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A FURTHER LOOK AT A COMPLETE CHARACTERIZATION OF RAMANUJAN-TYPE CONGRUENCES MODULO 16 FOR OVERPARTITIONS

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Abstract. In 2016, X. Xiong provided a complete determination of the overpartition function $\overline{p}(n)$ modulo 16 by relating it to some binary quadratic forms. In this paper, we approach the characterization of $\overline{p}(n)$ modulo 16 considering the relations of the form

$$\overline{p}\left(2^{\alpha}(8n+\ell)\right) \equiv r \pmod{16},$$

with $\alpha \geqslant 0$ and $\ell \in \{1,3,5,7\}$.

Key words: overpartitions, congruence relations, divisor functions.

1. INTRODUCTION

Recall [4] that an overpartition of the positive integer n is an ordinary partition of n where the first occurrence of parts of each size may be overlined. Let $\overline{p}(n)$ denote the number of overpartitions of n. For example, the overpartitions of the integer 3 are:

$$3, \overline{3}, 2+1, \overline{2}+1, 2+\overline{1}, \overline{2}+\overline{1}, 1+1+1 \text{ and } \overline{1}+1+1.$$

We see that $\overline{p}(3)=8$. It is well-known that the generating function of $\overline{p}(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{p}(n) q^n = \frac{(-q;q)_\infty}{(q,q)_\infty} = \left(\sum_{n=-\infty}^{\infty} (-q)^{n^2} \right)^{-1},$$

where

$$(a;q)_\infty = \lim_{n\to\infty} (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}).$$

Because the infinite product $(a;q)_\infty$ diverges when $a \neq 0$ and $|q| \geqslant 1$, whenever $(a;q)_\infty$ appears in a formula, we shall assume that $|q|<1$.

In the recent years many congruences for the number of overpartitions have been discovered. For more information and references, see Chen [1], Chen, Hou, Sun and Zhang [2], Chern and Dastidar [3], Dou and Lin [6], Fortin, Jacob and Mathieu [7], Hirschhorn and Sellers [8], Kim [10,11], Lovejoy and Osburn [12], Mahlburg [13], Xia [14], Xiong [15] and Yao and Xia [16].

It seems that the first Ramanujan-type congruences modulo power of 2 for $\overline{p}(n)$, was founded in 2003 by Fortin, Jacob and Mathieu [7]. For all n that cannot be written as a sum of s or less squares, they obtained that

$$\overline{p}(n) \equiv 0 \pmod{2^{s+1}}. \tag{1}$$

This result is meaningful only for $s<4$ since, by Lagrange's four-square theorem, all numbers can be written as a sum of four squares. So considering that $8n+7$ cannot be written as a sum of three or less squares, they derived the following congruence modulo 16:

$$\overline{p}(8n+7)\equiv 0 \pmod{16}. \tag{2}$$

The following Ramanujan-type congruence for $\overline{p}(n)$ modulo 16 was founded in 2013 by Yao and Xia [16] using dissection techniques:

$$\overline{p}(24n+17)\equiv 0 \pmod{16},$$

$$\overline{p}(48n+14)\equiv 0 \pmod{16}, \tag{3}$$

$$\overline{p}(96n+68)\equiv 0 \pmod{16},$$

$$\overline{p}(96n+92)\equiv 0 \pmod{16}, \tag{4}$$

$$\overline{p}(72n+21)\equiv 0 \pmod{16},$$

$$\overline{p}(72n+51)\equiv 0 \pmod{16},$$

and

$$\overline{p}(72n+3)\equiv \begin{cases} 8 \pmod{16}, & \text{if } n=G_k \\ 0 \pmod{16}, & \text{otherwise} \end{cases}$$

where

$$G_k=\frac{1}{2}\left\lceil\frac{k}{2}\right\rceil\left(3\left\lceil\frac{k}{2}\right\rceil+(-1)^k\right)$$

is either of the k-th generalized pentagonal numbers.

