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A NEW BERNSTEIN-STANCU TYPE OPERATOR WITH NEGATIVE PARAMETER

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Abstract. Using Pólya's urn model with negative replacement we introduce a new Bernstein-Stancu type operator and we show that the new operator improves the estimates for the classical Bernstein operator. We also provide numerical evidence showing that the new operator gives a better approximation when compared to some other classical Bernstein-type operators.

Key words: Bernstein operator, Bernstein-Stancu operator, Pólya urn model, positive linear operator, approximation theory.

1. INTRODUCTION

About a hundred years ago, in his beautiful and short paper ([1], 2 pages), Serge Bernstein gave a (probabilistic) proof of Weierstrass's theorem on uniform approximation by polynomials, with a constructive method of approximation, known nowadays as Bernstein's polynomials.

The probabilistic idea behind Bernstein's construction can be seen as follows. If X_n is random variable with a binomial distribution with parameters $n \in \mathbb{N}^*$ (number of trials) and $p \in [0,1]$ (probability of success), then $E(X_n/n) = p$. Choosing $p = x \in [0,1]$, we have $E(X_n/n) = x$, and since the variance $\sigma^2(X_n/n) = x(1-x)/n$ is small for n sufficiently large, heuristically we have $X_n/n \approx x$, and if $f:[0,1] \to R$ is continuous we also have $f(X_n/n) \approx f(x)$. Taking expectation leads to Bernstein's polynomials

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) C_n^k x^k (1-x)^{n-k} = Ef\left(\frac{X_n}{n}\right) \approx f(x), \tag{1}$$

and Bernstein's proof shows that this intuition is indeed correct: if f is continuous on $[0,1]$, then B_n converges uniformly to f on $[0,1]$ for $n \to \infty$.

Aside from their importance in Analysis, Bernstein's polynomials generated an important area of research in various fields of Mathematics and Computer Science, which continues to develop even today. The Bernstein polynomials were intensively studied in Operator Theory and Approximation Theory, where they were generalized by several authors, for example by F. Schurer (Bernstein-Schurer operator, [15]), D. D. Stancu (Bernstein-Stancu operator, [17]), A. Lupaş (Lupaş operator, [7], and q-Bernstein operator, [8]), G. M. Phillips (q-Bernstein operator, [14]), M. Mursaleen et. al. ((p,q)-Bernstein operator, [9]), and many others. See also [3] for a recent survey of Bernstein polynomials.

In the present paper we are concerned with a generalization of Bernstein polynomials based on Polya's urn distribution with (negative) replacement, the primary interest being the study of a new operator obtained for a particular choice of the parameters involved. Our main results (Theorem 5, Theorem 7, and Theorem 9) indicate that the new operator improves the approximation provided by the classical Bernstein operator, and the numerical results (Section 6) also show that the new operator gives a better approximation than some of the well-known Bernstein-type operators. The structure of the paper is as follows. In Section 2 we set up the notation and we review the basic properties of the Pólya's urn model, which will be used in the sequel.

In Section 3 we introduce the operator $P_n^{a,b,c}$ depending on $a,b\in\mathbb{R}_+$ and $c\in\mathbb{R}$ satisfying an additional hypothesis. Although for $c\ge 0$ the operator $P_n^{x,1-x,c}$ coincides with the classical Bernstein-Stancu operator ([17]), our primary interest in the present paper is to consider a particular negative value of c, for which the resulting operator R_n seems to have better approximation properties than other Bernstein-type operators (see Remark 2, and the results in Section 5 and Section 6).

The main properties of the operator R_n are given in Section 4, Theorem 3.

In Section 5 we give the error estimates for the operator R_n. Using a probabilistic lemma of independent interest (Lemma 4), in Theorem 5 we give a short proof of an error estimate for R_n using the modulus of continuity. The constant involved ($C=31/27\approx 1.14815$) is smaller than the corresponding constant obtained by Popoviciu ($C=1.5$) and Lorentz ($C=1.25$) in the case of Bernstein polynomial, but it is slightly larger than the optimal constant $C_{opt}\approx 1.08988$ obtained by Sikkema. In a subsequent paper ([10]), we will show that the actual value of the constant is in fact smaller than Sikkema's optimal constant. In Theorem 7 we give the error estimate in the case of a continuously differentiable function, and in Theorem 9 we give the asymptotic error estimate in the case of a twice continuously differentiable function. The paper concludes (Section 6) with some numerical results which suggest that the operator R_n provides a better approximation than other classical Bernstein-type operators.

