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HEUN SOLUTIONS FOR DIRAC FERMIONS IN BLACK STRINGS BACKGROUNDS

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Abstract. We study the solutions of the Dirac equation in the background of a static black string in four dimensions. We point out the presence of an additional term which was omitted in previous studies in the literature. We show how the Dirac equation can be separated for massive fermions. Finally, in the massless case, we obtain an exact solution of the Dirac equation in terms of the general Heun functions.

Key words: Dirac equation, Heun functions, cylindrical black holes.

1. INTRODUCTION

The cylindrically symmetric black hole solution of Einstein's field equations with a negative cosmological constant [1] has a rich geometrical structure, which can be explored both at the quantum and the classical level. Such black strings solutions have also been studied in the rotating and charged cases in [2]. These solutions can only exist in presence of a negative cosmological constant, such that their asymptotic structure in radial direction is that of the anti- de Sitter geometry.

The main purpose of the present paper is to extend the analysis developed in [3], devoted to the study of Dirac massless particles in black strings backgrounds, in the following three directions. Firstly, we are using a coordinate-free method, based on Cartan's formalism, in which the $\{\gamma^i\}$ matrices have the same form as on the Minkowskian background. We point out the presence of an additional term that is expressing the spin-connection and is missing in the analysis developed in [3,4,5]. Secondly, the Dirac equation is separated for massive particles into its angular and radial parts. Finally, in the case of massless fermions, it turns out that the radial equations can be exactly solved, their solutions being expressed in terms of the Heun general functions [6,7].

Such studies are currently a very active area of research due to their observational significance since the so-called quasinormal modes can be used for the detection of black holes [8,9,10]. One should note at this point that Heun functions [11] are often encountered when studying fields propagations on black holes backgrounds (with or without cosmological constant) (see for instance [12-17] and references therein).

Our paper is organized as follows: in the next section, we write down the Dirac equation using the Cartan formalism. We show that, in the background of a non-rotating black string, the Dirac equation is separable. In section 3, we prove that, for massless Dirac fermions, the solution of the Dirac equation can be further solved exactly in terms of the general Heun functions. The final section is dedicated to conclusions and avenues for further work.

2. THE $\mathrm{SO}(3,1)$ – GAUGE COVARIANT DIRAC EQUATION

In [14], for the general static metric of the form

$$\mathrm{d}s^2 = e^{2f}\mathrm{d}r^2 + a^2 e^{2p}\mathrm{d}\theta^2 + b^2 e^{2q}\mathrm{d}\varphi^2 - e^{2h}\mathrm{d}t^2 \,, \tag{1}$$

where the functions f, p, q and h depend only on the coordinates r and θ, we have defined the pseudo-orthonormal frame $\{E_a\}_{(a=\overline{1,4})}$, i.e.

$$E_1 = e^{-f}\partial_r,\ E_2 = \frac{e^{-p}}{a}\partial_\theta,\ E_3 = \frac{e^{-q}}{b}\partial_\varphi,\ E_4 = e^{-h}\partial_t, \tag{2}$$

whose corresponding dual base is

$$\omega^1 = e^f \mathrm{d}r,\ \omega^2 = a\,e^p \mathrm{d}\theta,\ \omega^3 = b\,e^q \mathrm{d}\varphi,\ \omega^4 = e^h \mathrm{d}t,$$

so that the metric (1) turns into the Minkowskian form $\mathrm{d}s^2 = \eta_{ab}\,\omega^a\omega^b$, where $\eta_{ab} = \mathrm{diag}[1,1,1,-1]$.