Three years later, Chen, Hou, Sun and Zhang [2] gave a 16-dissection of the generating function for $\overline{p}(n)$ modulo 16 and showed that:

$$\overline{p}(4n)\equiv(-1)^n\,\overline{p}(n) \pmod{16}$$

and

$$\overline{p}\left(4^{\alpha}(16n+14)\right)\equiv 0 \pmod{16}. \tag{5}$$

We see that this congruence is a generalization of (3). In addition, applying the 2-adic expansion of the generating function for $\overline{p}(n)$ according to Mahlburg, they obtain that

$$\overline{p}(\ell^2 n+r\ell)\equiv 0 \pmod{16},$$

where $\ell\equiv -1 \pmod 8$ is an odd prime and r is a positive integer coprime to ℓ.

In 2016, Xiong [15] considered some binary quadratic forms and provided a complete determination of overpartition function modulo 16. For $n\geqslant 1$, $r_2'(n)$ is the number of representations of n as sum of two squares m^2+l^2, with $m,l\geqslant 1$ and $m\neq l$. For $n\geqslant 1$, $e_2(n)$ is the number of representations of n as the form of m^2+2l^2, with $m,l\geqslant 1$.

THEOREM 1.1. *For $n\geqslant 1$, we have:*

$\overline{p}(n)\equiv 0 \pmod{16}$ *if n is neither a square nor a double square and* $e_2(n)\equiv r_2'(n) \pmod 2$,

$\overline{p}(n)\equiv 2 \pmod{16}$ *if n is a square of an odd number and* $e_2(n)\equiv r_2'(n) \pmod 2$,

$\overline{p}(n)\equiv 4 \pmod{16}$ *if n is a double of a square and* $e_2(n)\equiv r_2'(n) \pmod 2$,

$\overline{p}(n)\equiv 6 \pmod{16}$ *if n is a square of an even number and* $e_2(n)\not\equiv r_2'(n) \pmod 2$,

$\overline{p}(n)\equiv 8 \pmod{16}$ *if n is neither a square nor a double square and* $e_2(n)\not\equiv r_2'(n) \pmod 2$,

$\overline{p}(n) \equiv 10 \pmod{16}$ *if* n *is a square of an odd number and* $e_2(n) \not\equiv r_2'(n) \pmod{2}$,
$\overline{p}(n) \equiv 12 \pmod{16}$ *if* n *is a double of a square and* $e_2(n) \not\equiv r_2'(n) \pmod{2}$,
$\overline{p}(n) \equiv 14 \pmod{16}$ *if* n *is a square of an even number and* $e_2(n) \equiv r_2'(n) \pmod{2}$.

THEOREM 1.1 reduces the determination of overpartition function $\overline{p}(n)$ modulo 16 to the calculations of $r_2'(n)$ and $e_2(n)$. More details can be found in [15, Theorems 1.2 and 1.3].

In this paper, we shall provide a complete characterization of Ramanujan-type congruences modulo 16 for $\overline{p}(n)$ considering the identities of the form

$$\overline{p}\left(2^{\alpha}(8n+\ell)\right) \equiv r \pmod{16},$$

with $\alpha \geqslant 0$ and $\ell \in \{1,3,5,7\}$. Having

$$\mathcal{A}_{\ell} = \bigcup_{\alpha=0}^{\infty} \left\{ 2^{\alpha}(8n+\ell) \,\middle|\, n \in \mathbb{N}_0 \right\},$$

we note that $\left[\mathcal{A}_1, \mathcal{A}_3, \mathcal{A}_5, \mathcal{A}_7\right]$ is a partition of the set $\mathbb{N}$.

The first result is a generalization of (2), (4) and (5).

THEOREM 1.2. *For* $n, \alpha \geqslant 0$,

$$\overline{p}\left(2^{\alpha}(8n+7)\right) \equiv 0 \pmod{16}.$$

Surprisingly, this congruence went unobserved so far. It is clear that the congruence (5) is the case α odd of this theorem. Replacing n by $3n+2$ and α by 2 in Theorem 1.2, we obtain (4).

The following two results provide new Ramanujan-type congruences that combines the overpartition function $\overline{p}(n)$ and the divisor function $\tau_{\text{odd}}(n)$ that counts the odd positive divisors of n.

