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A MATHEMATICAL MODEL AND THE OPTIMAL STRATEGY IN THE TRANSACTIONS BETWEEN ONE BANK AND THE CENTRAL BANK

Elena Cristina CANEPA[1], Traian A. PIRVU[2]

[1] "Politehnica" University of Bucharest, Department of Mathematical Methods and Models, Romania
[2] McMaster University, Department of Mathematics & Statistics, Hamilton, Canada, E-mail: tpirvu@math.mcmaster.ca
Corresponding author: Elena Cristina Canepa, E-mail: cristinacanepa@yahoo.com

Abstract. We present a new mathematical model, based on stochastic control, for an economy formed by one bank and the Central Bank. In comparison to [6], the model allows for transactions to be discounted at different rates. We formulate and solve the bank problem of finding the optimal strategy when the underlying process is modeled by a Brownian motion with drift. We extend the model to involve the bank's asset size. In comparison to [1] and [3], we obtain that the optimal upper barrier for selling is a linear function of the asset size. As a consequence, using the double Skorokhod formula, the net purchase amount turns to be linear in the asset size.

Key words: Brownian motion with drift, Double Skorokhod formula, optimal control, barrier strategy.

1. INTRODUCTION

The model of this paper extends the optimal control problem of [2] and [6], by allowing for the transactions to be discounted at possibly different rates. We formulate and solve the problem by giving the optimal value function and the optimal strategy. A different approach is to use the entropy maximization as in [9, 10].

As in [1] and [2], our model gives an optimal net purchase amount as an output, using the martingale/ supermartingale principle (see e.g. [8]) and the double Skorokhod formula. The model is based on the asset size as in [3]. We obtain that the net purchase amount is increasing in the asset size. Anecdotally, large banks were known to be buyers of funds whereas small, risk-averse banks were known to be sellers of funds (see [4,11]). Therefore, our model puts in evidence this so-called small bank – large bank dichotomy.

The paper is structured as follows. In Section 2, we present the model and the main assumptions. In Section 3, we give the problem formulation and present the objective of the paper. In Section 4, we present the main results and discuss a particular interesting case. The paper ends with an appendix containing the proofs.

2. THE MODEL

Let us consider the problem of a bank which has an exogenously given demand deposit (net of withdrawals) and continuously sells and buys funds, thus lowering or increasing the excess reserves, defined as the difference between deposits and required reserves. We assume that the bank's only source of funds are demand deposits and funds from the Central Bank. We consider that during the business day, the bank can increase/ decrease its level of federal funds through direct transactions, which involve transaction costs.

The bank is thus characterized by the following stochastic processes:

1. A demand deposit process $\left(D_t\right)_{t\geq 0}$.
2. A required reserve process $\left(R_t\right)_{t\geq 0}$, where $R_t = qD_t$ and $q \in (0,1)$.

3. An excess reserve process $X_t = (1-q)D_t$.

Therefore, modeling the deposits D is equivalent to modeling the excess reserves X.

Let (Ω, F, P_x) be a probability space rich enough to accommodate a standard one-dimensional Brownian motion $B = (B_t, 0 \le t \le \infty)$ and such that $P_x(X_0 = x) = 1$, where $x \ge 0$. The excess reserve process is assumed to fluctuate over time as follows:

$$\mathrm{d}X_t = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}B_t, \tag{1}$$

where μ, σ are constants $(\mu \in R, \sigma > 0)$.

We consider $F = (F_t)_{t \ge 0}$ to be the completion of the augmented filtration generated by X (so that (F_t) satisfies the usual conditions).

Therefore, the bank observes nothing except the sample path of X.

Definition 2.1. A policy is defined as a pair of processes L and U such that

$$L,\ U \text{ are } \boldsymbol{F}-\text{adapted, continuous, increasing and positive.} \tag{2}$$

In the context of our idealized market, L_t and U_t are the cumulative funds purchases and funds sales (from the Central Bank) that the bank undertakes up to time *t*, in order to satisfy the reserve requirements and to maximize its profit.

Let us take λ_1 and $\lambda_2, \lambda_1 \ge \lambda_2$ be the interest rates at which the bank lends and borrows funds, respectively.