2. PRELIMINARIES

Pólya's urn model (also known as Pólya-Eggenberger urn model, see [2], [12]) generalizes the classical urn model in which one observes balls extracted from an urn containing balls of two colors, by requiring that the extracted ball to be returned to the urn together with c balls of the same color (the case of a negative integer $c\in\mathbb{Z}$ is interpreted as removing $|c|$ balls from the urn). When c is negative, the model breaks down if there are insufficient many balls of the desired color in the urn, the conditions for which the model is meaningful (also indicated in original Pólya's paper) being

$$a+(n-1)c\ge 0 \qquad \text{and} \qquad b+(n-1)c\ge 0. \tag{2}$$

The above model assumes a,b,c to be integers, but it still defines a distribution for $a,b\in\mathbb{R}_+$ and $c\in\mathbb{R}$ satisfying (2), hypotheses which we will assume in the sequel.

NOTATION 1. For $x,h\in\mathbb{R}$ and $n\in\mathbb{N}$ we set $x^{(n,h)}=x(x+h)(x+2h)\cdot\ldots\cdot(x+(n-1)h)$ for the generalized (rising) factorial with increment h. We use the convention $x^{(0,h)}=1$ for any $x,h\in\mathbb{R}$.

A random variable $X_n^{a,b,c}$ with Pólya's urn distribution with parameters $n\ge 1$, $a,b\in\mathbb{R}_+$, and $c\in\mathbb{R}$ satisfying (2) is given by (see for example [4])

$$P\left(X_n^{a,b,c}=k\right)=p_{n,k}^{a,b,c}=C_n^k\frac{a^{(k,c)}b^{(n-k,c)}}{(a+b)^{(n,c)}}, \qquad k\in\{0,1,\ldots,n\}. \tag{3}$$

It is known (see for example [4]) that the mean and variance of $X_n^{a,b,c}$ are given by

$$E\left(X_n^{a,b,c}\right)=\frac{na}{a+b} \qquad \text{and} \qquad \sigma^2\left(X_n^{a,b,c}\right)=\frac{nab}{(a+b)^2}\left(1+\frac{(n-1)c}{a+b+c}\right). \tag{4}$$

3. A NEW BERNSTEIN-STANCU TYPE OPERATOR

Denote by $\mathscr{F}([0,1])$ the set of real-valued functions defined on $[0,1]$, and by $\mathscr{C}([0,1])$ the subset of continuous functions on $[0,1]$. Consider the operator $P_n^{a,b,c}:\mathscr{F}([0,1])\to\mathscr{F}([0,1])$, defined by

$$P_n^{a,b,c}(f;x)=Ef\left(\frac{1}{n}X_n^{a,b,c}\right)=\sum_{k=0}^{n}p_{n,k}^{a,b,c}f\left(\frac{k}{n}\right),\qquad f\in\mathscr{F}([0,1]),\quad x\in[0,1], \tag{5}$$

where the parameters a,b,c may depend on n and x, and satisfy $a,b\ge 0$ and the condition (2). If the parameters a,b,c depend continuously on $x\in[0,1]$, from (3) it follows the operator $P_n^{a,b,c}$ maps $\mathscr{F}([0,1])$ to $\mathscr{C}([0,1])$, and in particular it maps $\mathscr{C}([0,1])$ to $\mathscr{C}([0,1])$.

Note that in the case $a=x$, $b=1-x$ and $c=\alpha\ge 0$ the above is the probabilistic representation of the Bernstein-Stancu operator (introduced in [17])

$$P_n^{<\alpha>}(f;x)=P_n^{x,1-x,\alpha}(f;x)=\sum_{k=0}^{n}C_n^k\frac{x^{(k,\alpha)}(1-x)^{(n-k,\alpha)}}{1^{(n,\alpha)}}f\left(\frac{k}{n}\right), \tag{6}$$

which generalizes the classical Bernstein operator $B_n(f;x)$ (the case $\alpha=0$), and the choice $a=x$, $b=1-x$, and $c=1/n$ gives the probabilistic representation of the Lupaş operator (introduced in [7])

$$P_n^{<1/n>}(f;x)=P_n^{x,1-x,1/n}(f;x)=\sum_{k=0}^{n}C_n^k\frac{x^{(k,1/n)}(1-x)^{(n-k,1/n)}}{1^{(n,1/n)}}f\left(\frac{k}{n}\right). \tag{7}$$

Remark 2. As noted above, for $c\ge 0$ the operator $P_n^{x,1-x,c}$ is just the classical Bernstein-Stancu operator; however, our main interest in the present paper is to consider the case $c<0$, which does not seem to have been properly addressed in the literature. To be precise, in [17] Stancu indicates that the choice $\alpha=-1/n$ in (6) gives the Lagrange interpolating polynomial, which cannot be used for the uniform approximation of every continuous function on $[0,1]$, and concludes with "We will henceforth assume that the parameter α is non-negative". In a subsequent paper ([18]), Stancu considered a certain negative choice $\alpha=\alpha(n,\varepsilon)<0$ of the parameter ($0\le\varepsilon<1/2$), but the resulting operator is defined just on a proper subset $[\varepsilon,1-\varepsilon]$ of $[0,1]$.

