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THE JACOBSTHAL-PADOVAN p-SEQUENCES AND THEIR APPLICATIONS

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Abstract. In this paper, we define the Jacobsthal-Padovan p-numbers. Then we obtain the generating matrix, permanental representation, the Binet formula, the generating function, the exponential representation and the combinatorial representations of the Jacobsthal-Padovan p-numbers. Also, we study the Jacobsthal-Padovan p-numbers modulo m and then, we obtain the cyclic groups which are generated by reducing the multiplicative orders of the generating matrix and the auxiliary equation of the Jacobsthal-Padovan p-numbers modulo m. Finally, we give the relationships among orders of these cyclic groups and the periods of the Jacobsthal-Padovan p-sequences.

Key words: Jacobsthal-Padovan p-numbers, matrix, sequence, period.

1. INTRODUCTION AND PRELIMINARIES

Suppose that the $(n+k)$th term of a sequence is defined recursively by a linear combination of the preceding k terms:

$$a_{n+k} = c_0 a_n + c_1 a_{n+1} + \cdots + c_{k-1} a_{n+k-1},$$

where $c_0, c_1, \ldots, c_{k-1}$ are real constants. In [9], Kalman developed a number of closed-form formulas for this generalized sequence by the companion matrix method as follows:

$$\mathbf{A}_k = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ c_0 & c_1 & c_2 & \cdots & c_{k-2} & c_{k-1} \end{bmatrix}. \qquad \mathbf{A}_k^n \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{bmatrix}.$$

It is known that the Jacobsthal sequence $\{J_n\}$ is defined by a second-order recurrence equation:

$$J_n = J_{n-1} + 2J_{n-2}$$

for $n \geq 2$, where $J_0 = 0$ and $J_1 = 1$.

The Padovan sequence $\{P(n)\}$ is defined by a third-order recurrence equation:

$$P(n) = P(n-2) + P(n-3)$$

for $n \geq 3$, where $P(0) = P(1) = P(2) = 1$. For more information on this sequence, see [8].

The Jacobsthal-Padovan sequence $\{J(n)\}$ is defined [4] by a third-order recurrence equation:

$$J(n+2) = J(n) + 2J(n-1)$$

for $n \geq 0$, where $J(-1) = 0$ and $J(0) = J(1) = 1$.

Number theoretic properties such as these obtained from homogeneous linear recurrence relations relevant to this paper have been studied by many authors [2,6,10,12–20]. In [4–7,11], the authors obtained the cyclic groups via some special matrices. In this paper, we define the Jacobsthal-Padovan p-numbers. Then we obtain their miscellaneous properties such as the generating matrix, permanental representation, the Binet formula, the generating function, the exponential representation and the combinatorial representations. Also, we consider the multiplicative orders of the generating matrix and the auxiliary equation of the Jacobsthal-Padovan p-numbers according to modulo m and then, we produce the cyclic groups. Furthermore, we study the Jacobsthal-Padovan p-sequence modulo m and then, we obtain the relationships among orders of the produced cyclic groups and the periods of the Jacobsthal-Padovan p-sequences.

2. THE JACOBSTHAL-PADOVAN p-NUMBERS

We next define the Jacobsthal-Padovan p-sequence as

$$JPa_p(n+p+2) = JPa_p(n+p) + 2JPa_p(n), \qquad \text{for } n > 0 \tag{1}$$

where $p \geq 2$ and $JPa_p(1) = JPa_p(2) = \cdots = JPa_p(p) = 0$, $JPa_p(p+1) = 1$ and $JPa_p(p+2) = 0$.
When $p = 2$ in (1), we obtain $JPa_2(2n+1) = J_n$.

