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MATHEMATICS

NILPOTENT SINGULAR POINTS AND SMOOTH PERIODIC WAVE SOLUTIONS

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Abstract. In this paper, dynamical system theory is applied to the b-family equation. The results demonstrate that there exists close connection between nilpotent singular points and smooth periodic waves. Moreover, an abundance of smooth periodic wave solutions, which are expressed in terms of elliptic functions, trigonometric functions, and hyperbolic functions, are obtained when their corresponding orbits have contact points with the singular straight line.

***Key words*:** b-family equation, nilpotent point, periodic wave solution.

1. INTRODUCTION

In this paper, we consider the following b-family equation

$$m_t = m_x u + bmu_x, m = u - u_{xx}, \tag{1}$$

where b is an arbitrary constant. When $b = 2$, Eq. (1) reduces to the well-known Cammasa–Holm (CH) equation, which was derived by Cammasa and Holm [1] as a shallow water wave model. The CH equation was found to be completely integrable with a Lax pair and associated bi-Hamiltonian structure [1, 2]. The most remarkable feature of the CH equation is to admit peakon solutions [1, 3]. When $b = 3$, Eq. (1) reduces to the well-known Degasperis-Procesi (DP) equation, which is another important nonlinear model possessing peakon solutions [4]. Eq. (1) and its generalizations have attracted many research attentions over years and have been studied successfully by many authors [5–13]. In [10], the bifurcations of traveling wave solutions of Eq. (1) are studied. Moreover, the authors obtained the conditions under which smooth and non-smooth traveling wave solutions exist. In [13], the authors showed that Eq. (1) admits both N-peakon and N-kink solutions in the case of $b = 0$. In fact, it is very important to understand the qualitative behavior of solutions for traveling wave equations. The present paper aims at studying the smoothness property of traveling wave solutions in the case of degenerate singular points. We apply the dynamical systems theory to Eq. (1) which can then be reduced to a planar polynomial differential system by transformation of variables. By phase space analytical technique, we explore the inner correlation between nilpotent points and smooth periodic wave solutions and obtain uncountably infinite many smooth periodic wave solutions of Eq. (1) under some proper parameter conditions. In addition, we find some new elliptic functions and hyperbolic functions of smooth periodic wave solutions instead of well-known trigonometric functions of smooth periodic wave solutions by analyzing the nilpotent points. Our results improve the previous results of [10].

2. VECTOR FIELDS AND NILPOTENT POINTS

To study the qualitative behavior of the b-family equation (1), we need to introduce some notations and propositions.

We denote by $P_n(\mathbf{R}^2)$ the set of polynamial vector fields on $\mathbf{R}^2$ of the form $X(x, y) = (P(x, y), Q(x, y))$ where P and Q are real polynomials in the variables x and y of degree at most n ($n \in \mathbf{N}^\star$). A point $p \in \mathbf{R}^2$ is said to be a singular point of the vector fields $X = (P, Q)$ if

$P(p)=Q(p)=0$. Letting $\Delta = P_x(p)Q_y(p)-P_y(p)Q_x(p)$ and $T=P_x(p)+Q_y(p)$, then the singular point p is said to be non-degenerate if $\Delta\neq 0$. Moreover, p is a saddle if $\Delta<0$, a node if $T^2>\Delta>0$ (stable if $T<0$, unstable if $T>0$), a focus if $4\Delta>T^2>0$ (stable if $T<0$, unstable if $T>0$), and either a weak focus or a center if $T=0<\Delta$. When $\Delta=T=0$ but the Jacobian matrix at p is not the zero matrix and p is isolated in the set of all singular points, we say that p is nilpotent. Now we quote Cairó and Llibre's results on nilpotent singular points that we shall need.

