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NUMERICAL SOLUTION OF TWO DIMENSIONAL REACTION-DIFFUSION EQUATION USING OPERATIONAL MATRIX METHOD BASED ON GENOCCHI POLYNOMIAL – PART I: GENOCCHI POLYNOMIAL AND OPPERATORIAL MATRIX

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Abstract. In this article, the aim is to find the solution of fractional order non-linear reaction-diffusion equation using collocation method through deriving the operational matrix of fractional derivative. For this purpose the required definitions of fractional order derivatives, Genocchi polynomial and properties of Kronecker product of matrices used for the approximation of arbitrary functions are discussed.

Key words: fractional PDE, diffusion equation, operational matrix, Genocchi polynomial, collocation method.

1. INTRODUCTION

Fractional calculus is an ancient topic of mathematics with history like as ordinary or integer calculus. It is developing progressively now. Theory of fractional calculus has been developed by N.H. Abel and J. Liouville. The details can be found in [1,2]. In last few years fractional calculus has attracted attentions of the researchers of medical physics, chemistry, biology, engineering and mathematics. Fractional calculus and fractional differential equation are found in many applications in different fields. Due to increasing applications, the researchers have paid their attention to find numerical and exact solutions of the fractional differential equations (FDEs). As there are many difficulties to solve a FDE by analytic method so there is need of seeking numerical solutions. There are many numerical methods available in literature viz., eigen-vector expansion, Adomain decomposition method [3], fractional differential transform method [4,5,6], homotopy perturbation method [7,8,9], predictor-corrector method [10] and generalized block pulse operational matrix method [11] etc. Some numerical methods based upon operational matrices of fractional order differentiation and integration with Legendre wavelets [12], Chebyshev wavelets [13,14,15], sine wavelets, Haar wavelets [16] have been developed to find the solutions of FDE and fractional order integro-differential equations. The functions which are commonly used include Legendre polynomial [17,18], Laguerre polynomial [19], Chebyshev polynomial and semi-orthogonal polynomial as Genocchi polynomial [20].

Many complicated natural phenomena, such as the spreading of bush fires and epidemics, and the nonlinear evolution of a population in a two-dimensional habitat (in which the balance of reaction and diffusion are concerned) can be modeled by a two-dimensional reaction-diffusion equation as

$$\frac{\partial u}{\partial t} = \nabla(D.\nabla u) + \mu f(u)\,. \tag{1}$$

Here $u(x,t)$ is a dimensionless temperature or population, $\frac{\partial u}{\partial t}$ is the rate of increase of u with time t, ∇ is the gradient operator in two-dimensional space, D is a constant which is second order tensor measuring the diffusivity of the media, and $f(u)$ is a nonlinear function of u representing the effect of reaction or multiplication. A constant coefficient μ has been placed before $f(u)$ for convenience.

The reaction-diffusion equation is one of the most important partial differential equations (PDEs), and its applications can be found in many fields, including biology, chemistry, physics, finance, and so on. In classical reaction-diffusion equations, the diffusion is described by the standard Laplace operator ∇, characterizing the transport mechanics due to the Brownian motion. Recently, it has been suggested that many complex (biological and chemical) systems are indeed characterized by the Levy motion, rather than the Brownian motion [21,22,23,24]. Hence, the classical reaction-diffusion models fail to describe properly the phenomena in those systems. To circumvent such issues, the fractional order reaction-diffusion equations were proposed, where the classical Laplace operator is replaced by the fractional Laplacian operator. Advective-dispersive theory is used in many physical situations viz., flow through porous media, mass transfer in fluids, relaxation in polymer systems, tracer dynamics in polymer networks, spread of contaminations in fluids [25,26]. Contaminations occur on the land of surface and permeate into the surface through pores. Finally, contaminants are transported into groundwater.

The following equation represents the solute transport in aquifers,

$$\frac{\partial c(x,t)}{\partial t} = -v\frac{\partial c}{\partial x} + d\frac{\partial^2 c}{\partial x^2}, \tag{2}$$

where $c(x,t)$ is solute concentration, $v > 0$ represents average fluid velocity and d represents dispersion coefficient. Equation (2) is also called advection-dispersion equation. This equation also describes probability function for location of particles in a continuum. The equation (2) is used in groundwater hydrology in which the transport of passive tracers is carried by fluid flow in porous media. Reaction-diffusion process has been investigated since a long time. In the process of reaction- diffusion, reacting molecules are used to move through space due to diffusion. This definition excludes other modes of transports as convection, drift etc. as those may arise due to presence of externally imposed fields.