Definition 2.2. A controlled process associated to the policy (L, U) is a process $Z = X + L - U$. Using the formula for X, we obtain the decomposition of Z into its continuous part and its finite variation part:

$$\mathrm{d}Z_t = \mu\mathrm{d}t + \sigma\,\mathrm{d}B_t + \mathrm{d}L_t - \mathrm{d}U_t. \tag{3}$$

In our model Z_t is the amount of excess funds in the bank's reserve account at time *t*.

Definition 2.3. The policy (L, U) is said to be feasible if

$$L_{0-} = U_{0-} = 0, \tag{4}$$

$$P_x\{Z_t \ge 0, \forall t\} = 1,\ \ \forall x \ge 0, \tag{5}$$

$$E_x\left[\int_0^\infty \mathrm{e}^{-\lambda_i t}\mathrm{d}L\right] < \infty,\ \ \forall x \ge 0,\ \ i = 1,2, \tag{6}$$

$$E_x\left[\int_0^\infty \mathrm{e}^{-\lambda_1 t}\mathrm{d}U\right] < \infty,\ \ \forall x \ge 0. \tag{7}$$

We denote by $S(x)$ the set of all feasible policies associated with the continuous process X that starts at $x \ge 0$.

We assume that the bank can continuously sell and buy funds, thus lowering or increasing its excess reserve account.

It is considered, as in [3], that there are three types of transaction costs: a proportional transaction cost α of buying funds, a proportional transaction cost β of selling funds and a continuous holding cost, incurred at the rate *h*.

3. THE PROBLEM FORMULATION

3.1. The cost function

The *cost function* associated to the feasible policy (*L, U*) is defined to be

$$k_{L,U}(x) \equiv E_x\left[\int_0^\infty \left[\mathrm{e}^{-\lambda_1 t}\left(hZ_t\mathrm{d}t+\beta\mathrm{d}U\right)+\left(n\,\mathrm{e}^{-\lambda_1 t}+\left(1-n\right)\mathrm{e}^{-\lambda_2 t}\right)\alpha\mathrm{d}L\right]\right], \quad x\geq 0, \tag{8}$$

with $n\in\left[0,1\right]$.

Remark 1. We consider that the cumulative lent funds and the held funds are discounted at the same rate. The cumulative funds purchases and funds sales are discounted at possible different rates. If $n=1$ then the discounting occurs at the same rate λ_1. The discount function $n\mathrm{e}^{-\lambda_1 t}+(1-n)\mathrm{e}^{-\lambda_2 t}$, $n\in\left[0,1\right]$ was considered in [5] and leads to a time-changing discount rate in the interval $\left[\lambda_2,\lambda_1\right]$.

3.2. The objective

The bank's reserve management and profit-making problem is to find the optimal strategy $\left(L^\wedge,U^\wedge\right)$ which minimizes the cost.

The control $\left(L^\wedge,U^\wedge\right)$ is said to be *optimal* if $k_{L^\wedge,U^\wedge}(x)$ is minimal among the cost functions $k_{L,U}(x)$ associated with feasible policies (*L, U*), for each fixed $x\geq 0$.

The problem of minimizing the cost can be translated to the task of maximizing a value function. This function is easier to work with and it turns out to have particular characteristics, when the policy is of a barrier type. Further on, we present the relation between the cost function and the gain function. The particular case *n* = 1 was discussed in [6].

3.3. The gain function

The *gain function* is defined by

$$v_{L,U}(x) \equiv E_x\left\{\int_0^\infty \mathrm{e}^{-\lambda_1 t}\left(r\,\mathrm{d}U-c\,\mathrm{d}L\right)\right\}-E_x\left\{\int_0^\infty \mathrm{e}^{-\lambda_2 t}(1-n)\,\alpha\,\mathrm{d}L\right\}, \quad x\geq 0, \tag{9}$$

where $r\equiv h/\lambda_1-\beta$ and $c\equiv h/\lambda_1+n\alpha$.

Then extending the arguments from [6] one gets the following Lemma.

LEMMA 1. *The relation between the cost function and the gain function is*

$$k_{L,U}(x)=hx/\lambda_1+h\mu/\lambda_1^2-v_{L,U}(x),\;\; x\geq 0. \tag{10}$$

4. THE OPTIMAL POLICIES

4.1. The barrier policies

Let $b>0$ be a real fixed number. We consider that $X_0=x\in\left[0,b\right]$.

