. The Laplace exponent of X_1 can be presented via Π as follows

$$\psi(u) = au + \frac{1}{2}b^2u^2 + \int_{\mathbb{R}} (e^{ux} - 1 - ux\mathbf{1}_{\{|x| \le 1\}})\Pi(\mathrm{d}x),$$

for some $a, b \in \mathbb{R}$. If $\Pi((0, +\infty)) = 0$, i.e., the process moves continuously to the right, we said that the Lévy process X is spectrally negative. If $\Pi((-\infty, 0)) = 0$, i.e., the process moves continuously to the left, we said that X is spectrally positive. Since M_T and $\hat{T}_T$ are independent random variables, so we have the interesting formula (see, [27, Proposition 2.16])

$$Q_n^{(X_T)}(x+y) = \sum_{k=0}^{n} \binom{n}{k} Q_k^{(M_T)}(x) Q_{n-k}^{(I_T)}(y).$$

PROPOSITION 3.1. *It holds*

$$Q_n^{(M_T)}(x) = \mathbb{E}_{\hat{I}_T} Q_n^{(X_T)}(x + \hat{I}_T) = \int_{-\infty}^{0} Q_n^{(X_T)}(x+y)\mathbb{P}(I_T \in \mathrm{d}y). \tag{7}$$

If X is spectrally positive, then polynomial $Q_n^{(M_T)}$ can be derived from the Laplace transform of $Q_n^{(X_T)}$ as follows

$$Q_n^{(M_T)}(x) = \hat{\rho}(r) \int_0^{\infty} Q_n^{(X_T)}(x-y) e^{-\hat{\rho}(r)y} \mathrm{d}y, \tag{8}$$

where $\hat{\rho}(r)$ is the unique positive root of the equation $\psi(-\theta) = r$.

Proof. It is seen that identity (7) can be obtained from the Wiener-Hopf factorization. In case Lévy process X is spectrally positive then I_T has exponential distribution with parameter $\hat{\rho}(r)$, and, hence, follows (8). □

In the next proposition we have the behaviour of the Appell polynomial $Q_n^{(M_T)}$. More precisely, polynomial $Q_n^{(M_T)}$ has a unique positive root (see also, [11, 14]).

PROPOSITION 3.2. *Fix $n \in \{1, 2, \dots\}$ and suppose that*

$$\int_{(1,\infty)} x^n \Pi(\mathrm{d}x) < \infty.$$

Then $Q_n^{(M_T)}$ has a unique strictly positive root x^* and is negative on $(0, x^*)$, positive, increasing on (x^*, ∞).

The following result provides a useful property of Appell polynomials associated with running maximum M_T which is applied to study an optimal stopping problem for Lévy processes with the power reward $(x^+)^n, n = 1,2\dots$.

PROPOSITION 3.3. *It holds that the function*

$$f(x) := \mathbb{E}\Big(Q_n^{(M_T)}(M_T + x)1_{\{M_T + x \geq x^*\}}\Big)$$

is r-excessive, i.e., for all $x \in \mathbb{R}, t \geq 0$

$$e^{-rt}\mathbb{E}(f(X_t + x)) \leq f(x),$$

where x^* *is a positive root of* $Q_n^{(M_T)}(x)$. *Hence, the process* $(e^{-rt}f(X_t))_{t\geq 0}$ *is a non-negative supermartingale and* f *is a majorant of* x^n.

Proof. From Proposition 3.2, polynomial $Q_n^{(M_T)}(x)$ is non-negative and increasing on $[x^*, \infty)$, and, hence,

$$f(x) = \mathbb{E}\Big(Q_n^{(M_T)}(\sup_{0\leq t\leq T}(X_t + x))1_{\{X_t + x \geq x^*\}}\Big) = \mathbb{E}\Big(\sup_{0\leq t\leq T} Q_n^{(M_T)}(X_t + x)1_{\{X_t + x \geq x^*\}}\Big).$$

From Lemma 2.2 in [3], it follows that the function f is r-excessive. Therefore, $(e^{-rt}f(X_t))_{t\geq 0}$ is a non-negative supermartingale. Furthermore we have f is majorant of x^n, i.e.,

$$f(x) = \mathbb{E}(Q_n^{(M_T)}(M_T + x)) - \mathbb{E}\Big(Q_n^{(M_T)}(M_T + x)1_{\{M_T + x \leq x^*\}}) \geq x^n.$$

□

In the next result we prove that in case the process X is spectrally positive we obtain an expectation identity for Appell polynomials associated with X_T and Appell polynomials associated with maximum M_T. This result will be exploited to investigate optimal stopping problems in the next section.