We consider the particular choice $a=x$, $b=1-x$ and $c=-\min\{x,1-x\}/(n-1)$ of the operator $P_n^{a,b,c}$ defined above (for this choice of parameters the inequality (2) is satisfied for all $n>1$ and $x\in[0,1]$), and denote by $R_n:\mathscr{F}([0,1])\to\mathscr{F}([0,1])$ the operator which maps $f\in\mathscr{F}([0,1])$ to

$$R_n(f;x)=Ef\left(\frac{1}{n}X_n^{x,1-x,-\min\{x,1-x\}/(n-1)}\right)=\sum_{k=0}^{n}C_n^k\frac{x^{(k,-\min\{x,1-x\}/(n-1))}(1-x)^{(n-k,-\min\{x,1-x\}/(n-1))}}{1^{(n,-\min\{x,1-x\}/(n-1))}}f\left(\frac{k}{n}\right). \tag{8}$$

The only downsize in considering the negative value $c=-\min\{x,1-x\}/(n-1)$ above is that the operator R_n is no longer a polynomial operator in x, but rather a rational operator: on each of the intervals $[0,1/2]$ and $[1/2,1]$, $R_n(f;x)$ is a ratio of a polynomial of degree at most n in x and the polynomial $1^{(n,-\min\{x,1-x\}/(n-1))}$ of degree $n-1$ in x, which does not depend on f. However, the advantages of our choice are that it produces better approximation results than other classical operators (see the various error estimates for the operator R_n given in Section 5 and the numerical results in Section 6), and from the point of view of applications the operator R_n is as easily computable as a polynomial operator.

4. SOME PROPERTIES OF THE OPERATOR R_n

THEOREM 3. *For any* $n>1$, $R_n : \mathcal{F}([0,1]) \to \mathcal{C}([0,1])$ *is a positive linear operator, which maps the test functions* $e_0(x)\equiv 1$, $e_1(x)\equiv x$, *and* $e_2(x)\equiv x^2$ *respectively to*

$$R_n(e_0;x)=1,\ R_n(e_1;x)=x,\ R_n(e_2;x)=\frac{(n-1)x^2}{n}+\frac{x((n-1)-n\min\{x,1-x\})}{n(n-1-\min\{x,1-x\})}.$$

In particular, if $f:[0,1]\to\mathbb{R}$ *is continuous, then* $R_n(f;x)$ *converges to* $f(x)$ *uniformly on* $[0,1]$ *as* $n\to\infty$. *Further, if* $f:[0,1]\to\mathbb{R}$ *is a convex function,* $R_n(f;x)\ge f(x)$ *for any* $x\in[0,1]$.

Proof. The first claim follows easily from the definition (8) of R_n, using the linearity and positivity of the expected value. Using again the probabilistic representation of R_n and (4), we obtain:

$$R_n(e_0;x)=E(1)=1,\quad R_n(e_1;x)=\frac{1}{n}E\left(X_n^{x,1-x,-\min\{x,1-x\}/(n-1)}\right)=x,$$

$$R_n(e_2;x)=\frac{1}{n^2}E\left(\left(X_n^{x,1-x,-\min\{x,1-x\}/(n-1)}\right)^2\right)=\frac{(n-1)x^2}{n}+\frac{x\left(n-1-n\min\{x,1-x\}\right)}{n\left(n-1-\min\{x,1-x\}\right)}.$$

Since $R_n(e_i;x)\xrightarrow[n\to\infty]{} e_i(x)$ uniformly on $[0,1]$ for $i=0,1,2$, the second part follows now from preceding part of the theorem using the classical Bohman-Korovkin theorem. Finally, if f is convex, by Jensen's inequality we have $R_n(f;x)\ge f\left(E\left(X_n^{x,1-x,-\min\{x,1-x\}/(n-1)}/n\right)\right)=f(x)$.