It is clear that

$$\begin{bmatrix} JPa_p(n+p+2) \\ JPa_p(n+p+1) \\ \vdots \\ JPa_p(n+2) \\ JPa_p(n+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 2 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & & \ddots & \cdots & \cdots \\ 0 & 0 & & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} JPa_p(n+p+1) \\ JPa_p(n+p) \\ \vdots \\ JPa_p(n+1) \\ JPa_p(n) \end{bmatrix}$$

letting

$$\mathbf{M} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 2 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & & \ddots & \cdots & \cdots \\ 0 & 0 & & 0 & 1 & 0 \end{bmatrix}.$$

The matrix $\mathbf{M}$ is said to be the Jacobsthal-Padovan p-matrix. It can be readily established by mathematical induction that for $n \geq 1$,

$$\mathbf{M}^n = \begin{bmatrix} JPa_p(n+p+1) & JPa_p(n+p+2) & 2JPa_p(n+1) & 2JPa_p(n+2) & \cdots & 2JPa_p(n+p) \\ JPa_p(n+p) & JPa_p(n+p+1) & 2JPa_p(n) & 2JPa_p(n+1) & \cdots & 2JPa_p(n+p-1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ JPa_p(n+1) & JPa_p(n+2) & 2JPa_p(n-p+1) & 2JPa_p(n-p+2) & \cdots & 2JPa_p(n) \\ JPa_p(n) & JPa_p(n+1) & 2JPa_p(n-p) & 2JPa_p(n-p+1) & \cdots & 2JPa_p(n-1) \end{bmatrix}. \tag{2}$$

We easily derive that

$$\det \mathbf{M} = (-1)^{p+1} \cdot 2.$$

Now we consider a permanental representation for the Jacobsthal-Padovan p-numbers.

Definition 2.1. Let $\mathbf{C}=\left[c_{i,j}\right]$ be an $n\times m$ real matrix. If column (resp. row) α contains exactly two nonzero entries, the matrix **C** is called a contractible matrix column (resp. row) α.

Let $u_1, u_2, \ldots, u_n$ be row vectors of the matrix **C** and let **C** be contractible on column α with $c_{i,\alpha}\neq 0,\ c_{j,\alpha}\neq 0$ and $i\neq j$.

Then the $(n-1)\times(m-1)$ matrix $\mathbf{C}_{i,j,\alpha}$ obtained from **C** by replacing row i with $c_{i,\alpha}\mathbf{u}_j+c_{j,\alpha}\mathbf{u}_i$ and deleting row j and column α is called the contraction on column α relative to rows i and j.

In [1], Brualdi and Gibson showed that $\operatorname{per}(\mathbf{A})=\operatorname{per}(\mathbf{B})$ if $\mathbf{A}$ is a real matrix of order $m>1$ and $\mathbf{B}$ is a contraction of $\mathbf{A}$.

Let p be a fixed integer such that $p\geq 2$ and let $\mathbf{M}_p^n=\left[m_{i,j}\right]$ be the $n\times n$ super-diagonal matrix with $m_{i,i+1}=2,\ m_{i+1,i}=m_{i,i+p+1}=1$ for all i and 0 otherwise, that is,

$$\mathbf{M}_p^n=\begin{array}{c} \qquad\qquad\qquad (p+2)\text{th} \\ \qquad\qquad\qquad\downarrow \\ \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 2 & 0 & \cdots & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 & 2 & 0 & \\ 0 & 1 & 0 & 1 & 0 & \cdots & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & \cdots & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \ddots & \ddots & \ddots & \ddots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \end{bmatrix}\end{array}$$

where $M_p^1=0$.

THEOREM 2.1. *For* $n\geq 1$, $\operatorname{per}\mathbf{M}_p^n$ *is the* $(n+p+1)$th *Jacobsthal-Padovan p-number,* $JPa_p(n+p+1)$.

Proof. We prove this by mathematical induction. Firstly, us consider the case $n<p+2$. From the definitions of the matrix $\mathbf{M}_p^n$ and the Jacobsthal-Padovan p-numbers it is clear that $\operatorname{per}\mathbf{M}_p^1=JPa_p(p+2)=0$ and $\operatorname{per}\mathbf{M}_p^2=JPa_p(p+3)=1$. Also, we have the following matrix for $3\leq k\leq p+1$

$$M_p^k=\begin{bmatrix} 0 & 1 & & & 0 \\ 1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 0 & 1 \\ 0 & & & 1 & 0 \end{bmatrix}.$$