THEOREM 3.4. *If the Lévy process* X *is spectrally positive, then for all* x

$$\mathbb{E}\big(Q_n^{(M_T)}(M_T + x)1_{\{M_T + x \geq x^*\}}\big) = \mathbb{E}\big(Q_n^{(X_T)}(X_T + x)1_{\{X_T + x \geq x^*\}}\big), \tag{9}$$

where x^* *is some point in* $\mathbb{R}$.

Proof. Note that Lévy processes are spatially homogeneous, and, hence, identity (9) can be written as

$$\mathbb{E}_x(Q_n^{(M_T)}(M_T)1_{\{M_T \geq x^*\}}) = \mathbb{E}_x(Q_n^{(X_T)}(X_T)1_{\{X_T \geq x^*\}}),$$

where the notation $\mathbb{E}_x$ stands for the expectation associated with X when initiated from x. Since X is spectrally positive, $-I_T$ is exponentially distributed with mean $\hat{\rho}(r)$, where $\hat{\rho}(r)$ is the unique root of equation $\psi(-u) = r$. So we have Appell polynomials associated with I_T

$$Q_n^{(I_T)}(x) = (x + \frac{n}{\hat{\rho}(r)})x^{n-1}.$$

Consequently $Q_n^{(I_T)}(0) = 1/\hat{\rho}(r)$ and $Q_n^{(I_T)}(0) = 0$ for all $n \geq 2$. So we get

$$Q_n^{(X_T)}(x) = \sum_{k=0}^{n}\binom{n}{k}Q_k^{(M_T)}(x)Q_{n-k}^{(I_T)}(0) = Q_n^{(M_T)}(x) + \frac{1}{\hat{\rho}(r)}Q_{n-1}^{(M_T)}(x),$$

and, hence,

$$\mathbb{E}_x(Q_n^{(X_T)}(X_T)1_{\{X_T\geq x^*\}}) = \int_{x^*}^{\infty} Q_n^{(X_T)}(y)\mathbb{P}_x(X_T\in \mathrm{d}y) = \int_{x^*}^{\infty} Q_n^{(M_T)}(y)\mathbb{P}_x(X_T\in \mathrm{d}y) + \frac{1}{\hat{\rho}(r)}\int_{x^*}^{\infty} Q_{n-1}^{(M_T)}(y)\mathbb{P}_x(X_T\in \mathrm{d}y). \tag{10}$$

Consider in the first integration in (10) we have

$$\begin{aligned}
\int_{x^*}^{\infty} Q_n^{(M_T)}(y)\mathbb{P}_x(X_T\in \mathrm{d}y) &= \int_{x^*}^{\infty} Q_n^{(M_T)}(y)\int_{-\infty}^{0}\mathbb{P}_x(M_T+z\in dy)\hat{\rho}(r)e^{\hat{\rho}(r)z}\mathrm{d}z \\
&= \int_{-\infty}^{0}\hat{\rho}(r)e^{\hat{\rho}(r)z}\mathrm{d}z\int_{x^*}^{\infty} Q_n^{(M_T)}(y)\mathbb{P}_x(M_T+z\in \mathrm{d}y) \\
&= \int_{-\infty}^{0}\hat{\rho}(r)e^{\hat{\rho}(r)z}\mathbb{E}_x(Q_n^{(M_T)}(M_T+z)1_{\{M_T\geq x^*\}})\mathrm{d}z \\
&= \int_{-\infty}^{0}\hat{\rho}(r)e^{\hat{\rho}(r)z}\sum_{k=0}^{n}\binom{n}{k}\mathbb{E}_x(Q_k^{(M_T)}(M_T)1_{\{M_T\geq x^*\}})z^{n-k}\mathrm{d}z \\
&= \sum_{k=0}^{n}\binom{n}{k}\mathbb{E}_x(Q_k^{(M_T)}(M_T)1_{\{M_T\geq x^*\}})\int_{-\infty}^{0}\hat{\rho}(r)e^{\hat{\rho}(r)z}z^{n-k}\mathrm{d}z.
\end{aligned}$$

Using integration by parts, we have

$$\int_{-\infty}^{0}\hat{\rho}(r)e^{\hat{\rho}(r)z}z^{n-k}\mathrm{d}z = \begin{cases} 1, & \text{if } n=k; \\ -(n-k)\int_{-\infty}^{0} e^{\hat{\rho}(r)z}z^{n-k-1}\mathrm{d}z, & \text{otherwise.}\end{cases}$$