5. ERROR ESTIMATES FOR THE APPROXIMATION BY THE OPERATOR $R_n(f;x)$

T. Popoviciu [11] proved the following bound for the approximation error for the Bernstein polynomial in the case of an arbitrary continuous function $f:[0,1]\to\mathbb{R}$

$$\left|B_n(f;x)-f(x)\right|\le C\omega\left(n^{-1/2}\right),\qquad x\in[0,1],\ n=1,2,..., \tag{9}$$

with $C=3/2$, where $\omega(\delta)=\omega^f(\delta)=\max\left\{\left|f(x)-f(y)\right|: x,y\in[0,1], |x-y|\le\delta\right\}$ denotes the modulus of continuity of f. Lorentz [6, pp. 20 –21] improved the value of the constant C above to $\frac{5}{4}$, and showed that the constant C cannot be less than one. The optimal value of the constant C for which the inequality (9) holds true for any continuous function was given by Sikkema [16], who obtained the value

$$C_{opt}=\frac{4306+837\sqrt{6}}{5932}\approx 1.0898873..., \tag{10}$$

attained in the case $n=6$ for a particular choice of f. We will show that the operator R_n defined by (8) also satisfies a Popoviciu type inequality. In order to give the result, we begin with the following auxiliary lemma which may be of independent interest. We note that although related estimates appear in the literature, we could not find a reference for them in the present form. For example, a result in the same spirit with a) below appears in [5, Theorem 1], but there $\delta=n^{-1/2}$, and the right hand side is replaced by the supremum of the corresponding inequality in (5.3).

Lemma 4. Let X *be a discrete random variable taking values in an interval* $[a,b]\subset\mathbb{R}$*, with finite mean* $E(X)=x$ *and variance* $\sigma^2(X)$*, and let* $f:[a,b]\to\mathbb{R}$ *for which* $f(X)$ *has finite mean.*

a) If f *is continuous on* $[a,b]$*, the for any* $\delta>0$ *we have*

$$\left|Ef(X)-f(x)\right|\le\omega(\delta)\left(1+\sigma^2(X)/\delta^2\right), \tag{11}$$

where $\omega(\delta)=\omega^f(\delta)$ *denotes the modulus of continuity of* f.

b) If f *is continuously differentiable on* $[a,b]$*, we have*

$$\left|Ef(X)-f(x)\right|\le\omega_1(\delta)\left(\sigma^2(X)/\delta^2+\sigma(X)\right). \tag{12}$$

where $\omega_1(\delta)=\omega_1^f(\delta)$ *denotes the modulus of continuity of* f'.

c) Finally, if f *is twice continuously differentiable on* $[a,b]$*, we have*

$$Ef(X)=f(x)+\frac{1}{2}f''(x)\sigma^2(X)+R(X), \tag{13}$$

and there exists $M>0$ *such that for each* $\varepsilon>0$ *there exists* $\delta=\delta(\varepsilon)>0$ *such that*

$$\left|R(X)\right|\le\varepsilon\sigma^2(X)+(b-a)^2MP\left(|X-x|>\delta\right), \tag{14}$$

where $M>0$ *and* $\delta=\delta(\varepsilon)>0$ *depend on* f*, but not on* X *or* x.

Proof. Denoting by $F:\mathsf{R}\to\mathsf{R}$ the distribution function of X, we have

$$\left|Ef(X)-f(x)\right|=\left|\int_a^b f(y)-f(x)\mathrm{d}F(y)\right|\le\int_a^b\left|f(y)-f(x)\right|\mathrm{d}F(y).$$

It is not difficult to see that two arbitrary points $x,y\in[a,b]$ are at most $\left[|y-x|/\delta\right]+1$ intervals of length $\delta>0$ apart ($\left[|y-x|/\delta\right]\in\mathbb{N}$ denotes here the integer part of $|y-x|/\delta$). Using this, the definition of the modulus of continuity $\omega(\delta)$ of f, and the above, we obtain

$$\begin{aligned}\left|Ef(X)-f(x)\right|&\le\omega(\delta)\left(1+\int_a^b\left[|\xi-x|/\delta\right]\mathrm{d}F(y)\right)\le\omega(\delta)\left(1+\int_a^b\left[|\xi-x|/\delta\right]^2\mathrm{d}F(y)\right)\\&=\omega(\delta)\left(1+\int_a^b|y-x|^2/\delta^2\mathrm{d}F(y)\right)=\omega(\delta)\left(1+\sigma^2(X)/\delta^2\right)\end{aligned}$$

since by hypothesis $E(X)=x$. To prove the second part, applying the mean value theorem, we have

$$f(\alpha)-f(\beta)=f'(\gamma)(\alpha-\beta)=f'(\beta)(\alpha-\beta)+\left(f'(\gamma)-f'(\beta)\right)(a-\beta),$$

for arbitrary points $\alpha,\beta\in[a,b]$, where γ is an intermediate point between α and β. Using this with $\alpha=y$ and $\beta=x$, we obtain

$$Ef(X)-f(x)=\int_a^b\left(f(y)-f(x)\right)\mathrm{d}F(y)=\int_a^b f'(x)(y-x)+\left(f'(\xi)-f'(x)\right)(y-x)\mathrm{d}F(y),$$

