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ALL FRACTIONAL (g, f)-FACTORS IN GRAPHS

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Abstract. Let G a graph, and $g, f : V(G) \to N$ be two functions with $g(x) \le f(x)$ for each vertex x in G. We say that G has all fractional (g, f)-factors if G includes a fractional r-factor for every $r : V(G) \to N$ with $g(x) \le r(x) \le f(x)$ for each vertex x in G. Let H be a subgraph of G. We say that G admits all fractional (g, f)-factors including H if for every $r : V(G) \to N$ with $g(x) \le r(x) \le f(x)$ for each vertex x in G, G includes a fractional r-factor F_h with $h(e) = 1$ for any $e \in E(H)$, where $h : E(G) \to [0,1]$ is the indicator function of F_h. In this paper, we obtain a characterization for the existence of all fractional (g, f)-factors including H and pose a sufficient condition for a graph to have all fractional (g, f)-factors including H.

Key words: graph, fractional (g, f)-factor, all fractional (g, f)-factors.

1. INTRODUCTION

We consider finite undirected graphs which have neither multiple edges nor loops. Let G be a graph. We denote its vertex set and edge set by $V(G)$ and $E(G)$, respectively. For each $x \in V(G)$, the degree of x in G is defined as the number of edges which are adjacent to x and denoted by $d_G(x)$. For any $S \subseteq V(G)$, we use $G[S]$ to denote the subgraph of G induced by S, and use $G - S$ to denote the subgraph obtained from G by deleting vertices in S together with the edges incident to vertices in S. A subset S of $V(G)$ is said to be independent if $N_G(S) \cap S = \phi$. Let S and T be two disjoint vertex subsets of G. Then $e_G(S,T)$ denotes the number of edges joining S to T.

Let $g, f : V(G) \to N$ be two functions with $g(x) \le f(x)$ for each $x \in V(G)$. A spanning subgraph F of G is called a (g, f)-factor if one has $g(x) \le d_F(x) \le f(x)$ for each vertex x in G. An (f, f)-factor is said to be an f-factor. If G includes an r-factor for every $r : V(G) \to N$ which satisfies $g(x) \le r(x) \le f(x)$ for each vertex x in G and $r(V(G))$ is even, then we say that G admits all (g, f)-factors. Let $h : E(G) \to [0,1]$ be a function. For any $x \in V(G)$, we denote the set of edges incident with x by $E(x)$. If $g(x) \le \sum_{e \in E(x)} h(e) \le f(x)$ holds for each vertex x in G, then we call graph F_h with vertex set $V(G)$ and edge set E_h a fractional (g, f)-factor of G with indicator function h, where $E_h = \{e : e \in E(G),\ h(e) > 0\}$. A fractional (f, f)-factor is called a fractional f-factor. If G contains a fractional r-factor for every $r : V(G) \to N$ with $g(x) \le r(x) \le f(x)$ for each vertex x in G, then we say that G admits all fractional (g, f)-factors. If $g(x) \equiv a$, $f(x) \equiv b$ and G admits all fractional (g, f)-factors, then we say that G contains all fractional $[a,b]$-factors. Let H be a subgraph of G. If for every $r : V(G) \to N$ such that

$g(x) \le r(x) \le f(x)$ for each vertex x in G, G includes a fractional r-factor F_h with $h(e)=1$ for any $e \in E(H)$, then we say that G admits all fractional (g,f)-factors including H, where h is the indicator function of F_h. For any function $\varphi : V(G) \to N$, we define $\varphi(S) = \sum_{x \in S} \varphi(x)$ and $\varphi(\phi) = 0$. Especially, $d_G(S) = \sum_{x \in S} d_G(x)$.

Lu [3] first introduced the definition of all fractional (g,f)-factors, and obtained a necessary and sufficient condition for a graph to have all fractional (g,f)-factors, and posed a sufficient condition for the existence of all fractional $[a,b]$-factors in graphs. Zhou and Sun [4] showed a neighborhood condition for a graph to have all fractional $[a,b]$-factors, which is an extension of Lu's result [3]. Zhou, Bian and Sun [5] obtained a binding number condition for the existence of all fractional $[a,b]$-factors in graphs. The following results on fractional (g,f)-factors and all all fractional (g,f)-factors are known.

Anstee [1] gave a necessary and sufficient condition for graphs to have fractional (g,f)-factors. Liu and Zhang [2] posed a new proof.

THEOREM 1 (Anstee [1], Liu and Zhang [2]). *Let G be a graph, and $g,f : V(G) \to Z^+$ be two functions with $g(x) \le f(x)$ for each vertex x in G. Then G contains a fractional (g,f)-factor if and only if*

$$f(S) + d_{G-S}(T) - g(T) \ge 0$$

for any subset S of $V(G)$, where $T = \{x : x \in V(G) - S,\ d_{G-S}(x) < g(x)\}$.