Definition 4.1. The barrier policies are the set of policies $\left(L,U\right)\in S(x)$ that satisfy:

1. $\left(L,U\right)$ continuous on $\left(0,\infty\right)$, increasing, $L_{0-}=U_{0-}=0$,

2. $Z_t \equiv X_t + L_t - U_t \geq 0,\ \forall t \geq 0,$ almost surely, and
3. $\int_0^\infty I_{Z_t>0}\,\mathrm{d}L_t = 0,\ \int_0^\infty I_{Z_t<b}\,\mathrm{d}U_t = 0$ almost surely.

A barrier policy (*L, U*) satisfies:

$$L_t = \sup_{0\le s\le t}\left(X_s - U_s\right)^-,\quad U_t = \sup_{0\le s\le t}\left(b - X_s - L_s\right)^-, \tag{11}$$

where x^- denotes the negative part of *x*. Moreover, the Double Skorokhod Formula obtained in [7] can be translated into a formula for the bank's net transaction amount $L-U$, as shown in [2]:

$$L_t - U_t = -\left[\left(X_0 - b\right)^+ \wedge \inf_{u\in[0,t]} X_u\right] \vee \sup_{s\in[0,t]}\left[\left(X_s - b\right)\wedge \inf_{u\in[s,t]} X_u\right]. \tag{12}$$

4.2. The optimal policy

Let $-\gamma_1, \underline{\gamma_1}$ be the roots of $\sigma^2\gamma^2/2 + \mu\gamma - \lambda_1 = 0$,

$$\gamma_1 \equiv \frac{\sqrt{\mu^2 + 2\sigma^2\lambda_1} + \mu}{\sigma^2} > 0,\quad \underline{\gamma_1} \equiv \frac{\sqrt{\mu^2 + 2\sigma^2\lambda_1} - \mu}{\sigma^2} > 0. \tag{13}$$

Define

$$g(x) \equiv \gamma_1 \mathrm{e}^{\underline{\gamma_1}x} + \underline{\gamma_1}\mathrm{e}^{-\gamma_1 x}. \tag{14}$$

Then $g(0) > 0,\ g'(0) = 0$ and g is strictly decreasing and continuous on $(-\infty, 0]$. Hence there must be a point $-b < 0$ such that

$$g(-b) = g(0)c/r. \tag{15}$$

Let γ_2 be the positive root of $\sigma^2\gamma^2/2 + \mu\gamma - \lambda_2 = 0$

$$\gamma_2 \equiv \frac{\sqrt{\mu^2 + 2\sigma^2\lambda_2} - \mu}{\sigma^2} > 0. \tag{16}$$

Define

$$\begin{aligned} v_1(x) &\equiv \frac{r}{g'(b)} g(x) + \frac{c}{g'(-b)} g(x-b) \quad \text{if } 0 \le x \le b \\ v_1(x) &\equiv v_1(b) - (x-b)\, r \quad \text{if } x > b \end{aligned} \tag{17}$$

and

$$v_2(x) \equiv -\frac{(1-n)\alpha}{\gamma_2}\mathrm{e}^{-\gamma_2 x}. \tag{18}$$

PROPOSITION 1. *The barrier policy* $\left(L^\wedge, U^\wedge\right)$ *associated with b solution of* $g(-b) = g(0)c/r$ *(15) is feasible, i.e.* $\left(L^\wedge, U^\wedge\right) \in S\left(x\right)$*. Moreover*

$$v_1(x) \equiv E_x\left\{\int_0^\infty e^{-\lambda_1 t}\left(r\,\mathrm{d}U^\wedge - c\,\mathrm{d}L^\wedge\right)\right\},\quad x \ge 0, \tag{19}$$

$$v_2(x) \equiv -E_x\left\{\int_0^\infty e^{-\lambda_2 t}(1-n)\alpha\, \mathrm{d}L^\wedge\right\}, \quad x \geq 0, \tag{20}$$

Therefore

$$v_{L^\wedge,U^\wedge}(x) = v_1(x) + v_2(x). \tag{21}$$

4.3. Main result

The following is the main result of our paper.

THEOREM 1. *The barrier policy* $\left(L^\wedge, U^\wedge\right)$ *associated with* b *given by* $g(-b) = g(0)c/r$ *(15) is optimal, i.e., for every* $\left(L,U\right) \in S\left(x\right)$

$$v_{L,U}(x) \leq v_{L^\wedge,U^\wedge}(x). \tag{22}$$

4.4. Special case

Let us take $n = 1$ so that we have the same discount rate λ_1. Moreover, let the drift μ and the volatility σ depend on the bank size A. Inspired by [1] we take μ and σ linear in A, i.e., $\mu = k_1 A$, $\sigma = k_2 A$, with $k_1 \in R$, $k_2 > 0$.