where $\xi=\xi(x,y)$ is an intermediate point between y and x. Applying a similar argument as above to the modulus of continuity ω_1 of f', and using the Cauchy-Schwarz inequality, we obtain

$$\left|Ef\left(X\right)-f\left(x\right)\right|\le\left|f'\left(x\right)\int_a^b y-x\mathrm{d}F\left(y\right)\right|+\omega_1\left(\delta\right)\int_a^b\left(\left[\left|\xi-x\right|/\delta\right]+1\right)\left|y-x\right|\mathrm{d}F\left(y\right)$$
$$\le\left|f'\left(x\right)\left(M\left(X\right)-x\right)\right|+\omega_1\left(\delta\right)\left(\int_a^b\left[\left|\xi-x\right|/\delta\right]\left|y-x\right|\mathrm{d}F\left(y\right)+\int_a^b\left|y-x\right|\mathrm{d}F\left(x\right)\right)$$
$$\le\omega_1\left(\delta\right)\left(\frac{1}{\delta}\int_a^b\left|y-x\right|^2\mathrm{d}F\left(y\right)+\left(\int_a^b\left|y-x\right|^2\mathrm{d}F\left(x\right)\right)^{1/2}\right)=\omega_1\left(\delta\right)\left(\sigma^2\left(X\right)/\delta+\sigma\left(X\right)\right).$$

For the last part of the lemma, using Taylor's theorem we obtain

$$f\left(y\right)-f\left(x\right)=f'\left(x\right)\left(y-x\right)+\frac{1}{2}f''\left(x\right)\left(y-x\right)^2+\alpha\left(y\right)\left(y-x\right)^2,\qquad y\in\left[a,b\right],$$

where $\alpha:\left[a,b\right]\to\mathsf{R}$ is bounded on $\left[a,b\right]$, say by $M>0$, and satisfies $\lim_{y\to x}\alpha\left(y\right)=0$. In particular, for any $\varepsilon>0$ there exists $\delta=\delta\left(\varepsilon\right)>0$ such that $\left|\alpha\left(y\right)\right|<\varepsilon$ for $\left|y-x\right|<\delta$. We obtain

$$Ef\left(X\right)-f\left(x\right)=\int_a^b\left(f\left(y\right)-f\left(x\right)\right)\mathrm{d}F\left(y\right)=$$
$$=f'\left(x\right)\int_a^b\left(y-x\right)\mathrm{d}F\left(y\right)+\frac{1}{2}f''\left(x\right)\int_a^b\left(y-x\right)^2\mathrm{d}F\left(y\right)+\int_a^b\alpha\left(y\right)\left(y-x\right)^2\mathrm{d}F\left(y\right)$$
$$=f'\left(x\right)\left(M\left(X\right)-x\right)+\frac{1}{2}f''\left(x\right)\sigma^2\left(X\right)+\int_a^b\alpha\left(y\right)\left(y-x\right)^2\mathrm{d}F\left(y\right)=\frac{1}{2}f''\left(x\right)\sigma^2\left(X\right)+\int_a^b\alpha\left(y\right)\left(y-x\right)^2\mathrm{d}F\left(y\right).$$

With $R\left(X\right)$ denoting the last integral above, we have

$$\left|R\left(X\right)\right|=\left|\int_a^b\alpha\left(y\right)\left(y-x\right)^2\mathrm{d}F\left(y\right)\right|\le\varepsilon\sigma^2\left(X\right)+M\int_{\substack{y\in\left[a,b\right]:\\ \left|y-x\right|\ge\delta\left(\varepsilon\right)}}\left(y-x\right)^2\mathrm{d}F(y)\le\varepsilon\sigma^2\left(X\right)+M\left(b-a\right)^2P\left(\left|X-x\right|>\delta\right).$$

With this preparation we can now prove the first result, as follows.

THEOREM 5. *If* $f:\left[0,1\right]\to\mathbb{R}$ *is a continuous function, then for any* $n>1$ *we have*

$$\left|R_n\left(f;x\right)-f\left(x\right)\right|\le\omega\left(n^{-1/2}\right)\left(1+x\left(1-x\right)\left(1-\min\left\{x,1-x\right\}\right)\right),\ x\in\left[0,1\right],\tag{15}$$

where $\omega\left(\delta\right)=\omega^f\left(\delta\right)$ *denotes the modulus of continuity of* f*. In particular, we have*

$$\left|R_n\left(f;x\right)-f\left(x\right)\right|\le\frac{31}{27}\omega\left(n^{-1/2}\right),\qquad x\in\left[0,1\right].\tag{16}$$