So we obtain

$$\int_{x^*}^{\infty} Q_n^{(M_T)}(y)\mathbb{P}_x(X_T\in \mathrm{d}y) = \mathbb{E}_x(Q_n^{(M_T)}(M_T)1_{\{M_T\geq x^*\}}) - \sum_{k=0}^{n-1}\binom{n-1}{k}\mathbb{E}_x(Q_k^{(M_T)}(M_T)1_{\{M_T\geq x^*\}})\int_{-\infty}^{0} e^{\hat{\rho}(r)z}z^{n-k-1}\mathrm{d}z. \tag{11}$$

Similarly the second integration gives

$$\frac{1}{\hat{\rho}(r)}\int_{x^*}^{\infty} Q_{n-1}^{(M_T)}(y)\mathbb{P}_x(X_T\in \mathrm{d}y) = \sum_{k=0}^{n-1}\binom{n-1}{k}\mathbb{E}_x(Q_k^{(M_T)}(M_T)1_{\{M_T\geq x^*\}})\int_{-\infty}^{0} e^{\hat{\rho}(r)z}z^{n-k-1}\mathrm{d}z. \tag{12}$$

From (10), (11) and (12) the claim (9) is proved. □

The following result is an immediate consequence of Theorem 3.4 and the mean value property.

COROLLARY 3.5. *It holds*

$$\mathbb{E}(Q_n^{(X_T)}(X_T+x)1_{\{X_T+x<x^*\}}) = \mathbb{E}(Q_n^{(M_T)}(M_T+x)1_{\{M_T+x<x^*\}}).$$

Proof. For all x we have

$$x^n = \mathbb{E}(Q_n^{(X_T)}(X_T+x)) = \mathbb{E}(Q_n^{(X_T)}(X_T+x)1_{\{X_T+x\geq x^*\}}) + \mathbb{E}(Q_n^{(X_T)}(X_T+x)1_{\{X_T+x<x^*\}}), \tag{13}$$

and

$$x^n = \mathbb{E}(Q_n^{(M_T)}(M_T+x)) = \mathbb{E}(Q_n^{(M_T)}(M_T+x)1_{\{M_T+x\geq x^*\}}) + \mathbb{E}(Q_n^{(M_T)}(M_T+x)1_{\{M_T+x<x^*\}}). \tag{14}$$

From (9), (13) and (14) the proof is complete. □

4. APPLICATIONS TO OPTIMAL STOPPING PROBLEMS

In this section, we present and discuss some applications of Appell polynomials to optimal stopping problems. Consider a real-valued Lévy process $X = \{X_t, t \geq 0\}$, initiating from x. Denoted $\mathscr{F}$ the natural filtration generated by X and by $\mathscr{M}$ the set of all stopping times with respect to $\mathscr{F}$. We consider the problem of finding a function V and a stopping time τ^* such that

$$V(x) = \sup_{\tau} \mathbb{E}_x(e^{-r\tau}G(X_\tau)1_{\{\tau<\infty\}}) = \mathbb{E}_x(e^{-r\tau^*}G(X_{\tau^*})), \tag{*}$$

where supremum taking all stopping times τ, G is a non-negative, measurable function and discount factor $r > 0$. The problem (*) is the so-called optimal stopping problem for Lévy process X. Stopping time τ^* and the function V are called optimal stopping time and value function, respectively. A very important result in optimal stopping theory is that if the reward function G is lower semicontinuous then the value function V is characterized as the smallest r-excessive majorant of G (see, e.g., [6, 15]). In the following, we are only interested in studying one-sided optimal stopping problem, i.e., optimal stopping time τ^* has the form $\tau^* = \inf\{t \geq 0 : X_t \geq x^*\}$ for some stopping point x^*. So we assume that the reward function G satisfies the condition

$$\lim_{x\to-\infty} G(x) = 0.$$

Recall that $M_t := \sup_{0\leq s\leq t} X_s$ is the running maximum process and T is an exponentially distributed random variable with parameter r, independent of X. It is proved in [3, 11, 14] that if the reward (gain) function $G(x) = (x^+)^n, n = 1, 2, \ldots$ then the value function V of the problem (*) is expressed by Appell polynomials $Q_n^{(M_T)}$ associated with M_T as follows

$$V(x) = \mathbb{E}_x(Q_n^{(M_T)}(M_T)1_{\{M_T\geq x^*\}}), \tag{15}$$

and optimal stopping $\tau^* = \inf\{t : X_t \geq x^*\}$, where x^* is a unique positive root of the polynomial $Q_n^{(M_T)}$.