Then

$$\operatorname{per}\mathbf{M}_p^k=\begin{cases} 1, & \text{if } k \text{ is even,} \\ 0, & \text{if } k \text{ is odd.} \end{cases}$$

Since also

$$JPa_p(\lambda+p+1)=\begin{cases} 1 & \text{if } \lambda \text{ is even,} \\ 0, & \text{if } \lambda \text{ is odd,} \end{cases} \text{ for } 3\leq\lambda\leq p+1,$$

$\operatorname{per}\mathbf{M}_p^n = JPa_p(n+p+1)$ for $1 \le n \le p+1$.

Now, let us consider the case $n \ge p+2$. Suppose that the equation holds for $n \ge p+2$. Then we show that the equation holds for $n+1$. If we expand the $\operatorname{per}\mathbf{M}_p^n$ by Laplace expansion of the permanent according to the first row, we obtain

$$\operatorname{per}\mathbf{M}_p^{n+1} = \operatorname{per}\mathbf{M}_p^{n-1} + 2\operatorname{per}\mathbf{M}_p^{n-p-1}.$$

Since $\operatorname{per}\mathbf{M}_p^{n-1} = JPa_p(n+p)$ and $\operatorname{per}\mathbf{M}_p^{n-p-1} = JPa_p(n)$, we get $\operatorname{per}\mathbf{M}_p^{n+1} = JPa_p(n+p+2)$. Thus this proof is complete.

LEMMA 2.1. *The characteristic equation of the Jacobsthal-Padovan p-numbers* $x^{p+2} - x^p - 2 = 0$ *does not have multiple roots.*

Proof. Let $g(x) = x^{p+2} - x^p - 2$. Suppose that α is a multiple root of $g(x)$. Then $g(\alpha) = 0$ and $g'(\alpha) = 0$. It is clear that 0 is not a root of $g(x)$. So, we obtain

$$g'(\alpha) = (p+2)\alpha^{p+1} - p\,\alpha^{p-1} = \alpha^{p-1}\left((p+2)\alpha^2 - p\right) = 0,$$

and hence $\alpha = \pm\left(\dfrac{p}{p+2}\right)^{1/2}$. Also, we have

$$-g(\alpha) = -\alpha^{p+2} + \alpha^p + 2 = \alpha^p(-\alpha^2+1) + 2 = 0.$$

Thus, two cases occur.

If α is negative and p is odd, then we obtain $-\left(\dfrac{p}{p+2}\right)^{\frac{p}{2}}\left(\dfrac{2}{p+2}\right) + 2 = 0$. So we may write $(p)^{\frac{p}{2}} = (p+2)^{\frac{p+2}{2}}$. Since $p \ge 2$, it is a contradiction.

If p is even or α is positive, then we obtain $\left(\dfrac{p}{p+2}\right)^{\frac{p}{2}}\left(\dfrac{2}{p+2}\right) + 2 = 0$. Since $p \ge 2$, it is a contradiction. Thus the proof is complete.

Let $g(x)$ be the characteristic polynomial of the Jacobsthal-Padovan p-matrix $\mathbf{M}$, then $g(x) = x^{p+2} - x^p - 2$. If $x_1, x_2, \ldots, x_{p+2}$ are eigenvalues of the matrix $\mathbf{M}$, then by Lemma 2.1, we have already known that $x_1, x_2, \ldots, x_{p+2}$ are distinct. Let $\mathbf{V}$ be a $(p+2)\times(p+2)$ Vandermonde matrix such that

$$\mathbf{V} = \begin{bmatrix} x_1^{p+1} & x_2^{p+1} & \cdots & x_{p+2}^{p+1} \\ x_1^{p} & x_2^{p} & \cdots & x_{p+2}^{p} \\ \vdots & \vdots & \cdots & \vdots \\ x_1 & x_2 & \cdots & x_{p+2} \\ 1 & 1 & \ldots & 1 \end{bmatrix}.$$