THEOREM 1.3. *For* $n, \alpha \geqslant 0$,

$$\overline{p}\left(2^{\alpha}(8n+3)\right) \equiv \begin{cases} 8 \pmod{16}, & \text{if } \tau_{\text{odd}}(8n+3)/2 \text{ is odd} \\ 0 \pmod{16}, & \text{if } \tau_{\text{odd}}(8n+3)/2 \text{ is even.} \end{cases}$$

THEOREM 1.4. *For* $n, \alpha \geqslant 0$,

$$\overline{p}\left(2^{\alpha}(8n+5)\right) \equiv \begin{cases} 8 \pmod{16}, & \text{if } \tau_{\text{odd}}(8n+5)/2 \text{ is odd} \\ 0 \pmod{16}, & \text{if } \tau_{\text{odd}}(8n+5)/2 \text{ is even.} \end{cases}$$

If n is a square or twice of a square, then the following result shows that $\overline{p}(n)$ is congruent to 2, 4, 6, 10, 12 or 14 $\pmod{16}$.

THEOREM 1.5. *Let* n *and* α *be nonnegative integers.*

i. If $8n+1$ *is not a square, then*

$$\overline{p}\left(2^{\alpha}(8n+1)\right) \equiv 0 \pmod{16}.$$

ii. If $8n+1$ *is a square, then it is of the form* $(8k \pm 1)^2$ or $(8k \pm 3)^2$. *We have*

$$\overline{p}\left(2^{\alpha}(8n \pm 1)^2\right) \equiv \begin{cases} 2 \pmod{16}, & \text{for } \alpha = 0 \\ 4 \pmod{16}, & \text{for } \alpha \text{ odd} \\ 14 \pmod{16}, & \text{for } \alpha > 0 \text{ even} \end{cases}$$

and

$$\overline{p}\left(2^{\alpha}(8n\pm 3)^2\right)\equiv\begin{cases}10 \pmod{16}, & \textit{for } \alpha=0\\ 12 \pmod{16}, & \textit{for } \alpha \textit{ odd}\\ 6 \pmod{16}, & \textit{for } \alpha>0 \textit{ even.}\end{cases}$$

The following linear homogeneous recurrence relation [7, Corollary 4]

$$\overline{p}(n)=2\sum_{j=1}^{\lfloor\sqrt{n}\rfloor}(-1)^{j+1}\overline{p}(n-j^2),$$

with $\overline{p}(0)=1$, provides a simple and reasonably efficient way to compute the value of $\overline{p}(n)$. In order to prove Theorems 1.3-1.5, we consider this recurrence relation and the following characterization of Ramanujan-type congruences modulo 8 for the overpartition function $\overline{p}(n)$ provided by Kim [11, Theorem 3]:

$$\overline{p}(n)\equiv\begin{cases}2 \pmod{8}, & \text{if } n \text{ is a square of an odd number,}\\ 4 \pmod{8}, & \text{if } n \text{ is a double of a square,}\\ 6 \pmod{8}, & \text{if } n \text{ is a square of an even number,}\\ 0 \pmod{8}, & \text{otherwise.}\end{cases} \tag{6}$$

2. PROOF OF THEOREM 1.2

We need to prove only the case α even. First we point out that $2^{2\alpha}(8n+7)$ is not a square.

The fundamental theorem on sums of two squares claims that a natural number N is a sum of two squares if and only if all prime factors of N of the form $4m+3$ have even exponent in the prime factorization of N. It is clear that

$$2^{2\alpha}(8n+7)=2^{2\alpha}\left(4(2n+1)+3\right)$$

cannot be written as a sum of two squares.

On the other hand, Legendre's three-square theorem states that a natural number N can be represented as the sum of three squares of integers if and only if N is not of the form $2^{2\alpha}(8n+7)$.

Thus we deduce that $2^{2\alpha}(8n+7)$ cannot be written as a sum of three or less squares. Considering (1), we obtain

$$\overline{p}\left(2^{2\alpha}(8n+7)\right)\equiv 0 \pmod{16}.$$

This concludes the proof.

3. PROOF OF THEOREM 1.3

We remark that an integer of the form $8n+3$ cannot be a square. For all integers a and b, we have

$$a^2+b^2\equiv 0,1 \text{ or } 2 \pmod{4}.$$

Thus we deduce that $8n+3$ cannot be written as a sum of two squares.

Let $R(n)$ be the number of nonnegative integer solutions to the equation

$$x^2+2y^2=8n+3.$$

Moreover, if (x,y) is an integer solution of this equation, then x and y are odd integers.