Proof. Applying part a) of Lemma 4 with $\delta=n^{-1/2}$ and $X=\frac{1}{n}X_n^{x,1-x,\min\{x,1-x\}/n-1}$, and using (4), we obtain

$$\left|R_n\left(f;x\right)-f\left(x\right)\right|\le\omega\left(n^{-1/2}\right)\left(1+x\left(1-x\right)\left(1-\frac{\min\left\{x,1-x\right\}}{1-\dfrac{\min\left\{x,1-x\right\}}{n-1}}\right)\right)\le\omega\left(n^{-1/2}\right)\left(1+x\left(1-x\right)\left(1-\min\left\{x,1-x\right\}\right)\right).$$

The expression $E\left(x\right)=x\left(1-x\right)\left(1-\min\left\{x,1-x\right\}\right)$ is a symmetric function of x with respect to $1/2$. For $x\in\left[0,1/2\right]$ we have $E\left(x\right)=x\left(1-x\right)^2$, with a maximum of $E\left(1/3\right)=4/27$ at $x=1/3$. This shows that $E\left(x\right)\le4/27$ for $x\in\left[0,1\right]$, concluding the proof.

Remark 6. Note that the estimate (15) improves the known estimate for the classical Bernstein operator (see, e.g. [26]) $\left|B_n\left(f;x\right)-f\left(x\right)\right|\le\omega\left(n^{-1/2}\right)\left(1+x\left(1-x\right)\right),\ x\in\left[0,1\right],$ *by the factor* $\min\left\{x,1-x\right\}\le1/2<1$.

Secondly, note that the value of the constant $C = 31/27 = 1.14815$ in (16) above is smaller than the constants obtained by Popoviciu ($3/2$), respectively by Lorentz ($5/4$), in the case of classical Bernstein polynomials, but it is slightly larger than the optimal constant $C_{opt} \approx 1.0898873...$ obtained by Sikkema [16]. However, the bound in (16) is not optimal, and we chose to present it in this form due to the simplicity of the proof. In a subsequent paper [10] we will show that the constant C for which Popoviciu's type inequality (16) holds for any continuous function is actually smaller than Sikkema's optimal constant for Bernstein polynomials. In turn, this shows that the operator R_n improves the well-known estimate for the classical Bernstein operator.

The next result gives the error estimate for R_n in the case of a continuously differentiable function.

THEOREM 7. *If* $f:[0,1]\to\mathbb{R}$ *is continuously differentiable on* $[0,1]$*, we have*

$$\left|R_n\left(f;x\right)-f\left(x\right)\right| \le n^{-1/2}\omega_1\left(n^{-1/2}\right)\left(x(1-x)(1-\min\{x,1-x\})+\sqrt{x(1-x)(1-\min\{x,1-x\})}\right), \qquad (17)$$

for any $n>1$ *and* $x\in[0,1]$*, where* $\omega_1(\delta)$ *denotes the modulus of continuity of* f'*. In particular, we have*

$$\left|R_n\left(f;x\right)-f\left(x\right)\right| \le \frac{4+6\sqrt{3}}{27} n^{-1/2}\omega_1\left(n^{-1/2}\right), \qquad x\in[0,1]. \qquad (18)$$

Proof. Applying part b) of Lemma 4 with $X=\frac{1}{n}X_n^{x,1-x,-\min\{x,1-x\}/(n-1)}$ and $\delta = n^{-1/2}$, and using (4), we obtain

$$\left|R_n\left(f;x\right)-f\left(x\right)\right| \le n^{-1/2}\omega_1\left(n^{-1/2}\right)\left(x\left(1-x\right)\left(1-\min\{x,1-x\}\right)+\sqrt{x\left(1-x\right)\left(1-\min\{x,1-x\}\right)}\right).$$

The same argument used in the last part of the proof of Theorem 5 shows that the expression in parentheses above has a maximum over $[0,1]$ equal to $\frac{4+6\sqrt{3}}{27}\approx 0.533$.

Remark 8. The estimate corresponding to (18) in the case of Bernstein operator B_n ([12], p. 21) is given by

$$\left|B_n\left(f;x\right)-f\left(x\right)\right| \le \frac{3}{4} n^{-1/2}\omega_1\left(n^{-1/2}\right), \qquad x\in[0,1],$$

and comparing to (18) we see that the operator R_n *improves this estimate.*

The following result gives the precise asymptotic of the error estimate for the operator R_n in the case of a twice continuously differentiable function.