PROPOSITION 1 (see [14]). *Let* $(0,0)$ *be a nilpotent singular point of the vector field* $(y+F(x,y),G(x,y))$, *where* F *and* G *are analytic functions in a neighborhood of the origin at least with quadratic terms in the variables* x *and* y. *Let* $y=f(x)$ *be the solution of the equation* $y+F(x,y)=0$ *in a neighborhood of* $O(0,0)$. *Assume that the development of the function* $G(x,f(x))$ *is of the form* $\alpha x^k+o(x^k)$ *and* $\Phi(x)\equiv(\partial F/\partial x+\partial G/\partial y)(x,f(x))=\beta x^n+o(x^n)$ *with* $\alpha\neq 0, k\geq 2$ *and* $n\geq 1$, *Then the following statements hold.*

(1) If k is even and

(1.a) $k>2n+1$, then the origin is a saddle-node. Moreover the saddle-node has one separatrix tangent to the semi-axis $x<0$, and other two separatrices tangent to the semi-axis $x>0$.

(1.b) $k<2n+1$ or $\Phi\equiv 0$, then the origin is a cusp, i.e. a singular point formed by the union of two hyperbolic sectors. Moreover, the cusp has two separatrices tangent to the positive x-axis.

(2) If k is odd and $\alpha>0$, then the origin is a saddle. Moreover, the saddle has two separatrices tangent to the semi-axis $x<0$, and other two separatrices tangent to the semi-axis $x>0$.

(3) If k is odd and $\alpha<0$ and

(3.a) n even, $k=2n+1$ and $\beta^2+4(n+1)\alpha\geq 0$, or n even and $k>2n+1$, then the origin is a stable (unstable) node if $L<0 (L>0)$, having all the orbits tangent to the x-axis at $(0,0)$.

(3.b) n odd, $k=2n+1$ and $\beta^2+4(n+1)\alpha\geq 0$, or n odd and $k>2n+1$, then the origin is an elliptic-saddle, i.e. a singular point formed by the union of one hyperbolic sector and one elliptic sector. Moreover, one separatrix of the elliptic-saddle is tangent to the semi-axis $x<0$, and the other to the semi-axis $x>0$.

(3.c) $k=2n+1$ and $\beta^2+4(n+1)\alpha<0$, or $k<2n+1$, then the origin is a focus or a center, and if $\Phi(x)\equiv 0$ then the origin is a center.

3. SMOOTH PERIODIC WAVE SOLUTIONS

We look for traveling wave solutions of Eq. (1) in the form of

$$u(x,t)=\varphi(\xi)-c,\ \xi=x-ct, \tag{2}$$

where c is the wave speed. Substituting it into Eq. (1), we have

$$-c\varphi_\xi+c\varphi_{\xi\xi\xi}-(b+1)(\varphi-c)\varphi_\xi=-b\varphi_\xi\varphi_{\xi\xi}-(\varphi-c)\varphi_{\xi\xi\xi}, \tag{3}$$

where φ_ξ is the derivative with respect to ξ. Integrating Eq. (3) once, we get

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}\xi^2}=\frac{-bc\varphi+\frac{b+1}{2}\varphi^2+\frac{b-1}{2}c^2+g-\frac{b-1}{2}\varphi_\xi^2}{\varphi}, \tag{4}$$

where g is an integration constant. Letting $g\to\frac{1-b}{2}c^2$ in (4) generates a two-dimensional system

$$\begin{cases}\dfrac{\mathrm{d}\varphi}{\mathrm{d}\xi}=\psi,\\[2mm] \dfrac{\mathrm{d}\psi}{\mathrm{d}\xi}=\dfrac{-bc\varphi+\frac{b+1}{2}\varphi^2-\frac{b-1}{2}\psi^2}{\varphi}.\end{cases} \tag{5}$$