Using the first Cartan's equation,

$$\mathrm{d}\omega^a = \Gamma^{a}_{.[bc]}\,\omega^b \wedge \omega^c, \tag{3}$$

with $1 \le b < c \le 4$ and $\Gamma^{a}_{.[bc]} = \Gamma^{a}_{.bc} - \Gamma^{a}_{.cb}$, we obtain the following connection one-forms $\Gamma_{ab} = \Gamma_{abc}\omega^c$, where $\Gamma_{abc} = -\Gamma_{bac}$, namely

$$\begin{aligned}
\Gamma_{12} &= \frac{e^{-p}}{a}\frac{\partial f}{\partial \theta}\omega^1 - e^{-f}\frac{\partial p}{\partial r}\omega^2,\\
\Gamma_{13} &= -\frac{e^{-p}}{a}\frac{\partial q}{\partial \theta}\omega^3,\\
\Gamma_{14} &= e^{-f}\frac{\partial h}{\partial r}\omega^4,\\
\Gamma_{24} &= \frac{e^{-p}}{a}\frac{\partial h}{\partial \theta}\omega^4.
\end{aligned} \tag{4}$$

In the case under consideration in this paper, we start from the following metric:

$$\mathrm{d}s^2 = \frac{1}{F}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + \alpha^2 r^2 \mathrm{d}z^2 - F\,\mathrm{d}t^2 \tag{5}$$

where $F = F(r)$, so that the relations (2) and (4) lead to the following simple expressions:

$$E_1 = \sqrt{F}\partial_r,\ E_2 = \frac{1}{r}\partial_\theta,\ E_3 = \frac{1}{\alpha r}\partial_z,\ E_4 = \frac{1}{\sqrt{F}}\partial_t, \tag{6}$$

and

$$\Gamma_{122} = \Gamma_{133} = -\frac{\sqrt{F}}{r},\quad \Gamma_{144} = \frac{F'}{2\sqrt{F}}, \tag{7}$$

where F' means the derivative with respect to r.

The massive spinor of mass μ, minimally coupled to gravity, is described by the SO(3,1) gauge-covariant Dirac equation

$$\gamma^a E_a \Psi + \frac{1}{4}\Gamma_{bca}\gamma^a\gamma^b\gamma^c\,\Psi + \mu\Psi = 0, \tag{8}$$

which, in view of the relations (6) and (7), gets the explicit form

$$\sqrt{F}\gamma^1\left[\Psi' + \left(\frac{1}{r} + \frac{F'}{4F}\right)\Psi\right] + \frac{1}{r}\gamma^2\frac{\partial\Psi}{\partial\theta} + \frac{1}{\alpha r}\gamma^3\frac{\partial\Psi}{\partial z} + \frac{1}{\sqrt{F}}\gamma^4\frac{\partial\Psi}{\partial t} + \mu\Psi = 0. \tag{9}$$

In order to simplify the above equation, we take the wave function as

$$\Psi = F^{-1/4}\frac{\phi}{r}, \tag{10}$$

and, since the cylindrically symmetric spacetime admits ∂_z and ∂_t as Killing vectors, we can assume that

$$\phi = G(r,\theta)e^{i(k\alpha z-\omega t)}, \tag{11}$$

so that one gets the following equation

$$\sqrt{F}\gamma^1\frac{\partial G}{\partial r}+\frac{1}{r}\gamma^2\frac{\partial G}{\partial\theta}+\left[\frac{ik}{r}\gamma^3-\frac{i\omega}{\sqrt{F}}\gamma^4+\mu\right]G=0. \tag{12}$$

For the bi-spinor G written in terms of two components spinors as

$$G(r,\theta)=\begin{bmatrix}\zeta(r,\theta)\\ \chi(r,\theta)\end{bmatrix}, \tag{13}$$

in the Weyl's representation for the γ^i matrices,

$$\gamma^1=-i\beta\alpha^3,\ \gamma^2=-i\beta\alpha^1,\ \gamma^3=-i\beta\alpha^2,\ \gamma^4=-i\beta, \tag{14}$$

with

$$\alpha^\mu=\begin{pmatrix}\sigma^\mu & 0\\ 0 & -\sigma^\mu\end{pmatrix},\ \beta=\begin{pmatrix}0 & -I\\ -I & 0\end{pmatrix},\ \gamma^5=\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix},$$