THEOREM 9. *If* $f:[0,1]\to\mathbb{R}$ *is twice continuously differentiable on* $[0,1]$*, we have*

$$\lim_{n\to\infty} n\left(R_n\left(f;x\right)-f\left(x\right)\right)=\frac{1}{2}f''\left(x\right)x\left(1-x\right)\left(1-\min\{x,1-x\}\right), \qquad x\in[0,1]. \qquad (19)$$

Proof. Using the recursion formula for the centered moments μ_k of Pólya's distribution $X_n^{a,b,c}$ with parameters $a=x$, $b=1-x$, $c=-\min\{x,1-x\}/(n-1)$ and n (see e.g. [4, p. 179], we obtain:

$$\mu_3=\gamma+2\left(\min\{x,1-x\}+\delta\right)\sigma^2\left(X_n^{x,1-x,-\min\{x,1-x\}/(n-1)}\right)=O(n),$$

since $\gamma = nx(1-x)\left(1-n\min\{x,1-x\}/(n-1)\right) = O(n)$, $\delta = -n(1-2x)\min\{x,1-x\}/(n-1) - x = O(1)$, and by (4) we have $\sigma^2\left(X_n^{x,1-x,-\min\{x,1-x\}/(n-1)}\right) = O(n)$. Similarly,

$$\mu_4 = \gamma + 3\sigma^2\left(X_n^{x,1-x,-\frac{\min\{x,1-x\}}{n-1}}\right)\left(-\frac{\min\{x,1-x\}}{n-1}+\delta+\gamma\right)+3\mu_3\left(-\frac{\min\{x,1-x\}}{n-1}+\delta\right) = O\left(n^2\right).$$

Using part c) of Lemma 4 with $X = \frac{1}{n}X_n^{x,1-x,-\min\{x,1-x\}/(n-1)}$, $[a,b]=[0,1]$, and (4), we obtain

$$n\left(R_n(f;x)-f(x)\right) = \frac{1}{2}f''(x)x(1-x)\left(1-\frac{\min\{x,1-x\}}{1-\frac{\min\{x,1-x\}}{n-1}}\right)+nR\left(\tfrac{1}{n}X_n^{x,1-x,-\min\{x,1-x\}/(n-1)}\right),$$

where for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ (which depends on f, but not on n or x) such that

$$\left|nR\left(\tfrac{1}{n}X_n^{x,1-x,-\min\{x,1-x\}/(n-1)}\right)\right| \le \frac{\varepsilon}{n}\sigma^2\left(X_n^{x,1-x,-\min\{x,1-x\}/(n-1)}\right)+\frac{M}{n^3\delta^4}\mu_4 = \varepsilon O(1)+\frac{1}{\delta^4}O\left(\frac{1}{n}\right),$$

which can be made arbitrarily small for n large, concluding the proof.

Remark 10. The result in the previous theorem improves the corresponding result in the case of the Bernstein operator see [6, p. 22] by the factor $1-\min\{x,1-x\}$.

6. NUMERICAL RESULTS

For comparison, we will use the following well-known Bernstein-type operators: the classical Bernstein operator B_n, given by (1), the Lupaş operator $L_n = P_n^{<1/n>}$ given by (7), the q-Bernstein operator $B_{n,q}$ (see [13], [14]), and the (p,q)-Bernstein operator $S_{n,p,q}$ (see [9]). For the comparison of the operator R_n with these operators, we considered three representative choices of functions: a smooth, highly varying function (Figure 1), a continuous, but only piecewise smooth function (Figure 2), and a discontinuous function (Figure 3). The graphical analysis of Figures 1, 2, and 3 below clearly indicates that the operator R_n provides the best approximation of the function f in all three cases, followed by the Bernstein operator B_n. The ranking of the remaining operators is as follows: for small values of n it appears that $B_{n,q}$ provides a better approximation of f, while for larger values of n, L_n appears to be better. Although $S_{n,p,q}$ provides a reasonably good approximation for small values of n, this situation changes for larger values of n.