The following theorem is equivalent to Theorem 1.

THEOREM 2. *Let G be a graph, and $g,f : V(G) \to Z^+$ be two functions with $g(x) \le f(x)$ for each vertex x in G. Then G contains a fractional (g,f)-factor if and only if*

$$f(S) + d_{G-S}(T) - g(T) \ge 0$$

for all disjoint subsets S and T of $V(G)$.

Lu [3] showed a characterization of graphs having all fractional (g,f)-factors.

THEOREM 3 (Lu [3]). *Let G be a graph, and $g,f : V(G) \to Z^+$ be two functions with $g(x) \le f(x)$ for each vertex x in G. Then G admits all fractional (g,f)-factors if and only if*

$$g(S) + d_{G-S}(T) - f(T) \ge 0$$

for any subset S of $V(G)$, where $T = \{x : x \in V(G) - S,\ d_{G-S}(x) < f(x)\}$.

Some other results on factors and fractional factors of graphs see [6-21]. In this paper, we study the existence of all fractional (g,f)-factors including any given subgraph in graphs, and pose some new results which are shown in the following.

THEOREM 4. *Let G be a graph, and $g,f : V(G) \to Z^+$ be two functions such that $g(x) \le f(x)$ for each vertex x in G. Let H be a subgraph of G. Then G has all fractional (g,f)-factors including H if and only if*

$$g(S) + d_{G-S}(T) - f(T) \ge d_H(S) - e_H(S,T)$$

for all disjoint subsets S and T of $V(G)$.

THEOREM 5. *Let G be a graph, H be a subgraph of G, and $g,f:V(G)\to Z^+$ be two functions with $d_H(x)\le g(x)\le f(x)\le d_G(x)$ for each vertex x in G. If*

$$(g(x)-d_H(x))d_G(y)\ge(d_G(x)-d_H(x))f(y)$$

holds for any $x,y\in V(G)$, then G has all fractional (g,f)-factors including H.

If $E(H)=\phi$ in Theorem 5, then we obtain the following corollary.

COROLLARY 6. *Let G be a graph, and $g,f:V(G)\to Z^+$ be two functions with $g(x)\le f(x)\le d_G(x)$ for each vertex x in G. If*

$$g(x)d_G(y)\ge d_G(x)f(y)$$

holds for any $x,y\in V(G)$, then G contains all fractional (g,f)-factors.

2. THE PROOF OF THEOREM 4

Proof of Theorem 4. We first verify this sufficiency. Let $r:V(G)\to Z^+$ be an arbitrary integer-valued function such that $g(x)\le r(x)\le f(x)$ for each $x\in V(G)$. According to the definition of all fractional (g,f)-factors including H, we need only to verify that G admits a fractional r-factor including H, that is, we need only to verify that G admits a fractional r'-factor excluding H, where $r'(x)=d_G(x)-r(x)$. Let $G'=G-E(H)$. Thus, we need only to prove that G' admits a fractional r'-factor.

For any disjoint subsets S and T of $V(G)$,

$$g(S)+d_{G-S}(T)-f(T)\ge d_H(S)-e_H(S,T),$$

and so,

$$g(T)+d_{G-T}(S)-f(S)-d_H(T)+e_H(S,T)\ge 0. \tag{1}$$

It follows from (1) that

$$\begin{aligned}r'(S)+d_{G'-S}(T)-r'(T)&=r'(S)+d_{G-S}(T)-r'(T)-d_H(T)+e_H(S,T)\\&=d_G(S)-r(S)+d_{G-S}(T)-d_G(T)+r(T)-d_H(T)+e_H(S,T)\\&\ge d_G(S)-f(S)+d_{G-S}(T)-d_G(T)+g(T)-d_H(T)+e_H(S,T)\\&=g(T)+d_{G-T}(S)-f(S)-d_H(T)+e_H(S,T)\ge 0.\end{aligned}$$

In terms of Theorem 2, G' admits a fractional r'-factor, that is, G has all fractional (g,f)-factors including H.