System (5) is called a singular traveling wave system since the second equation of (5) is not continuous on the singular straight line $\varphi = 0$. Making the transformation $\frac{\mathrm{d}\xi}{\mathrm{d}\zeta} = \varphi$, for $\varphi \neq 0$, system (5) becomes

$$\begin{cases} \dfrac{\mathrm{d}\varphi}{\mathrm{d}\zeta} = \varphi\psi, \\ \dfrac{\mathrm{d}\psi}{\mathrm{d}\zeta} = -bc\varphi + \dfrac{b+1}{2}\varphi^2 - \dfrac{b-1}{2}\psi^2, \end{cases} \tag{6}$$

which is called the associated regular system of (5). Obviously, the singular line $\varphi = 0$ now becomes an invariant line solution of system (6). System (6) has two singular points $E(\frac{2bc}{b+1}, 0)$ and $O(0,0)$ on the φ-axis when $b \neq -1$. Corresponding to the singular point $E(\frac{2bc}{b+1}, 0)$ it is easy to know that $\Delta = -\dfrac{2(bc)^2}{b+1}$ and $T = 0$, so $E(\frac{2bc}{b+1}, 0)$ is a saddle when $b > -1$, a center when $b < -1$. For the singular point $O(0,0)$, $\Delta = T = 0$ and the Jacobian matrix is not the zero matrix, therefore it is a nilpotent point. From the equation $bc\varphi - \dfrac{b+1}{2}\varphi^2 + \dfrac{b-1}{2}\psi^2 = 0$, we have

$$\varphi = f(\psi) = \frac{1-b}{2bc}\psi^2 + \mathrm{HOT}, \tag{7}$$

$$G(\psi, \varphi(\psi)) = \frac{b-1}{2b^2c^2}\psi^3 + \mathrm{HOT}, \tag{8}$$

$$\Phi(\psi) = \frac{b-2}{bc}\psi. \tag{9}$$

Hence, $k = 3,\ n = 1,\ \alpha = \dfrac{b-1}{2b^2c^2}$ and $\beta = \dfrac{b-2}{bc}$. Based on the statement (3.b) of Proposition 1, we know that $O(0,0)$ is a nilpotent saddle.

On the other hand, we point out that near the singular straight line $\varphi = 0$, even though system (5) has the same level curves as system (6), the smoothness property of orbits of system (5) with respect to the "time variable" ξ should be studied carefully since the straight line $\varphi = 0$ is not an orbit of (5). In addition, for the singular traveling wave system (5), the following conclusion holds in [15–18].

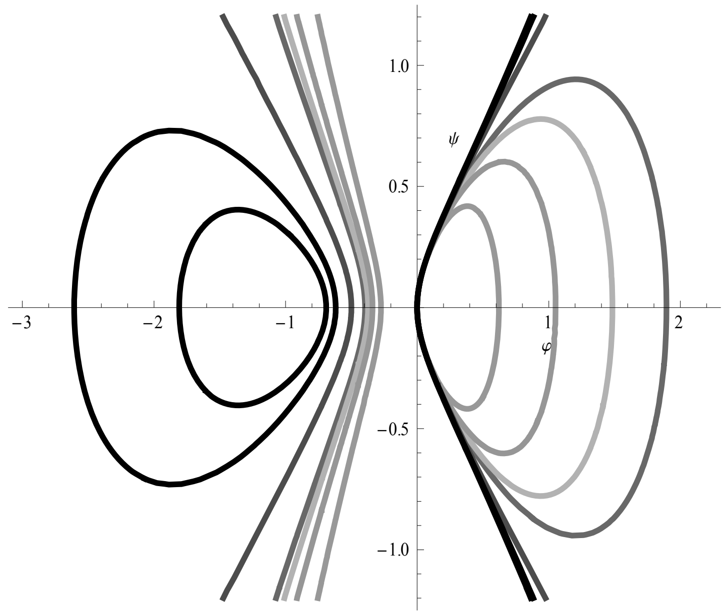

Fig. 1 – Phase portrait of system (10).

THEOREM 1. *Let* $(\varphi(\xi),\ \psi(\xi))$ *be the parametric representation of an orbit* γ *of the singular traveling wave system (5). If along the orbit* γ *, as* ξ *increases or (and) decreases,* γ *is in contact with the* ψ *-axis at the point* $O(0,0)$ *, then, it only takes a finite time interval for the phase point* $(\varphi(\xi),\ \psi(\xi))$ *to arrive at the singular straight line* $\varphi=0$.