When a reaction occurs within an element of space, molecules can be created or consumed. These events are added to the diffusion equation and lead to reaction-diffusion equation of the form

$$\frac{\partial c}{\partial t} = D\nabla^2 c + R(c,t), \tag{3}$$

where $R(c,t)$ denotes reaction term at time t. The extension of the reaction-diffusion equation in fractional order system can be found in the articles [27,28,29,30,31,32,33].

Many analytical and numerical techniques on two dimensional diffusion equation have been developed [34,35] e.g., Finite Element method [36], Legendre Collocation Method [37] etc. Chen has developed method for finding exact analytical solution of two-dimensional advection-diffusion equation in the cylindrical coordinates [38]. The numerical methods for the numerical solution of Volterra integral equation [39], Hirota-Satsuma coupled-KdV equation [40], Burgers and Sharma-Tasso-Olver equations [41] and weak solutions to Dirichlet problem [42] are available in literature. In this present article Genocchi polynomials [43,44] have been introduced in collocation method to solve non-linear fractional reaction-diffusion equation. After finding the operational matrix of fractional differentiation, we collocate the given non-linear fractional equation model and boundary conditions. By collocating a non-linear system of algebraic equations is obtained which are solved by using an iteration method called Newton method. The article is organized as follows.

In the section 2, the definitions, mathematical preliminaries of fractional calculus, Genocchi numbers, Genocchi polynomial, their properties and Kronecker product of two matrices are given. It also continues the function order approximation for operational matrix of fractional differentiation by Genocchi polynomial. The error bound and stability analysis have been done in section 3. In section 4, a drive has been taken to solve the proposed model using the operational matrix with Genocchi polynomials. The validation of the method through a comparison of the numerical results with the existing analytical results for two particular cases and also illustrations of numerical results of the proposed model through graphical presentations are given in section 5. The conclusion of overall work is presented in section 6.

2. PRELIMINARIES

Here, few definitions and important properties of fractional calculus have been introduced. It is well known that the Riemann-Liouville definition has disadvantages when it comes for modeling of real world problems. But definition of fractional differentiation given by M. Caputo is more reliable regarding application point of view.

2.1. Definition of R-L order derivative and integration

Fractional order integration of Riemann-Liouville type of a given order ϑ of a function $f(t)$ is given by

$$I^{\vartheta} f(t) = \frac{1}{\Gamma(\vartheta)} \int_0^t (t-\omega)^{\vartheta-1} f(\omega)\,\mathrm{d}\omega, \quad t>0, \quad \vartheta \in R^{+}. \tag{4}$$

Fractional order derivative of the Riemann-Liouville type of order $\vartheta > 0$ is defined as

$$D_l^{\vartheta} = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m (I^{m-\vartheta} f)(t), \quad (\vartheta>0, \quad m-1<\vartheta<m). \tag{5}$$

2.2. Definition of Caputo derivative

Fractional derivative of order $\vartheta > 0$ in Caputo sense is defined as

$$D_c^{\vartheta} = \begin{cases} \dfrac{\mathrm{d}^l f(t)}{\mathrm{d}t^l} & \vartheta = l \in N \\ \dfrac{1}{\Gamma(\vartheta)} \displaystyle\int_0^t (t-\eta)^{l-\vartheta-1} f^l(\eta)\,\mathrm{d}\eta & l-1<\vartheta<l. \end{cases} \tag{6}$$

Here, l is an integer, $t > 0$.

Basic properties of caputo fractional derivative are

$$D_c^{\vartheta} C = 0, \tag{7}$$

where C is a constant and

$$D_c^{\vartheta} t^{\sigma} = \begin{cases} 0, & \sigma \in N \cup 0 \quad \text{and} \quad \sigma < \vartheta \\ \dfrac{\Gamma(1+\sigma)}{\Gamma(1-\vartheta+\sigma)} t^{-\vartheta+\sigma}, & \sigma \in N \cup 0 \quad \text{and} \quad \sigma \geq \vartheta \quad \text{or} \quad \sigma \notin N \quad \text{and} \quad \sigma > \lfloor \vartheta \rfloor, \end{cases} \tag{8}$$

where $\lfloor \vartheta \rfloor$ is floor function.

The operator D_c^{ϑ} is linear, since

$$D_c^{\vartheta}(Af(t) + BG(t)) = AD_c^{\vartheta} f(t) + BD_c^{\vartheta} g(t), \tag{9}$$

where A and B are constants.