Let $x_1,x_2,\ldots,x_{R(n)}$ and $y_1,y_2,\ldots,y_{R(n)}$ be nonnegative integers such that

$$(2x_k+1)^2+2(2y_k+1)^2=8n+3, \qquad k=1,2,\ldots,R(n).$$

Considering (6), the expression

$$\overline{p}(8n+3)=2\sum_{j=1}^{\lfloor\sqrt{8n+3}\rfloor}(-1)^{j+1}\overline{p}(8n+3-j^2),$$

can be reduced modulo 16 as follows:

$$\overline{p}(8n+3)\equiv 2\sum_{k=1}^{R(n)}\overline{p}\left(8n+3-(2x_k+1)^2\right)\equiv 2\sum_{k=1}^{R(n)}\overline{p}\left(2(2y_k+1)^2\right)\equiv 2\sum_{k=1}^{R(n)}4\equiv 8R(n) \pmod{16}.$$

On the other hand, due to Dirichlet [5], we know that the number of representation of $8n+3$ as the sum of a square and twice a square is given by

$$2\left(d_1(n)+d_3(n)-d_5(n)-d_7(n)\right),$$

where $d_\ell(n)$ is the number of positive divisors of $8n+3$ of the form $8k+\ell$. It is clear that

$$R(n)=\frac{d_1(n)+d_3(n)-d_5(n)-d_7(n)}{2}.$$

Moreover, we see that $R(n)$ and $\tau_{\text{odd}}(8n+3)/2$ have the same parity. In addition, having $R(2^\alpha n)=R(n)$, we obtain

$$\overline{p}\left(2^\alpha(8n+3)\right)\equiv 8R(n) \pmod{16}$$

and the proof is finished.

4. PROOF OF THEOREM 1.4

Firstly we remark that the equations of the form

$$x^2+2y^2=2^\alpha(8n+5)$$

do not have integer solutions. Let $R(n)$ be the number of positive integer solutions to the equation

$$x^2+y^2=8n+5.$$

If (x,y) is an integer solution of this equation, then we remark that x and y have different parities.

Let $x_1,x_2,\ldots,x_{R(n)}$ and $y_1,y_2,\ldots,y_{R(n)}$ be nonnegative integers such that

$$(2x_k+1)^2+(2y_k)^2=8n+5, \qquad k=1,2,\ldots,R_1(n).$$

Considering (6), the expression

$$\overline{p}(8n+5)=2\sum_{j=1}^{\lfloor\sqrt{8n+5}\rfloor}(-1)^{j+1}\overline{p}(8n+5-j^2),$$

can be reduced modulo 16 as follows:

$$\begin{aligned}\overline{p}(8n+5)&\equiv 2\sum_{k=1}^{R(n)}\left(\overline{p}\left(8n+5-(2x_k+1)^2\right)-\overline{p}\left(8n+5-(2y_k)^2\right)\right)\\&\equiv 2\sum_{k=1}^{R(n)}\left(\overline{p}\left((2y_k)^2\right)-\overline{p}\left((2x_k+1)^2\right)\right)\equiv 2\sum_{k=1}^{R(n)}(6-2)\equiv 8R(n) \pmod{16}.\end{aligned}$$

Due to Jacobi [9], we know that the number of representation of $8n+5$ as the sum of two squares is

$$4\left(d_1(n)-d_3(n)\right),$$

where $d_\ell(n)$ is the number of positive divisors of $8n+5$ of the form $4k+\ell$.

Thus we obtain that

$$R(n)=\frac{d_1(n)-d_3(n)}{2}.$$

Moreover, we see that $R(n)$ and $\tau_{\text{odd}}(8n+5)/2$ have the same parity. In a similar way, considering that $R(2^\alpha n)=R(n)$, we obtain

$$\overline{p}\left(2^\alpha(8n+5)\right)\equiv 8R(n) \pmod{16}.$$

This concludes the proof.