In order to study the dynamical behaviors of the orbits of system (5) which are in contact with the singular straight line $\varphi=0$, let us seperate three cases to discuss: (i) $b=-2$, (ii) $b=-1$, and (iii) $b=-\frac{1}{2}$. Without loss of generality, along these cases we assume $c<0$.

3.1. Case I: $b=-2$

When $b=-2$, system (6) becomes

$$\begin{cases}\dfrac{\mathrm{d}\varphi}{\mathrm{d}\zeta}=\varphi\psi,\\[2mm] \dfrac{\mathrm{d}\psi}{\mathrm{d}\zeta}=2c\varphi-\dfrac{1}{2}\varphi^2+\dfrac{3}{2}\psi^2.\end{cases}\tag{10}$$

which has the first integral

$$H(\varphi,\psi)=\varphi^{-3}(\psi^2+2c\varphi-\varphi^2)=h.\tag{11}$$

There exist a family of closed orbits

$$\psi^2=h\varphi(\varphi-\varphi_M)(\varphi-\varphi_m),\tag{12}$$

which are tangent to the singular straight line $\varphi=0$ at the nilpotent point $O(0,0)$ (see Fig. 1), where

$$\varphi_M=\frac{-1-\sqrt{1+8ch}}{2h},\ \varphi_m=\frac{-1+\sqrt{1+8ch}}{2h}.\tag{13}$$

Integrating the first equation of (5) along the closed orbits (12) yields the following implicit solutions

$$\begin{aligned}\xi&=\frac{1}{\sqrt{-h}}\int_{\varphi}^{\varphi_M}\frac{1}{\sqrt{\varphi(\varphi_M-\varphi)(\varphi-\varphi_m)}}\,\mathrm{d}\varphi\\ &=\frac{2}{\sqrt{h(\varphi_m-\varphi_M)}}F\left(\arcsin\sqrt{\frac{\varphi_m-\varphi}{\varphi_M}},\ k\right),\end{aligned}\tag{14}$$

Where $F(\cdot\ ,\ \cdot)$ is the elliptic integral of the first kind and $k=\sqrt{\varphi_M/(\varphi_M-\varphi_m)}$. It implies that

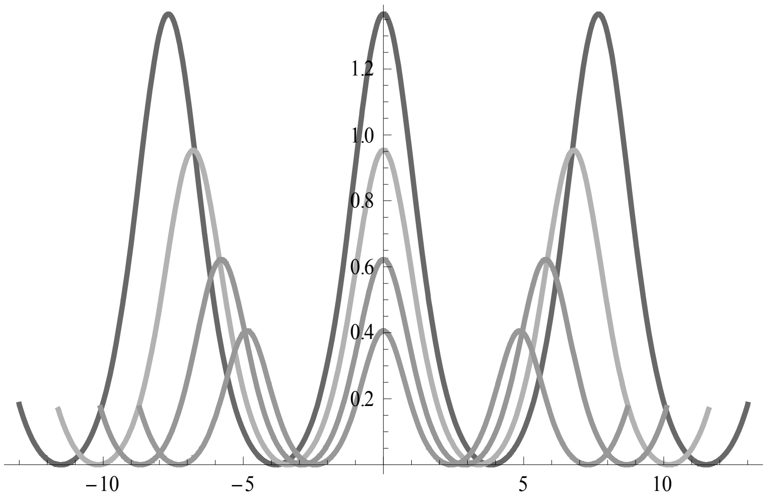

Fig. 2 – Profile of smooth periodic waves for Eq. (1) with $b=-2$. The meganta wave corresponds to $h=-1$. The cyan wave corresponds to $h=-1.6$. The green wave corresponds to $h=-2.9$. The orange wave corresponds to $h=-5.5$.

$$\varphi(\xi) = \varphi_M \mathrm{cn}^2(\omega\xi,\ k), \tag{15}$$

where $\omega = \frac{\sqrt{h(\varphi_m - \varphi_M)}}{2}$. This is the explicit parametric representation of the nondenurable smooth periodic wave solutions (see Fig. 2).