COROLLARY 1. *The barrier b is linear in the bank size A. Consequently,* $L - U$ *is increasing in the bank size A.*

Remark 2. This result can be used by a bank to develop a strategy for selling funds when its controlled excess reserve process hits this upper optimal barrier b, i.e. a certain percent of its assets' size. This corollary is consistent with the so-called small bank-large bank dichotomy, meaning that the bigger the size of the bank, the larger the net purchase amount that the bank undertakes.

APPENDICES

Appendix A: Proof of Theorem 1

The idea of the proof is based on the martingale/ supermartingale principle. In a first step we show that some processes are supermartingales.

LEMMA A. *For every* $\left(L,U\right) \in S\left(x\right)$ *with* $Z = X + L - U$ *the process*

$$\mathrm{e}^{-\lambda_1 t} v_1\left(Z_t\right) + \int_0^t \mathrm{e}^{-\lambda_1 s}\left[r\,\mathrm{d}U - c\,\mathrm{d}L\right], \quad t \geq 0$$

is supermartingale. Moreover with $\overline{Z} = X + L$ *the process*

$$\mathrm{e}^{-\lambda_2 t} v_2\left(\overline{Z}_t\right) + \int_0^t \mathrm{e}^{-\lambda_2 s}\left[(n-1)\right]\alpha\,\mathrm{d}L, \quad t \geq 0$$

is supermartingale.

Therefore for a fixed $T > 0$, by taking expectations we get

$$E_x\left\{\int_0^T \mathrm{e}^{-\lambda_1 t}\left(r\,\mathrm{d}U - c\,\mathrm{d}L\right)\right\} \leq v_1(x) - E_x\left[e^{-\lambda_1 T} v_1\left(Z_T\right)\right].$$

Next, the positivity of Z, the linearity of $v_1(z)$ for large z, the integrability conditions (6), (7) and the Dominated Convergence Theorem yield that

$$E_x\left\{\int_0^\infty \mathrm{e}^{-\lambda_1 t}\left(r\,\mathrm{d}U - c\,\mathrm{d}L\right)\right\} \le v_1(x).$$

Similarly

$$E_x\left\{\int_0^T \mathrm{e}^{-\lambda_2 t}(n-1)\,\mathrm{d}L\right\} \le v_2(x) - E_x\left[\mathrm{e}^{-\lambda_2 T} v_2\left(\bar{Z}_T\right)\right].$$

The positivity of $\bar{Z}$, the boundedness of $v_2(z)$ for positive z, the integrability condition (6) from Definition 2.3 and the Dominated Convergence Theorem yield that

$$E_x\left\{\int_0^\infty \mathrm{e}^{-\lambda_2 t}(n-1)\,\mathrm{d}L\right\} \le v_2(x).$$

By adding these inequalities we get

$$E_x\left\{\int_0^\infty \mathrm{e}^{-\lambda_1 t}\left(r\,\mathrm{d}U - c\,\mathrm{d}L\right)\right\} - E_x\left\{\int_0^T \mathrm{e}^{-\lambda_2 t}(1-n)\,\mathrm{d}L\right\} \le v_1(x) + v_2(x)$$

However, by Proposition 1

$$v_{L^\wedge, U^\wedge}(x) = v_1(x) + v_2(x)$$

which proves optimality of $\left(L^\wedge, U^\wedge\right)$.

Proof of Lemma A. Recall that

$$\Gamma v_1(z) = \lambda_1 v_1(z),\ z \in [0,b],\quad v_1'(0) = c,\quad v_1'(b) = r, \tag{23}$$

Moreover

$$\Gamma v_1(z) \le \lambda_1 v_1(z),\quad r \le v_1'(z) \le c,\quad z \ge 0. \tag{24}$$