where σ^μ denotes the usual Pauli matrices, the equation (12) leads to the following system of coupled equations for the spinors ζ and χ:

$$\begin{aligned}&r\sqrt{F}\sigma^3\frac{\partial\zeta}{\partial r}+\sigma^1\frac{\partial\zeta}{\partial\theta}+ik\sigma^2\zeta-\frac{i\omega r}{\sqrt{F}}\zeta-i\mu r\eta=0,\\ &r\sqrt{F}\sigma^3\frac{\partial\eta}{\partial r}+\sigma^1\frac{\partial\eta}{\partial\theta}+ik\sigma^2\eta-\frac{i\omega r}{\sqrt{F}}\eta+i\mu r\zeta=0.\end{aligned} \tag{15}$$

Next, by applying the separation ansatz

$$\zeta_1=R_1(r)T_1(\theta),\ \zeta_2=R_2(r)T_2(\theta),\ \eta_1=R_2(r)T_1(\theta),\ \eta_2=R_1(r)T_2(\theta), \tag{16}$$

one gets the system

$$\begin{aligned}&\left[r\sqrt{F}\partial_r-\frac{i\omega r}{\sqrt{F}}\right]R_1T_1+\left[\partial_\theta+k\right]R_2T_2-i\mu rR_2T_1=0\\ &\left[r\sqrt{F}\partial_r+\frac{i\omega r}{\sqrt{F}}\right]R_2T_2-\left[\partial_\theta-k\right]R_1T_1+i\mu rR_1T_2=0\\ &\left[r\sqrt{F}\partial_r+\frac{i\omega r}{\sqrt{F}}\right]R_2T_1+\left[\partial_\theta+k\right]R_1T_2+i\mu rR_1T_1=0\\ &\left[r\sqrt{F}\partial_r-\frac{i\omega r}{\sqrt{F}}\right]R_1T_2-\left[\partial_\theta-k\right]R_2T_1-i\mu rR_2T_2=0,\end{aligned} \tag{17}$$

which can be separated in the radial and angular parts as:

$$\begin{aligned}&\left[r\sqrt{F}\partial_r-\frac{i\omega r}{\sqrt{F}}\right]R_1=(\lambda+i\mu r)R_2,\ \left[r\sqrt{F}\partial_r+\frac{i\omega r}{\sqrt{F}}\right]R_2=(\lambda-i\mu r)R_1,\\ &\left[\partial_\theta-k\right]T_1=\lambda T_2,\ \left[\partial_\theta+k\right]T_2=-\lambda T_1,\end{aligned} \tag{18}$$

where λ is a separation constant.

The solutions for the first-order angular equations being given by

$$T_1 = Ae^{im\theta}, \; T_2 = -\frac{k-im}{\lambda} Ae^{im\theta}, \tag{19}$$

where $\lambda^2 = m^2 + k^2$, the radial equations in (18) lead to the following second-order differential equation for R_1,

$$\frac{\mathrm{d}^2 R_1}{\mathrm{d}r^2} + \left[\frac{1}{r} + \frac{F'}{2F} - \frac{i\mu}{\lambda + i\mu r}\right]\frac{\mathrm{d}R_1}{\mathrm{d}r} + \frac{1}{F}\left[\frac{\omega^2}{F} - i\omega\left(\frac{1}{r} - \frac{F'}{2F}\right) - \frac{\mu\omega}{\lambda + i\mu r} - \frac{\lambda^2}{r^2} - \mu^2\right]R_1 = 0 \tag{20}$$

and, similarly, for R_2. The general equation (20) cannot be analytically solved in general. However, in the massless case we show in the next section that one can find an exact solution.

3. THE MASSLESS CASE

For the massless case, the equation (20) gets the simpler form:

$$\frac{\mathrm{d}^2 R_1}{\mathrm{d}r^2} + \left[\frac{1}{r} + \frac{F'}{2F}\right]\frac{\mathrm{d}R_1}{\mathrm{d}r} + \frac{1}{F}\left[\frac{\omega^2}{F} - i\omega\left(\frac{1}{r} - \frac{F'}{2F}\right) - \frac{m^2 + k^2}{r^2}\right]R_1 = 0, \tag{21}$$

which depends on the explicit form of the function $F(r)$.