3.2. Case II: $b = -1$

For $b = -1$, system (6) becomes

$$\begin{cases} \dfrac{\mathrm{d}\varphi}{\mathrm{d}\zeta} = \varphi\psi, \\ \dfrac{\mathrm{d}\psi}{\mathrm{d}\zeta} = c\varphi + \psi^2. \end{cases} \tag{16}$$

with the first integral

$$H(\varphi,\psi) = \varphi^{-2}(\psi^2 + 2c\varphi) = h. \tag{17}$$

System (16) has a nilpotent saddle at the origin and a family of elliptic orbits

$$\psi^2 = \varphi(h\varphi - 2c), \tag{18}$$

which are tangent to the singular straight line $\varphi = 0$ at the nilpotent point $O(0,0)$ (see Fig. 3). These elliptic orbits bring about a family of smooth periodic wave solutions

$$\varphi(\xi) = \frac{c}{h}(1 + \cos(\sqrt{-h}\,\xi)). \tag{19}$$

The profile of smooth periodic waves (19) is similar to Fig. 2, we omit it here.

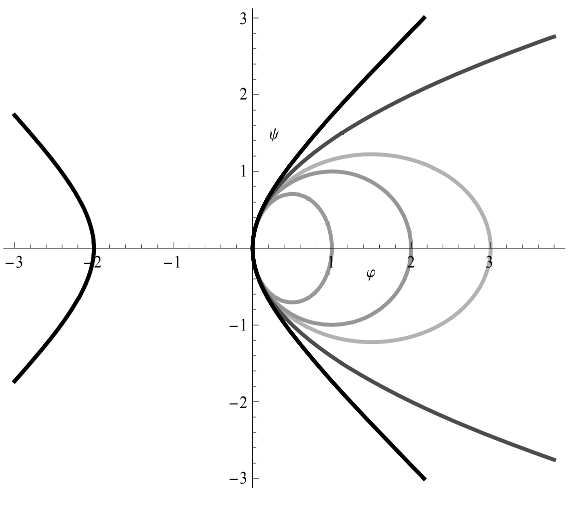

Fig. 3 – Phase portrait of system (16).

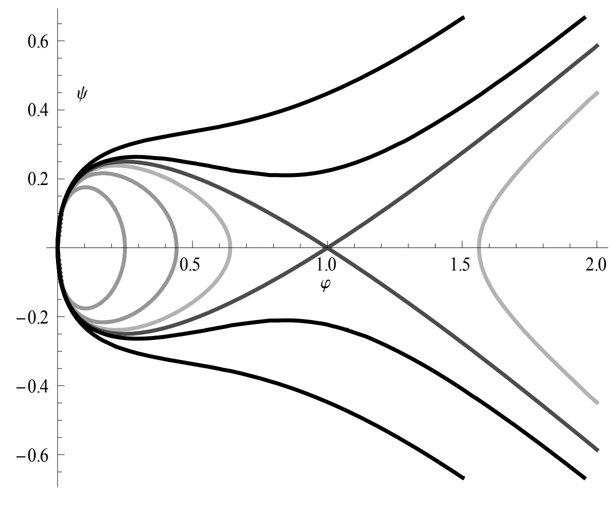

Fig. 4 – Phase portrait of system (20).

3.3. Case III: $b = -\frac{1}{2}$

In the above subsections, we show that the b-family equation (1) admits uncountable smooth periodic wave solutions which are expressed in the form of elliptic functions and trigonometric functions. This inspire us to look for new types of smooth periodic wave solutions.