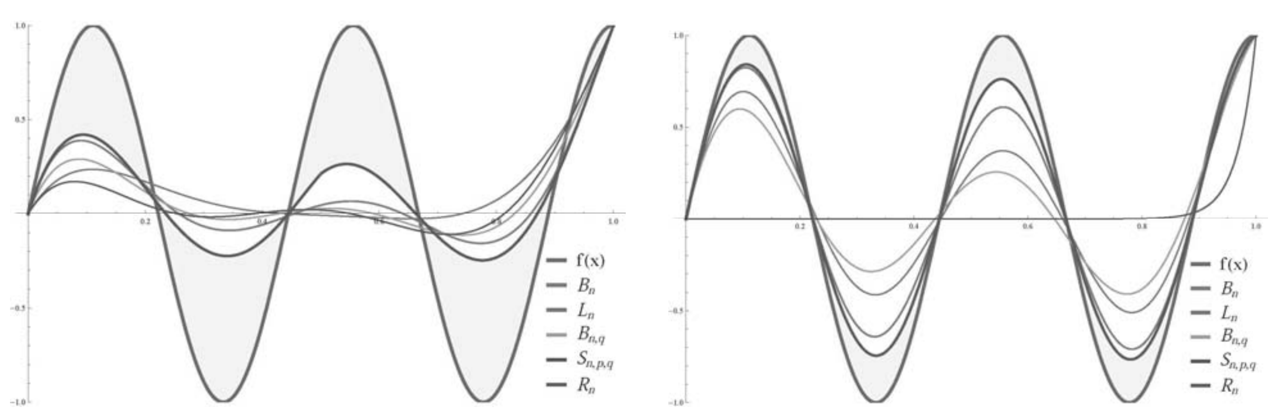

Fig. 1 – The approximation of $f(x)=\sin\left(\frac{9\pi}{2}x\right)$, in the case $n=10$ (left) and $n=50$ (right).

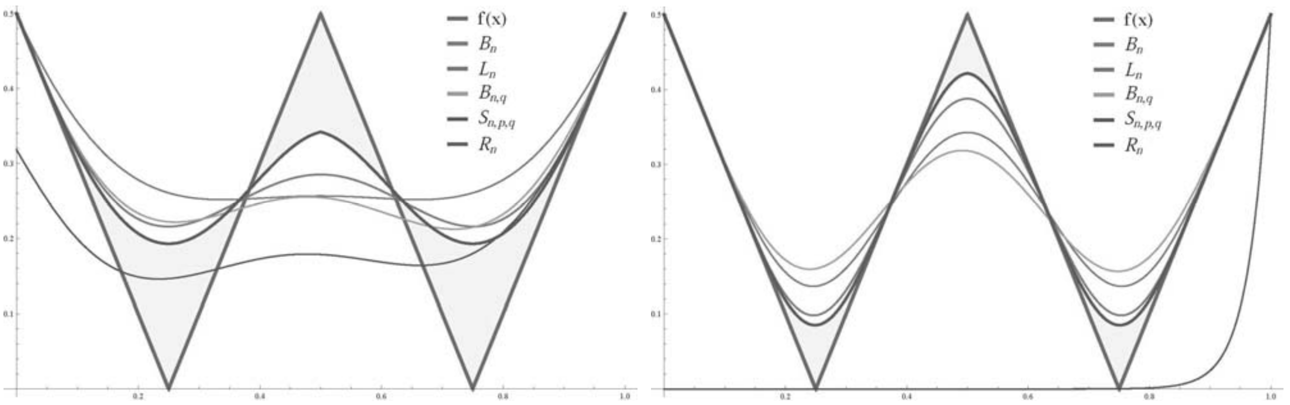

Fig. 2 – The approximation of $f(x) = \left|2\,|\,x - 0.5\,| - 0.5\right|$, in the case $n = 10$ (left) and $n = 50$ (right).

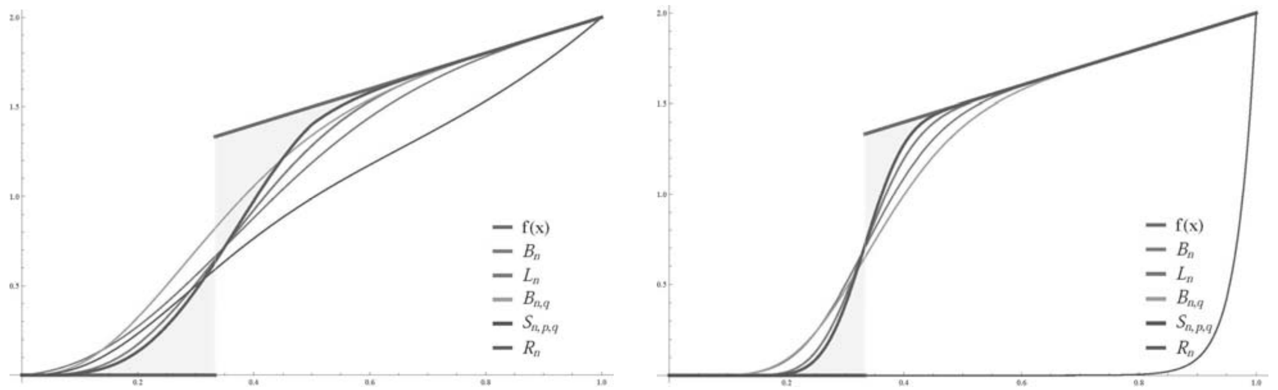

Fig. 3 – The approximation of $f(x) = (x+1)1_{[1/3,1]}(x)$ in the case $n = 10$ (left) and $n = 50$ (right).

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