In [13], a new approach to optimal stopping problems (*) is presented. The approach is based on the Riesz decomposition of r-excessive functions. More precisely, there exists a Radon measure σ with support on the stopping region $[x^*, \infty)$ such that V has the representation

$$V(x) = \int_{[x^*,\infty)} G_r(x,y)\sigma(\mathrm{d}y),$$

where $G_r(x,y)$ is the Green kernel of X. The measure σ is called representing measure. It is also proved that (see [13, Proposition 4.3]) there exists a function H on stopping region $[x^*, \infty)$ such that V can be expressed in terms of running maximum M_T and the function H as follows

$$V(x) = \mathbb{E}_x(H(M_T)1_{\{M_T\geq x^*\}}), \quad x \leq x^*.$$

Moreover, the function H can be represented via the representing measure σ and the density of running minimum I_T

$$H(x) = \frac{1}{r}\int_{x^*}^{x} f_I(y-x)\sigma(\mathrm{d}y), \quad x \geq x^*.$$

Applying this approach to optimal stopping problem with the power reward $G(x) = (x^+)^n, n = 1,2,\ldots,$ it is shown in [18] that Appell polynomial $Q_n^{(M_T)}$ coincides with the function H and there is a measure σ_n such that $Q_n^{(M_T)}$ has the explicit representation

$$Q_n^{(M_T)}(x) = \frac{1}{r}\int_{x^*}^{x} f_I(y-x)\sigma_n(\mathrm{d}y), \quad x \geq x^*. \tag{16}$$

Furthermore, the Laplace transform of σ_n can be represented in terms of the series of $Q_n^{(I_T)}(0)$ and $Q_k^{(M_T)}(x^*)$, $k = 0,1,2,\ldots,n$. We refer to [18, pp. 9-10] for more details. Therefore, we are able to obtain the following nice result for the special case when X is spectrally positive Lévy process (see [18, p.10-11]). However, we will show that this result can be proved directly and simply.

PROPOSITION 4.1. *If process X is spectrally positive, then*

(i) $\sigma_n(\mathrm{d}x) = rQ_n^{(X_T)}(x)\mathrm{d}x, \quad x \geq x^*.$

(ii) $V(x) = \mathbb{E}_x(Q_n^{(X_T)}(X_T)1_{\{X_T \geq x^*\}}).$

Proof. From (8) we have

$$Q_n^{(M_T)}(x) = \hat{\rho}(r)e^{-\hat{\rho}(r)x}\int_{-\infty}^{x} e^{\hat{\rho}(r)y}Q_n^{(X_T)}(y)\mathrm{d}y,$$

and, hence, for $x \geq x^*$

$$Q_n^{(M_T)}(x) = e^{-\hat{\rho}(r)x}\int_{-\infty}^{x^*}\hat{\rho}(r)e^{\hat{\rho}(r)y}Q_n^{(X_T)}(y)\mathrm{d}y + \hat{\rho}(r)e^{-\hat{\rho}(r)x}\int_{x^*}^{x} e^{\hat{\rho}(r)y}Q_n^{(X_T)}(y)\mathrm{d}y. \tag{17}$$

Consider the first integration in (17), changing variable $z = y - x^*$ we have

$$\begin{aligned}
\int_{-\infty}^{x^*}\hat{\rho}(r)e^{\hat{\rho}(r)y}Q_n^{(X_T)}(y)\mathrm{d}y &= e^{\hat{\rho}(r)x^*}\int_{-\infty}^{0}\hat{\rho}(r)e^{\hat{\rho}(r)z}Q_n^{(X_T)}(z+x^*)\mathrm{d}z \\
&= e^{\hat{\rho}(r)x^*}\mathbb{E}(Q_n^{(X_T)}(x^* + \hat{I}_I)) \\
&= e^{\hat{\rho}(r)x^*}Q_n^{(M_T)}(x^*) = 0,
\end{aligned}$$

where $\hat{I}_T$ is an independent copy of I_T. So we obtain

$$Q_n^{(M_T)}(x) = \hat{\rho}(r)e^{-\hat{\rho}(r)x}\int_{x^*}^{x} e^{\hat{\rho}(r)y}Q_n^{(X_T)}(y)\mathrm{d}y.$$

From (16) we get

$$Q_n^{(M_T)}(x) = \frac{1}{r}\hat{\rho}(r)e^{-\hat{\rho}(r)x}\int_{x^*}^{x} e^{\hat{\rho}(r)y}\sigma_n(\mathrm{d}y).$$

Hence, for all x

$$\int_{x^*}^{x} e^{\hat{\rho}(r)y}rQ_n^{(X_T)}(y)\mathrm{d}y = \int_{x^*}^{x} e^{\hat{\rho}(r)y}\sigma_n(\mathrm{d}y).$$

Consequently,

$$\sigma_n(\mathrm{d}x) = rQ_n^{(X_T)}(x)\mathrm{d}x, \quad x \geq x^*.$$

The proof of (ii) is directly implied from Theorem 3.4. □

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Received August 14, 2019