Let $\mathbf{V}(i,j)$ be a $(p+2)\times(p+2)$ matrix obtained from $\mathbf{V}$ by replacing the jth column of $\mathbf{V}$ by $\mathbf{X}_{p+2}^i$ where $\mathbf{X}_{p+2}^i$ is a $(p+2)\times 1$ such that

$$\mathbf{X}_{p+2}^{i}=\begin{bmatrix} x_1^{n+p+2-i} \\ x_2^{n+p+2-i} \\ \vdots \\ x_{p+2}^{n+p+2-i} \end{bmatrix}.$$

THEOREM 2.2. *Let* $\mathbf{M}^n=\left[m_{i,j}^{(n)}\right]$. *Then,*

$$m_{i,j}^{(n)}=\frac{\det\left(\mathbf{V}(i,j)\right)}{\det(\mathbf{V})}.$$

Proof. Since $x_1, x_2, \ldots, x_{p+2}$ are distinct, the matrix $\mathbf{M}$ is diagonalizable. Suppose that $\mathbf{D}=\left(x_1, x_2, \ldots, x_{p+2}\right)$, then we can write $\mathbf{MV}=\mathbf{VD}$. Since also $\mathbf{V}$ is invertible, $\left(\mathbf{V}\right)^{-1}\mathbf{M}\,\mathbf{V}=\mathbf{D}$. So we get that the matrix $\mathbf{M}$ is similar to $\mathbf{D}$. Therefore, we obtain the matrix equation $\mathbf{M}^n\mathbf{V}=\mathbf{VD}^n$ for $n\geq 1$. Since $\mathbf{M}^n=\left[m_{i,j}^{(n)}\right]$, we can write the following linear system of equations for $n\geq 1$:

$$\begin{aligned}
&m_{i,1}^{(n)}x_1^{p+1}+m_{i,2}^{(n)}x_1^{p}+\cdots+m_{i,p+2}^{(n)}=x_1^{n+p+2-i}\\
&m_{i,1}^{(n)}x_2^{p+1}+m_{i,2}^{(n)}x_2^{p}+\cdots+m_{i,p+2}^{(n)}=x_2^{n+p+2-i}\\
&\vdots\\
&m_{i,1}^{(n)}x_{p+2}^{p+1}+m_{i,2}^{(n)}x_{p+2}^{p}+\cdots+m_{i,p+2}^{(n)}=x_{p+2}^{n+p+2-i}.
\end{aligned}$$

From which we obtain

$$m_{i,j}^{(n)}=\frac{\det\left(\mathbf{V}\left(i,j\right)\right)}{\det(\mathbf{V})}\ \text{ for each } i,\ j=1,2,\ldots,p+2.$$

This yields the Binet-type formula for the Jacobsthal-Padovan *p*-numbers, namely:

COROLLARY 2.1. *Let* $JPa_p(n)$ *be the n th Jacobsthal-Padovan p-number, then*

$$JPa_p(n)=\frac{\det\left(\mathbf{V}\left(p+2,1\right)\right)}{\det(\mathbf{V})}=\frac{\det\left(\mathbf{V}\left(2,3\right)\right)}{2\cdot\det(\mathbf{V})}=\frac{\det\left(\mathbf{V}\left(p+1,p+2\right)\right)}{2\cdot\det(\mathbf{V})}.$$

Suppose that $f(x)$ is generating function of the Jacobsthal-Padovan *p*-numbers. Then
$f(x)=JPa_p(p+1)+JPa_p(p+2)x+JPa_p(p+3)x^2+\cdots+JPa_p(p+n+1)x^n+JPa_p(p+n+2)x^{n+1}+\cdots$.
So, we can write

$$f(x)-x^2f(x)-2x^{p+2}f(x)=JPa_p(p+1)=1.$$

From which we obtain

$$f(x)=\frac{1}{1-x^2-2x^{p+2}}. \tag{3}$$

for $0\leq x^2+2x^{p+2}<1$.

Now we can give an exponential representation for the Jacobsthal-Padovan *p*-numbers.