5. PROOF OF THEOREM 1.5

Let $R_1(n)$ be the number of positive integer solutions of the equation

$$x^2+y^2=8n+1.$$

If (x,y) is an integer solution of this equation, the x and y have different parities. Let $x_1,x_2,\ldots,x_{R_1(n)}$ and $y_1,y_2,\ldots,y_{R_1(n)}$ be positive integers such that

$$(2x_k+1)^2+(2y_k)^2=8n+1, \qquad k=1,2,\ldots,R_1(n).$$

Let $R_2(n)$ be the number of positive integer solutions of the equation

$$z^2+2w^2=8n+1.$$

If (z,w) is an integer solution of this equation, the z is odd. Let $z_1,z_2,\ldots,z_{R_2(n)}$ and $w_1,w_2,\ldots,w_{R_2(n)}$ be positive integers such that

$$(2z_k+1)^2+2w_k^2=8n+1, \qquad k=1,2,\ldots,R_2(n).$$

If $8n+1$ is a square, then considering (6), the expression

$$\overline{p}(8n+1)=2\sum_{j=1}^{\lfloor\sqrt{8n+1}\rfloor}(-1)^{j+1}\overline{p}(8n+1-j^2),$$

can be reduced modulo 16 as follows:

$$\begin{aligned}\overline{p}(8n+1)&\equiv 2\sum_{k=1}^{R_1(n)}\left(\overline{p}\left(8n+1-(2x_k+1)^2\right)-\overline{p}\left(8n+1-(2y_k)^2\right)\right)+2\sum_{k=1}^{R_2(n)}\overline{p}\left(8n+1-(2z_k+1)^2\right)+2\overline{p}(0)\equiv\\ &\equiv 2\sum_{k=1}^{R_1(n)}\left(\overline{p}\left((2y_k)^2\right)-\overline{p}\left((2x_k+1)^2\right)\right)+2\sum_{k=1}^{R_2(n)}\overline{p}\left(2w_k^2\right)+2\equiv\\ &\equiv 2\sum_{k=1}^{R_1(n)}(6-2)+2\sum_{k=1}^{R_2(n)}4+2\equiv 8\left(R_1(n)+R_2(n)\right)+2 \pmod{16}.\end{aligned}$$

In a similar way, when $8n+1$ is not a square we obtain:

$$\overline{p}(8n+1)\equiv 8\left(R_1(n)+R_2(n)\right) \pmod{16}.$$

According to Dirichlet [5] and Jacobi [9], we have

$$d_1(n)-d_3(n)+d_5(n)-d_7(n)=\begin{cases}2R_1(n)+1, & \text{if } 8n+1 \text{ is a square}\\ 2R_1(n), & \text{otherwise}\end{cases}.$$

and

$$d_1(n)+d_3(n)-d_5(n)-d_7(n)=\begin{cases}2R_2(n)+1, & \text{if } 8n+1 \text{ is a square}\\ 2R_2(n), & \text{otherwise}\end{cases},$$

where $d_\ell(n)$ is the number of positive divisors of $8n+1$ of the form $8k+\ell$. Thus, we deduce

$$\overline{p}(8n+1)\equiv\begin{cases}8\tau_1(8n+1)+10 \pmod{16}, & \text{if } 8n+1 \text{ is a square}\\ 8\tau_1(8n+1) \pmod{16}, & \text{otherwise,}\end{cases}$$

where $\tau_1(n)$ counts the positive divisors of n congruent to $\pm 1 \mod 8$. In a similar way, we obtain the following two congruences:

$$\overline{p}\left(2^{2\alpha+1}(8n+1)\right)\equiv\begin{cases}8\tau_1(8n+1)+12 \pmod{16}, & \text{if } 8n+1 \text{ is a square}\\ 8\tau_1(8n+1) \pmod{16}, & \text{otherwise}\end{cases}$$

and

$$\overline{p}\left(2^{2\alpha+2}(8n+1)\right)\equiv\begin{cases}8\tau_1(8n+1)+6 \pmod{16}, & \text{if } 8n+1 \text{ is a square}\\ 8\tau_1(8n+1) \pmod{16}, & \text{otherwise.}\end{cases}$$

On the other hand, if $8n+1$ is a square, then it is of the form $(8k\pm1)^2$ or $(8k\pm3)^2$. It is not difficult to prove that $\tau_1(8n+1)$ is odd if and only if $8n+1$ is a square of the form $(8k\pm1)^2$. The proof follows easily.

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