Let us consider the solution of Einstein's field equations with a negative cosmological constant Λ, for which $\alpha = -\Lambda/3$, which describes the cylindrically symmetric black string spacetime, with the mass M the only parameter. In this case, $F(r)$ is given by the expression

$$F(r) = \alpha^2 r^2 - \frac{M}{r}, \tag{22}$$

pointing out the event horizon location at

$$r_* = \left(\frac{M}{\alpha^2}\right)^{1/3}.$$

Using the dimensionless variable $x = r/r_*$, the function (22) becomes

$$F(x) = \frac{M}{r_* x}\left(x^3 - 1\right) = \frac{M}{r_* x}(x-1)(x-a)(x-b) \tag{23}$$

with

$$a = \frac{-1 + i\sqrt{3}}{2}, \; b = \frac{-1 - i\sqrt{3}}{2},$$

and the explicit form of the equation (21) reads

$$\frac{\mathrm{d}^2 R_1}{\mathrm{d}x^2} + \frac{4x^3 - 1}{2x(x^3 - 1)}\frac{\mathrm{d}R_1}{\mathrm{d}x} + \frac{r_*}{M(x^3 - 1)}\left[\frac{\omega r_*}{x^3 - 1}\left(\frac{\omega r_*^2 x^2}{M} + \frac{3i}{2}\right) - \frac{m^2 + k^2}{x}\right]R_1 = 0. \tag{24}$$

Using Maple [11] one can check that the equation (24) can be analytically solved, its solutions being expressed in terms of Heun general functions [6,7] as:

$$\begin{gathered} R_1(x) = (x-1)^{\frac{i\omega r_*^2}{3M}} (x-a)^{-(i-\sqrt{3})\frac{\omega r_*^2}{6M}} (x-b)^{-\left(i+\sqrt{3}\right)\frac{\omega r_*^2}{6M}} \times \\ \times \left\{ C_1 \,\mathrm{Heun}G\left[a_1, q_1, \alpha_1, \beta_1, \gamma_1, \delta_1, \zeta\right] + C_2 \sqrt{\zeta}\, \mathrm{Heun}G\left[a_2, q_2, \alpha_2, \beta_2, \gamma_2, \delta_2, \zeta\right] \right\} \end{gathered} \tag{25}$$

of variable

$$\zeta = \frac{x(1-a)}{x-a},$$

and parameters:

$$\begin{aligned}
&a_1 = \frac{1-i\sqrt{3}}{2}, \quad q_1 = \frac{i\, r_* \left(m^2+k^2\right)}{\sqrt{3}M}, \\
&\alpha_1 = 0, \quad \beta_1 = \frac{1}{2} - \frac{(\sqrt{3}-i)}{3}\frac{\omega r_*^2}{M}, \quad \gamma_1 = \frac{1}{2}, \quad \delta_1 = \frac{1}{2} + \frac{2i}{3}\frac{\omega r_*^2}{M}.
\end{aligned} \tag{26}$$

and

$$\begin{aligned}
&a_2 = a_1, \quad q_2 = q_1 + \frac{\sqrt{3}}{8}(\sqrt{3}-i), \\
&\alpha_2 = \alpha_1 + \frac{1}{2}, \quad \beta_2 = \beta_1 + \frac{1}{2}, \quad \gamma_2 = \frac{3}{2}, \quad \delta_2 = \delta_1.
\end{aligned} \tag{27}$$

Due to the large number of applications in a variety of fields, the Heun differential equations in either the general or confluent forms have been intensively studied in the last two decades [12, 13]. However, it is difficult to find closed-form solutions of these equations in terms of simpler functions, especially for complex values of the parameters.