THEOREM 2.3. *Let $f(x)$ be generating function of the Jacobsthal-Padovan p-numbers, then*

$$f(x)=\exp\left(\sum_{i=1}^{\infty}\frac{x^{2i}}{i}\left(1+2x^{p}\right)^{i}\right).$$

Proof. From (3), we can write

$$\ln f(x)=\ln\left(1-x^{2}-2x^{p+2}\right)^{-1}=-\ln\left(1-x^{2}-2x^{p+2}\right)=$$

$$=-\left[-\left(x^{2}+2x^{p+2}\right)-\frac{1}{2}\left(x^{2}+2x^{p+2}\right)^{2}-\frac{1}{3}\left(x^{2}+2x^{p+2}\right)^{3}-\cdots-\frac{1}{n}\left(x^{2}+2x^{p+2}\right)^{n}-\cdots\right]$$

$$=x^{2}\left(1+2x^{p}\right)+\frac{x^{4}}{2}\left(1+2x^{p}\right)^{2}+\frac{x^{6}}{3}\left(1+2x^{p}\right)^{3}+\cdots+\frac{x^{2n}}{n}\left(1+2x^{p}\right)^{n}+\cdots=\sum_{i=1}^{\infty}\frac{x^{2i}}{i}\left(1+2x^{p}\right)^{i}.$$

Then,

$$f(x)=\exp\left(\sum_{i=1}^{\infty}\frac{x^{2i}}{i}\left(1+2x^{p}\right)^{i}\right).$$

Now we consider the combinatorial representations for the Jacobsthal-Padovan p-numbers. Let $h(x)=x^{2}+2x^{p+2}$ be such that $0\le x^{2}+2x^{p+2}<1$ and let u and n be positive integers. Since

$$\left(h(x)\right)^{u}=\left(x^{2}+2x^{p+2}\right)^{u}=\left(x^{2}(1+2x^{p})\right)^{u}=x^{2u}\sum_{i=0}^{u}\binom{u}{i}2^{i}x^{pi},$$

the coefficient of x^{n} in $\left(h(x)\right)^{u}$ is

$$\sum_{i=0}^{u}\binom{u}{i}2^{i},\quad \frac{n}{p+2}\le u\le\frac{n}{2}. \tag{4}$$

such that $2u+pi=n$.

Then we can give a combinatorial representation of the Jacobsthal-Padovan p-numbers by the following Theorem.

THEOREM 2.4. *Let m and n be positive integers, then*

$$JPa_{p}(n+p+1)=\sum_{\frac{n}{p+2}\le m\le\frac{n}{2}}\ \sum_{i=0}^{m}\binom{m}{i}2^{i}$$

where $2m+pi=n$.

Proof. It is easy to see that

$$f(x)=\frac{1}{1-x^{2}-2x^{p+2}}=\frac{1}{1-h(x)}=1+h(x)+\left(h(x)\right)^{2}+\cdots+\left(h(x)\right)^{n}+\cdots$$

$$=1+x^{2}\left(1+2x^{p}\right)+x^{4}\sum_{i=0}^{2}\binom{2}{i}2^{i}x^{pi}+\cdots+x^{2n}\sum_{i=0}^{n}\binom{n}{i}2^{i}x^{pi}+\cdots.$$

Since also

$$f(x)=JPa_{p}(p+1)+JPa_{p}(p+2)x+JPa_{p}(p+3)x^{2}+\cdots+$$
$$+JPa_{p}(p+n+1)x^{n}+JPa_{p}(p+n+2)x^{n+1}+\cdots,$$

the coefficient of x^{n} in $f(x)$ is $JPa_{p}(p+n+1)$. As we need the coefficient of x^{n}, we only consider the first $n+1$ terms on the right-side in the above equation. Then, the conclusion follows directly from (4).

Now we consider other combinatorial representations than the above for the Jacobstal-Padovan p-numbers.