Caputo operator and Riemann-Liouville operator have a relation given as follows

$$(I^{\vartheta} D_c^{\vartheta} g)(t) = g(t) - \sum_{k=0}^{l-1} g^k(0^{+}) \frac{t^k}{k!}, \quad l-1<\vartheta \leq l. \tag{10}$$

2.3. Kronecker product of two matrices

Suppose F is a field like as R and C. If $\mathbf{A} \in F^{m\times n}$ and $\mathbf{B} \in F^{p\times q}$ are any matrices then their Kronecker product denoted as $\mathbf{A} \otimes \mathbf{B}$ is defined as [22]

$$\mathbf{A}\otimes\mathbf{B}=\begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \cdots & a_{2n}\mathbf{B} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1}\mathbf{B} & a_{m2}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{bmatrix},$$

Some properties of Kronecker product are given as follows

- $(\alpha\mathbf{A})\otimes\mathbf{B}=\mathbf{A}\otimes(\alpha\mathbf{B})=\alpha(\mathbf{A}\otimes\mathbf{B})$
- $(\mathbf{A}+\mathbf{B})\otimes\mathbf{C}=\mathbf{A}\otimes\mathbf{C}+\mathbf{B}\otimes\mathbf{C}$
- $(\mathbf{A}\otimes\mathbf{B})\otimes\mathbf{C}=\mathbf{A}\otimes(\mathbf{B}\otimes\mathbf{C})$
- $\mathbf{A}\otimes\mathbf{B}=(\mathbf{A}\otimes\mathbf{I}_p)(\mathbf{I}_n\otimes\mathbf{B})=(\mathbf{I}_m\otimes\mathbf{B})(\mathbf{A}\otimes\mathbf{I}_q)$
- $(\mathbf{A}\otimes\mathbf{B})(\mathbf{C}\otimes\mathbf{D})=\mathbf{AC}\otimes\mathbf{BD}$
- $(\mathbf{A}\otimes\mathbf{B})^{-1}=\mathbf{A}^{-1}\otimes\mathbf{B}^{-1}$

where $\mathbf{A}\in F^{m\times n}$, $\mathbf{C}\in F^{n\times p}$, $\mathbf{B}\in F^{q\times r}$, $\mathbf{D}\in F^{r\times s}$.

2.4. Genocchi polynomial and its properties

Genocchi polynomials and numbers have been applied in lot of branches of physics and mathematics like homotopy theory, number theory, quantum physics, differential topology. Genocchi number G_n and Genocchi polynomial $G_n(x)$ can be derived respectively by the following exponential generating functions.

$$\frac{2x}{e^x+1}=\sum_{n=0}^{\infty}G_n\frac{x^n}{n!},\quad (|x|<\pi), \tag{11}$$

$$\frac{2x\,e^{tx}}{e^x+1}=\sum_{n=0}^{\infty}G_n(t)\frac{x^n}{n!},\quad (|x|<\pi), \tag{12}$$

where $G_n(t)$ is Genocchi polynomial of degree n given by

$$G_n(t)=\sum_{k=0}^{l}\binom{l}{k}G_{l-k}\,t^k. \tag{13}$$

Here, G_{l-k} is the Genocchi number.

Some properties of Genocchi polynomial are given below:

$$\int_0^1 G_n(t)G_m(t)\,\mathrm{d}t=\frac{2(-1)^l m!\,l!}{(m+l)!}G_{m+l},\quad l,m\ge 1, \tag{14}$$

$$G_n(1)+G_n(0)=0,\quad n>1, \tag{15}$$

$$\frac{\mathrm{d}G_l(t)}{\mathrm{d}t}=l\,G_{l-1}(t),\quad l\ge 1. \tag{16}$$

2.5. Approximation of an arbitrary function

Let us Suppose $\{G_1(t),G_2(t),.....,G_M(t)\}\subset L^2[0,1]$ is the set of Genocchi polynomials. A function $u(t)$ which belongs to $L^2[0,1]$ can be expressed as

$$u(t)=\sum_{l=1}^{M} c_l\, G_l(t)=C^{\mathrm{T}}G(t)\,, \tag{17}$$

where $c_l=(u(t),G_l(t))$ and (.) denotes the inner product. C and $G(t)$ are column vectors.