4. CONCLUSIONS

In the present paper we have used a coordinate-free approach, based on the Cartan formalism to write down the Dirac equation in the background of a static black string in four dimensions. We point out the presence of an additional term expressing the spin-connection, namely the second term in (8), which was missing in previous studies of the Dirac equation in this background. We showed that the massive Dirac equation can be separated into its angular and radial parts. In general, the radial equation cannot be solved analytically and it should be approached numerically. In the massless case, we have been able to find an exact solution, which can be expressed using the general Heun functions.

As avenues for further work, using the same method, one should be able to obtain and study new solutions of the Dirac equation in the background of a rotating black string, with or without electric charge [2]. Work on these issues is in progress and it will be published elsewhere.

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REFERENCES

1. J. P. S. LEMOS, *Cylindrical black hole in general relativity*, Phys. Lett. B, **353**, pp. 46-51, 1995.
2. J. P. S. LEMOS, V. T. ZANCHIN, *Rotating charged black strings in general relativity*, Phys.Rev. D, **54**, pp. 3840-3853, 1996.
3. J. AHMED, K. SAIFULLAH, *Hawking temperature of rotating charged black strings from tunneling*, Journal of Cosmology and Astroparticle Physics, **11**, p. 023, 2011.
4. W. JAVED, R. BABAR, *Fermions tunneling and quantum corrections for quintessential Kerr-Newman-AdS black hole*, Adv. High Energy Phys., article ID 2759641, 2019.
5. M. SHARIF, W. JAVED, *Fermions tunneling from charged accelerating and rotating black holes with NUT parameter*, Eur. Phys. J. C, **72**, p. 1997, 2012.
6. A. RONVEAUX (ed.), *Heun's differential equations*, Oxford Univ. Press, New York, 1995.
7. S. Y. SLAVYANOV, W. LAY, *Special functions, a unified theory based on singularities*, Oxford Mathematical Monographs, Oxford Univ. Press, New York, 2000.

8. H.S. VIEIRA, V.B. BEZERRA, *Confluent Heun functions and the physics of black holes: resonant frequencies, Hawking radiation and scattering of scalar waves*, Annals Phys., **373**, pp. 28-42, 2016.
9. J.P. MORAIS GRACA, H.S. VIEIRA. V. B. BEZERRA, *Scalar and spinor QNMs of a black hole with a global monopole in f (R) gravity*, Gen. Rel. Grav., **48**, *4*, p. 38, 2016.
10. R. SINI, V.C. KURIAKOSE, *Quasinormal modes of RN black hole space time with cosmic string in a Dirac field*, Mod. Phys. Lett. A, **24**, pp. 2025-2037, 2009.
11. Maplesoft, a division of Waterloo Maple Inc., *Maple 2018*, Waterloo, Ontario.
12. T. BIRKANDAN, M. HORTACSU, *Heun-type solutions for Schwarzschild metric with electromagnetic fields*, EPL (Europhysics Letters), **119**, p. 20002, 2017.
13. M. HORTACSU, *Heun functions and some of their applications in physics*, Adv. High Energy Phys., article ID 8621573, 2018.
14. C. DARIESCU, M.A. DARIESCU, C. STELEA, *The SO(3,1)×U(1)-gauge invariant approach to charged bosons in relativistic magnetars*, Gen. Rel. Grav., **49**, *12*, p. 153, 2017.
15. A.M. ISHKHANYAN, T.A. SHAHVERDYAN, T.A. ISHKHANYAN, *Thirty five classes of solutions of the quantum time-dependent two-state problem in terms of the general Heun functions*, Eur. Phys. J. D, **69**, p. 10, 2015.
16. A.M. ISHKHANYAN, T.A. SHAHVERDYAN, T.A. ISHKHANYAN, *Schrodinger potentials solvable in terms of the general Heun functions*, Ann. Phys., **388**, pp. 456-471, 2018.
17. P.P. FIZIEV, *Classes of Exact Solutions to the Teukolsky Master Equation*, Class. Quant. Grav., **27**, p. 135001, 2010.

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