Now we verify the necessity. Conversely, we assume that there exist disjoint subsets S and T of $V(G)$ such that

$$g(S)+d_{G-S}(T)-f(T)<d_H(S)-e_H(S,T).$$

Let $r(x)=g(x)$ for any $x\in S$ and $r(y)=f(y)$ for any $y\in V(G)\setminus S$. Thus, we have

$$0>g(S)+d_{G-S}(T)-f(T)-d_H(S)+e_H(S,T)=r(S)+d_{G-S}(T)-r(T)-d_H(S)+e_H(S,T).$$

Set $r'(x)=d_G(x)-r(x)$ and $G'=G-E(H)$. Thus,

$$\begin{aligned}0&>r(S)+d_{G-S}(T)-r(T)-d_H(S)+e_H(S,T)\\&=d_G(S)-r'(S)+d_{G'-S}(T)+d_H(T)-e_H(S,T)-d_G(T)+r'(T)-d_H(S)+e_H(S,T)\\&=d_{G'}(S)+d_H(S)-r'(S)+d_{G'-S}(T)+d_H(T)-d_{G'}(T)-d_H(T)+r'(T)-d_H(S)\\&=r'(T)+d_{G'-T}(S)-r'(S),\end{aligned}$$

which implies that G' has no fractional r'-factor. (Otherwise, $r'(A)+d_{G'-A}(B)-r'(B)\geq 0$ for all disjoint subsets A and B of $V(G)$ by Theorem 2. Set $A=T$ and $B=S$. Thus, we obtain $r'(T)+d_{G'-T}(S)-r'(S)\geq 0$, a contradiction.) And so, G has no fractional r'-factor excluding H, that is, G has no fractional r-factor including H. Hence, G has no all fractional (g,f)-factors excluding H, a contradiction. This finishes the proof of Theorem 4.

3. THE PROOF OF THEOREM 5

Proof of Theorem 5. According to Theorem 4, we need only to verify that

$$g(S)+d_{G-S}(T)-f(T)\geq d_H(S)-e_H(S,T)$$

for all disjoint subsets S and T of $V(G)$.

If $T=\phi$, then we have

$$g(S)+d_{G-S}(T)-f(T)=g(S)\geq d_H(S)=d_H(S)-e_H(S,T).$$

In the following, we assume that $T\neq\phi$. Note that $(g(x)-d_H(x))d_G(y)\geq(d_G(x)-d_H(x))f(y)$ holds for any $x,y\in V(G)$, that is, $g(x)d_G(y)\geq d_G(x)f(y)+d_H(x)(d_G(y)-f(y))$ holds for any $x,y\in V(G)$. Hence, we have

$$\left(\sum_{x\in S}g(x)\right)\left(\sum_{y\in T}d_G(y)\right)\geq\left(\sum_{x\in S}d_G(x)\right)\left(\sum_{y\in T}f(y)\right)+\left(\sum_{x\in S}d_H(x)\right)\left(\sum_{y\in T}(d_G(y)-f(y))\right),$$

that is,

$$g(S)d_G(T)\geq d_G(S)f(T)+d_H(S)(d_G(T)-f(T)). \tag{2}$$

We write $U=V(G)\setminus(S\bigcup T)$. Then we obtain

$$\begin{aligned}d_G(S)&=e_G(S,T)+e_G(S,S)+e_G(S,U)\geq\\&\geq e_G(S,T)+e_H(S,S)+e_G(S,U)=\\&=e_G(S,T)+d_H(S)-e_H(S,T)-e_H(S,U)+e_G(S,U)\geq\\&\geq e_G(S,T)+d_H(S)-e_H(S,T)=\\&=d_G(T)-d_{G-S}(T)+d_H(S)-e_H(S,T),\end{aligned}$$

which implies

$$d_G(S)-d_G(T)\geq -d_{G-S}(T)+d_H(S)-e_H(S,T). \tag{3}$$

In terms of (2) and (3), we have

$$\begin{aligned}&d_G(T)(g(S)+d_{G-S}(T)-f(T)-d_H(S)+e_H(S,T))=\\&\quad=d_G(T)g(S)+d_G(T)d_{G-S}(T)-d_G(T)f(T)-d_G(T)d_H(S)+d_G(T)e_H(S,T)\\&\quad\geq d_G(S)f(T)+d_H(S)(d_G(T)-f(T))+d_G(T)d_{G-S}(T)-d_G(T)f(T)-d_G(T)d_H(S)+d_G(T)e_H(S,T)\\&\quad=f(T)(d_G(S)-d_G(T))+d_G(T)d_{G-S}(T)-d_H(S)f(T)+d_G(T)e_H(S,T)\\&\quad\geq f(T)(-d_{G-S}(T)+d_H(S)-e_H(S,T))+d_G(T)d_{G-S}(T)-d_H(S)f(T)+d_G(T)e_H(S,T)\\&\quad=(d_{G-S}(T)+e_H(S,T))(d_G(T)-f(T))\geq 0.\end{aligned}$$

Combining this with $d_G(T)\geq f(T)\geq|T|\geq 1$, we obtain

$$g(S)+d_{G-S}(T)-f(T)\geq d_H(S)-e_H(S,T).$$

Theorem 5 is proved.

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