Similarly, an arbitrary function $u(x,y,t)$ belongs to $L^2[0,1]\times L^2[0,1]\times L^2[0,1]$ of three variables can be expressed in terms of Genocchi polynomials as

$$u(x,y,t)=\sum_{l=1}^{M}\sum_{m=1}^{M}\sum_{n=1}^{M} c_{lmn}G_m(t)G_l(x)G_n(y)=\boldsymbol{\psi}^{\mathrm{T}}(t).\mathbf{V}.(\boldsymbol{\psi}(x)\otimes\boldsymbol{\psi}(y))\,, \tag{18}$$

where $\mathbf{V}=[c_{lmn}]_{M\times M^2}$ and $\otimes$ denotes Kronecker product.

LEMMA [41]. *Let us consider* $G_j(x)$ *be the Genocchi polynomial, then* $D^{\vartheta}G_j(x)=0$ for $j=1,\cdots,\vartheta$, $\vartheta>0$.

2.6. Genocchi operational matrix of fractional derivative

THEOREM [41]. *Let* $\boldsymbol{\psi}(y)=(G_1(y),G_2(y),...G_N(y))^{\mathrm{T}}$ *is the Genocchi vector and* $\vartheta>0$. *Then*

$$D^{\vartheta}\boldsymbol{\psi}(y)=\mathbf{Q}^{\vartheta}\boldsymbol{\psi}(y)\,, \tag{19}$$

where $\mathbf{Q}^{\vartheta}$ *is an* $M\times M$ *operational matrix of fractional derivative of order* ϑ.

It is defined as,

$$\mathbf{Q}^{\vartheta}=\begin{bmatrix} 0 & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \cdots & \vdots\\ \sum_{k=\vartheta}^{\vartheta}\varsigma_{\vartheta,k,1} & \sum_{k=\vartheta}^{\vartheta}\varsigma_{\vartheta,k,2} & \cdots & \sum_{k=\vartheta}^{\vartheta}\varsigma_{\vartheta,k,M}\\ \vdots & \vdots & \cdots & \vdots\\ \sum_{k=\vartheta}^{i}\varsigma_{i,k,1} & \sum_{k=\vartheta}^{i}\varsigma_{i,k,2} & \cdots & \sum_{k=\vartheta}^{i}\varsigma_{i,k,M}\\ \vdots & \vdots & \cdots & \vdots\\ \sum_{k=\vartheta}^{M}\varsigma_{M,k,1} & \sum_{k=\vartheta}^{M}\varsigma_{M,k,2} & \cdots & \sum_{k=\vartheta}^{M}\varsigma_{M,k,M} \end{bmatrix}$$

where $\varsigma_{i,k,j}$ is

$$\varsigma_{i,k,j}=\frac{G_{i-k}\,i!}{\Gamma(1-\vartheta+k)(i-k)!}h_j\,. \tag{20}$$

Here G_{i-k} is Genocchi numbers and h_j can be obtained by

$$h_j=\frac{\mathrm{Gram}_j(G_1(y),G_2(y),.....G_N(y))}{\mathrm{Gram}(G_1(y),G_2(y),.....G_N(y))}\,, \tag{21}$$

where $\mathrm{Gram}(G_1(y),G_2(y),.....G_M(y))=\begin{vmatrix}\langle G_1(y),G_1(y)\rangle & \langle G_1(y),G_2(y)\rangle & \cdots & \langle G_1(y),G_M(y)\rangle\\ \langle G_2(y),G_1(y)\rangle & \langle G_2(y),G_2(y)\rangle & \cdots & \langle G_2(y),G_M(y)\rangle\\ \vdots & & & \\ \langle G_N(y),G_1(y)\rangle & \langle G_N(y),G_2(y)\rangle & \cdots & \langle G_M(y),G_M(y)\rangle\end{vmatrix}$.

Here, $\mathrm{Gram}_j(G_1(y),G_2(y),.....G_M(y))$ can be obtained by replacing the j-th column of $\mathrm{Gram}(G_1(y),G_2(y),.....G_M(y))$ by a column whose elements are

$$\langle G_1(t),f(y)\rangle,\ \langle G_2(y),f(y)\rangle,\ \cdots,\ \langle G_M(y),f(y)\rangle$$

3. CONCLUSION

The preliminary definitions of fractional order derivative duly supported by suffcient literature review on fractional caculus and diffusion equation have been provided. The approximation of arbitrary function of three variables in terms of Kronecker product of matrices with the help of Genocchi polynomials has also been dicussed. The derivation of operational matrix of fractional order derivative is the important feature of the present contribution. The error bound, method of the solution of space-time fractional two-dimensional non-linear reaction-diffusion model and also the study of this model for different particular cases will be dicussed in the following article (part II).

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