Let $\mathbf{E}$ be a $l \times l$ companion matrix such that

$$\mathbf{E}(e_1, e_2, \cdots, e_l) = \begin{bmatrix} e_1 & e_2 & e_3 & \cdots & e_l \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

THEOREM 2.5 (Chen and Louck [3]). *The* (i, j) *entry* $e_{ij}^{(n)}(e_1, e_2, \cdots, e_l)$ *in the matrix* $\mathbf{E}^n(e_1, e_2, \cdots, e_l)$ *is given by the following formula:*

$$e_{ij}^{(n)}(e_1, e_2, \cdots, e_l) = \sum_{(t_1, t_2, \cdots, t_l)} \frac{t_j + t_{j+1} + \cdots + t_l}{t_1 + t_2 + \cdots + t_l} \times \binom{t_1 + t_2 + \cdots + t_l}{t_1, t_2, \cdots, t_l} e_1^{t_1} \cdots e_l^{t_l}, . \tag{5}$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + lt_l = n - i + j$, *and the coefficients in* (5) *are defined to be* 1 if $n = i - j$.

Then we can give some combinatorial representations for the Jacobsthal-Padovan p-numbers by the following Corollary.

COROLLARY 2.2. (i) $JPa_p(n) = \sum_{(t_1, t_2, \ldots, t_{p+2})} \binom{t_1 + t_2 + \cdots + t_{p+2}}{t_1, t_2, \ldots, t_{p+2}} 2^{t_{p+2}}$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + (p+2)t_{p+2} = n - p - 1$.

$$\text{(ii)} \quad JPa_p(n) = \sum_{(t_1, t_2, \ldots, t_{p+2})} \frac{t_3 + t_4 + \cdots + t_{p+2}}{t_1 + t_2 + \cdots + t_{p+2}} \binom{t_1 + t_2 + \cdots + t_{p+2}}{t_1, t_2, \ldots, t_{p+2}} 2^{t_{p+2} - 1}$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + (p+2)t_{p+2} = n + 1$.

$$\text{(iii)} \quad JPa_p(n) = \sum_{(t_1, t_2, \ldots, t_{p+2})} \frac{t_{p+2}}{t_1 + t_2 + \cdots + t_{p+2}} \binom{t_1 + t_2 + \cdots + t_{p+2}}{t_1, t_2, \ldots, t_{p+2}} 2^{t_{p+2} - 1}$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + (p+2)t_{p+2} = n + 1$.

Proof. If we take $i = p+2,\ j = 1$ for the case (i), $i = 2,\ j = 3$ for the case (ii) and $i = p+1,\ j = p+2$ for the case (iii) in Theorem 2.5, then we can directly see the conclusions from (5).

3. THE JACOBSTHAL-PADOVAN *P*-NUMBERS MODULO *m*

In this section, we obtain the cyclic groups which are generated by reducing the multiplicative orders of the Jacobsthal-Padovan p-matrix $\mathbf{M}$ and the auxiliary equation of the Jacobsthal-Padovan p-sequence modulo m. Also, we consider the Jacobsthal-Padovan p-sequence modulo m. Then, we present the relationships among orders of these cyclic groups and the periods of the Jacobsthal-Padovan p-sequences.

For given a matrix $\mathbf{A} = [a_{ij}]$ with m_{ij} integers, $\mathbf{A}(\bmod m)$ means that each element of $\mathbf{A}$ is reduced modulo m, that is, $\mathbf{A}(\bmod m) = (a_{ij} (\bmod m))$. Let us consider the set $\langle \mathbf{A} \rangle_m = \{\mathbf{A}^i(\bmod m) \mid i \geq 0\}$. If

$\gcd(m, \det \mathbf{A}) = 1$, then the set $\langle \mathbf{A} \rangle_m$ is a cyclic group. Let the notation $\left|\langle \mathbf{A} \rangle_m\right|$ denote the order of the set $\langle \mathbf{A} \rangle_m$. Since $\det \mathbf{M} = (-1)^{p+1} \cdot 2$, the set $\langle \mathbf{M} \rangle_m$ is a cyclic group for every positive odd integer m.

Now we consider the cyclic groups which are generated by the Jacobsthal-Padovan p-matrix $\mathbf{M}$.