When $b = -\frac{1}{2}$, system (6) becomes

$$\begin{cases} \dfrac{d\varphi}{d\zeta} = \varphi\psi, \\ \dfrac{d\psi}{d\zeta} = \dfrac{c}{2}\varphi + \dfrac{1}{4}\varphi^2 + \dfrac{3}{4}\psi^2, \end{cases} \tag{20}$$

which has the first integral

$$H(\varphi,\psi) = \varphi^{-\frac{3}{2}}(\psi^2 + 2c\varphi - \varphi^2) = h. \tag{21}$$

System (20) has two singular points $E(-2c,0)$ and $O(0,0)$ on the φ-axis, in which $E(-2c,0)$ is a saddle and $O(0,0)$ is a nilpotent elliptic-saddle. In addition, there exist a family of closed orbits

$$\psi^2 = \varphi(\sqrt{\varphi_M} - \sqrt{\varphi})(\sqrt{\varphi_m} - \sqrt{\varphi}), \tag{22}$$

which are tangent to the singular straight line $\varphi = 0$ at the nilpotent point $O(0,0)$ (see Fig. 4), where

$$\varphi_M = \frac{-h + \sqrt{h^2 + 8c}}{2}, \quad \varphi_m = \frac{-h - \sqrt{h^2 + 8c}}{2}. \tag{23}$$

These closed orbits engender infinitely many hyperbolic functions of smooth periodic wave solutions

$$\varphi(\xi) = (\sqrt{\varphi_m}\cosh^2(\xi - 2nT) - \sqrt{\varphi_M}\sinh^2(\xi - 2nT))^2, \tag{24}$$

where $T = \tanh^{-1}\sqrt[4]{\dfrac{\varphi_m}{\varphi_M}}$, $(2n-1)T \le \xi \le (2n+1)T$, $n \in \mathbf{Z}$. The profile of smooth periodic waves (24) is shown in Fig. 5.

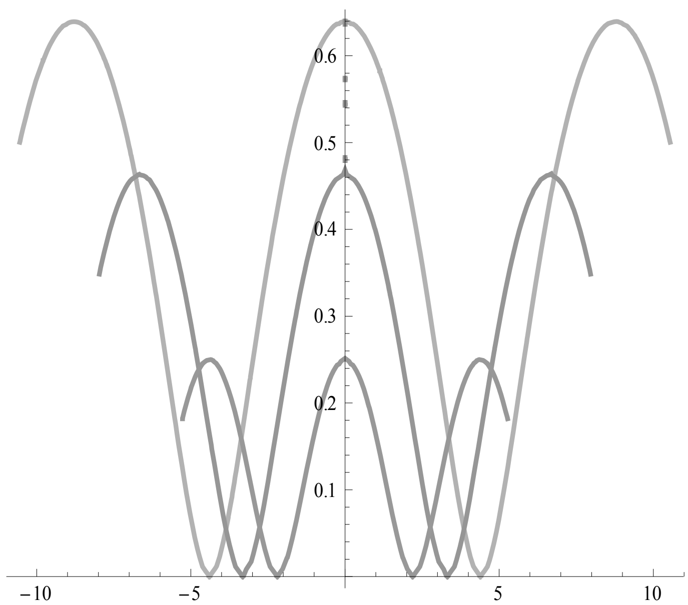

Fig. 5 – Profile of smooth periodic waves for Eq. (1) with $b = -\frac{1}{2}$. The cyan wave corresponds to $h = -2.05$. The green wave corresponds to $h = -2.17$. The orange wave corresponds to $h = -2.51$.

REMARK. The regular system (6) has the same first integral $H(\varphi,\psi)$ and the same topological phase portraits as (5) except for the straight line $\varphi = 0$. Hence, the analysis of the phase portraits of system (6) should be performed through three cases $b < -1, b = -1$ and $b > -1$, respectively. However, in order to conveniently obtain the exact explicit expressions of the smooth traveling wave solutions of Eq.(1), we choose three special cases $b = -2, b = -1$ and $b = -\frac{1}{2}$ as representatives.

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