By Ito's Lemma for processes with jumps

$$\mathrm{d}\left(\mathrm{e}^{-\lambda_1 t} v_1\left(Z_t\right) + \int_0^t \mathrm{e}^{-\lambda_1 s}\left[r\,\mathrm{d}U - c\,\mathrm{d}L\right]\right) = \mathrm{e}^{-\lambda_1 t}\left(\Gamma v_1 - \lambda v_1\right)\left(Z_t\right)\mathrm{d}t + \mathrm{e}^{-\lambda_1 t}\left(v_1'(Z_t) - c\right)\mathrm{d}\tilde{L} +$$
$$+ \mathrm{e}^{-\lambda_1 t}\left(r - v_1'(Z_t)\right)\mathrm{d}\tilde{U} + \sum_{0 \le s \le t} e^{-\lambda s}\left(\Delta v_1(Z_s) - c\Delta L_s + r\Delta U_s\right)$$

where $\mathrm{d}\tilde{L} = \mathrm{d}L - \Delta L$ and $\mathrm{d}\tilde{U} = \mathrm{d}U - \Delta U$. Using the boundedness of v_1' (24) the claim yields if we prove that

$$\sum_{0 \le s \le t} e^{-\lambda s}\left(\Delta v_1(Z_s) - c\,\Delta L_s + r\,\Delta U_s\right) \le 0.$$

Suppose that $\Delta L_t > 0$ and $\Delta U_t = 0$ (the other cases are similar). Then $\Delta Z_t = \Delta L_t$ and

$$\Delta v_1(Z_t) - c\Delta L_t + r\Delta U_t = v_1(Z_t) - v_1\left(Z_t - \Delta L_t\right) - c\Delta L_t.$$

The last quantity is negative because $v_1'(z) \le c,\ z \ge 0.$ Recall that

$$\Gamma v_2 = \lambda_2 v_2,\quad v_2'(0) = (1-n)\alpha. \tag{25}$$

Moreover

$$v_2'(z) \le (1-n)\alpha,\quad z \ge 0. \tag{26}$$

By Ito's Lemma for processes with jumps

$$\mathrm{d}\left(e^{-\lambda_2 t}v_2\left(\bar{Z}_t\right)+\int_0^t \mathrm{e}^{-\lambda_2 s}\left[(n-1)\alpha \mathrm{d}L\right]\right)=\mathrm{e}^{-\lambda_2 t}\left(\Gamma v_2-\lambda v_1\right)\left(\bar{Z}_t\right)\mathrm{d}t+$$
$$+\mathrm{e}^{-\lambda_2 t}\left(v_2'(\bar{Z}_t)-(1-n)\alpha\right)\mathrm{d}\tilde{L}+\sum_{0\le s\le t}\mathrm{e}^{-\lambda s}\left(\Delta v_2(\bar{Z}_s)-(1-n)\alpha\Delta L_s\right).$$

Using the boundedness of v_2' (26), the claim yields if we prove that

$$\sum_{0\le s\le t}\mathrm{e}^{-\lambda s}\left(\Delta v_2(\bar{Z}_s)-(1-n)\alpha\Delta L_s\right)\le 0.$$

Suppose that $\Delta L_t>0,$ then $\Delta\bar{Z}_t=\Delta L_t$ and

$$\Delta v_2(Z_t)-(1-n)\alpha\Delta L_t=v_2(Z_t)-v_2\left(Z_t-\Delta L_t\right)-(1-n)\alpha\Delta L_t$$

The last quantity is negative because $v_2'(z)\le(1-n)\alpha,\ z\ge 0$.

Appendix B: Proof of Corollary 1

In light of $\mu=k_1A,\ \sigma=k_2A,$ if we let γ_{11} and γ_{12} be the roots of

$$\sigma^2\gamma^2/2+\mu\gamma-\lambda_1=0$$

it follows that $\gamma_{11}=a_1/A,\ \gamma_{12}=a_2/A,$ for some constants a_1, a_2. Recall that

$$g(x)\equiv\gamma_{11}\mathrm{e}^{\gamma_{12}x}+\gamma_{12}\mathrm{e}^{-\gamma_{11}x},$$

whence

$$g(-b)/g(0)=F\left(b/A\right),$$

for some function F. It follows that b, the optimal upper barrier (Section 4.2), should solve

$$F\left(b/A\right)=c/r. \tag{27}$$

Therefore $b=kA,$ for some positive constant k.

Given $X_0\in\left[0,b\right],$ the Double Skorokhod Formula yields (see [2]) that $L-U$ is increasing in the barrier b. Since $b=kA,$ for some positive constant k, then $L-U$ is increasing in the bank size A.

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