THEOREM 3.1. *Let $u \neq 2$ be a prime and let α be the largest positive integer such that $\left|\langle \mathbf{M} \rangle_u\right| = \left|\langle \mathbf{M} \rangle_{u^\alpha}\right|$. Then $\left|\langle \mathbf{M} \rangle_{u^\lambda}\right| = u^{\lambda-\alpha} \cdot \left|\langle \mathbf{M} \rangle_u\right|$ for every $\lambda \geq \alpha$. In particular, if $\left|\langle \mathbf{M} \rangle_u\right| \neq \left|\langle \mathbf{M} \rangle_{u^2}\right|$, then $\left|\langle \mathbf{M} \rangle_{u^\lambda}\right| = u^{\lambda-1} \cdot \left|\langle \mathbf{M} \rangle_u\right|$ for every $\lambda \geq 2$.*

Proof. Let k be a positive integer and $\mathbf{I}$ be a $(p+2)\times(p+2)$ identity matrix. If $(\mathbf{M})^{\left|\langle \mathbf{M} \rangle_{u^{k+1}}\right|} \equiv \mathbf{I} \left(\bmod u^{k+1}\right)$, then $(\mathbf{M})^{\left|\langle \mathbf{M} \rangle_{u^{k+1}}\right|} \equiv \mathbf{I} \left(\bmod u^{k}\right)$. This yields that $\left|\langle \mathbf{M} \rangle_{u^k}\right|$ divides $\left|\langle \mathbf{M} \rangle_{u^{k+1}}\right|$. Also, writing $(\mathbf{M})^{\left|\langle \mathbf{M} \rangle_{u^k}\right|} = \mathbf{I} + \left(m_{i,j}^{(k)} \cdot u^k\right)$ we obtain

$$(\mathbf{M})^{\left|\langle \mathbf{M} \rangle_{u^k}\right| \cdot u} = \left(\mathbf{I} + \left(m_{i,j}^{(k)} \cdot u^k\right)\right)^u = \sum_{i=0}^{u} \binom{u}{i} \left(m_{i,j}^{(k)} \cdot u^k\right)^i \equiv \mathbf{I} \left(\bmod u^{k+1}\right)$$

by the binomial expansion. This means that $\left|\langle \mathbf{M} \rangle_{u^{k+1}}\right|$ divides $\left|\langle \mathbf{M} \rangle_{u^{k+1}}\right| \cdot u$. Then $\left|\langle \mathbf{M} \rangle_{u^{k+1}}\right| = \left|\langle \mathbf{M} \rangle_{u^k}\right|$ or $\left|\langle \mathbf{M} \rangle_{u^{k+1}}\right| = \left|\langle \mathbf{M} \rangle_{u^{k+1}}\right| \cdot u$. It is easy to see that $\left|\langle \mathbf{M} \rangle_{u^{k+1}}\right| = \left|\langle \mathbf{M} \rangle_{u^{k+1}}\right| \cdot u$ holds if and only if there is a $m_{i,j}^{(k)}$ which is not divisible by u. Since α is the largest positive integer such that $\left|\langle \mathbf{M} \rangle_u\right| = \left|\langle \mathbf{M} \rangle_{u^\alpha}\right|$, we have $\left|\langle \mathbf{M} \rangle_{u^\alpha}\right| \neq \left|\langle \mathbf{M} \rangle_{u^{\alpha+1}}\right|$. Then there exists an integer $m_{i,j}^{(\alpha+1)}$ which is not divisible by u. So we find that $\left|\langle \mathbf{M} \rangle_{u^{\alpha+1}}\right| \neq \left|\langle \mathbf{M} \rangle_{u^{\alpha+2}}\right|$. To complete the proof we use an inductive method on α.

Now we consider the Jacobsthal-Padovan p-numbers modulo m.

If we reduce the Jacobsthal-Padovan p-sequence $\left\{JPa_p(n)\right\}$ by a modulus m, then we get the repeating sequence, denoted by

$$\left\{JPa_p^m(n)\right\} = \left\{ JPa_p^m(1), JPa_p^m(2), \ldots, JPa_p^m(p+2), \ldots, JPa_p^m(i), \ldots\right\}$$

where $JPa_p^m(i) \equiv JPa_p(i) (\bmod m)$. It has the same recurrence relation as in (1).

It is well-known that a sequence is periodic if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the repeating subsequence is the period of the sequence. A sequence is simply periodic with period k if the first k elements in the sequence form a repeating subsequence.

Now we give some properties of the sequence $\left\{JPa_p^m(n)\right\}$ by the following theorems.

THEOREM 3.2. *The sequence $\left\{JPa_p^m(n)\right\}$ is periodic.*

Proof. Suppose that $Q = \left\{\left(q_1, q_2, \ldots, q_{p+2}\right) \,\middle|\, 0 \leq q_i \leq m-1\right\}$. Then we have $|Q| = m^{p+2}$. Since there are m^{p+2} distinct k-tuples of elements of Z_m, at least one of the $(p+2)$-tuples appears twice in the sequence $\left\{JPa_p^m(n)\right\}$. Thus, the subsequence following this $(p+2)$-tuple repeats and this implies that the sequence is periodic.

We denote the period of the sequence $\left\{JPa_p^m(n)\right\}$ by $l_p^J(m)$.

THEOREM 3.3. *Let m be a positive odd integer and let $m=\prod_{i=1}^{k} u_i^{e_i}$, $(k \geq 1)$ such that u_i's are distinct primes. Then* $l_p^J(m)=\operatorname{lcm}\left[l_p^J\left(u_1^{e_1}\right), l_p^J\left(u_2^{e_2}\right), \ldots, l_p^J\left(u_k^{e_k}\right)\right]$.

Proof. Since $l_p^J\left(u_i^{e_i}\right)$ is the length of the period of the sequence $\left\{JPa_p^{u_i^{e_i}}(n)\right\}$, the sequence $\left\{JPa_p^{u_i^{e_i}}(n)\right\}$ repeats only after blocks of length $k \cdot l_p^J\left(u_i^{e_i}\right)$, $(k \in N)$. Since also $l_p^J(m)$ is the length of the period $\left\{JPa_p^m(n)\right\}$, the sequence $\left\{JPa_p^{u_i^{e_i}}(n)\right\}$ repeats after $l_p^J(m)$ terms for all values i. This implies that $l_p^J(m)$ is the form $k \cdot l_p^J\left(u_i^{e_i}\right)$ for all values of i. We thus prove that $l_p^J(m)$ equals the least common multiple of $l_p^J\left(u_i^{e_i}\right)$'s.

We give the relationship between the period $l_p^J(m)$ and $\left|\langle\mathbf{M}\rangle_m\right|$ by the following theorem.

THEOREM 3.4. *If $u \neq 2$ is a prime and k is a positive integer, then* $l_p^J\left(u^k\right)=\left|\langle\mathbf{M}\rangle_{u^k}\right|$.

Proof. It is clear that $l_p^J\left(u^k\right)$ divides $\left|\langle\mathbf{M}\rangle_{u^k}\right|$. So we need only to prove that $l_p^J\left(u^k\right)$ is divisible by $\left|\langle\mathbf{M}\rangle_{u^k}\right|$. By (2), we know that $\mathbf{M}^{l_p^J\left(u^k\right)}\left(\bmod u^k\right) \equiv \mathbf{I}$, where $\mathbf{I}$ is the $(p+2)\times(p+2)$ identity matrix. From which we obtain that $\left|\langle\mathbf{M}\rangle_{u^k}\right|$ divides $l_p^J\left(u^k\right)$. Thus we have the conclusion.

Let $u \neq 2$ be a prime and let

$$A(u)=\left\{x^n(\bmod u): n \in Z^{+} \cup\{0\},\ x^{p+2}=x^p+2\right\}$$

Then, it is clear that the set $A(u)$ is a cyclic group.

Now we can give relationships among the auxiliary equation of the Jacobsthal-Padovan p-sequence and the period $l_p^J(m)$ by the following Corollary.

COROLLARY 3.1. *Let $u \neq 2$ be a prime. Then, the cyclic group $A(u)$ is isomorphic to the cyclic group $\langle\mathbf{M}\rangle_u$.*

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