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# CRITICAL POINTS OF THE NUCLEAR SHAPE PHASE TRANSITIONS AND ANHARMONIC OSCILLATOR POTENTIALS OF SIXTH ORDER

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**Abstract**. A retrospective discussion on the applications of anharmonic oscillator potentials of sixth order to the description of the critical points of some nuclear shape phase transitions is made by pointing out the main achievements, but also attracting the attention on the problems that still remain to be solved in order to improve the models and to get a better description of these phenomena and of the corresponding experimental data. Further possible developments, in this research area or related areas, are suggested.

***Key words***: nuclear shape phase, critical point, Bohr Hamiltonian, anharmonic oscillator potential.

## 1. INTRODUCTION

The quadrupole collective states of the ground band and of the first $\beta$ and $\gamma$ bands of the heavy nuclei are very well described in the frame of the Bohr-Mottelson model [1, 2] in terms of vibrations and rotations of their shape in the ground state. Depending on the intrinsic deformation variables, $\beta$ and $\gamma$, one can have spherical, axial symmetric (prolate and oblate), and asymmetric (triaxial, $\gamma$-unstable) shapes. Using an algebraic model, called the Interacting Boson Model [3], dynamical symmetries were associated with these shapes, called also phases, namely U(5) [4], SU(3) [5] and O(6) [6], respectively, while by using some coherent state functions, phase transitions were evidenced to take place between these shape phases of first and second order, respectively [7–11]. Some of these studies indicated that quartic and sextic potentials in $\beta$ would be more appropriate to describe the corresponding critical points, for which, unfortunately, exact analytical solutions are difficult to extract. A great progress in this field was done when analytical solutions have been proposed for critical points of the phase transitions from spherical vibrator to $\gamma$-unstable rotor and from spherical vibrator and axial symmetric rotor, called E(5) [12] and X(5) [13], respectively, by using an infinite square well for the $\beta$ variable and an oscillator one for the $\gamma$ variable. These two papers, basically, attracted the attention of the researchers for this research field, being followed quickly by an exponential increase of the number of publications [14–16].

In the present paper, the discussion is restricted only to a special class of these solutions, namely, those of the Bohr-Mottelson Hamiltonian with anharminic oscillator potentials of sixth order in the $\beta$ variable. These potentials are important here, because, depending on the choice of parameters, can have a spherical minimum or a deformed one, simultaneously spherical and deformed minima, a flat shape as an infinite square well, being therefore appropriate to cover a phase transition from a spherical shape to a deformed one by crossing the critical point. Besides their advantages, there is a real challenge to apply such potentials in studying nuclear shape phase transitions due to the difficulty in getting analytical solutions for them, while in some situations also numerical solutions are obtained after a hard work. Therefore, there is not a surprise, that frequently are preferred exactly solvable potentials as harmonic oscillator [1], Davidson [17, 18], Kratzer [19], Coulomb [19], Infinite square well [20], etc. A first application of a sextic potential, a quasi-exactly solvable one [21], to the Bohr-Mottelson Hamiltonian was done for the spherical vibrator to $\gamma$-unstable rotor shape phase transition [22, 23]. Then, applying similar technique as in [22], the same potential [21] was applied for $\gamma$-soft triaxial nuclei [24, 25], $\gamma$-soft prolate nuclei [26], $\gamma$-rigid triaxial nuclei

[27], and γ-rigid prolate nuclei [28]. All these solutions have been written also in a unified form [29], by pointing out the similarities and the differences between them on one hand, but also offering some simplifications concerning the fitting of the free parameters for experimental data, on the other hand. Particular cases for the sextic potential were also proposed in [30], an exact solution, and in [31] using some approximations for the final solution. Also, a more general form has been introduced in [32], which can recover in some limits the harmonic oscillator, Davidson and sextic potentials. Recently, a different solvability method has been proposed for a more general sextic potential [33], which allow us to have degenerated minima as spherical and deformed one, separated by a barrier (a maximum). The later, besides the properties of the quasi-exactly solvable sextic potential [21], can additionally describe shapes coexistence, which is of great interest at present in nuclear structure [34].

The present work is constructed more or less as a short review paper, trying to attract de attention of the research community on some special solutions for the Bohr-Mottelson model, by underlying their success, but also by specifying what are the drawbacks of each solution and by indicating new methods to improve them and possible directions of research. Therefore, the plan of the work is the following. After a brief introduction for the Bohr-Mottelson model in Sec. 2, the presentation is moved in Sec. 3 to its solutions for different types of sextic oscillator potentials, while then, in Sec. 4, a comprehensive discussion is carried out on these solutions by trying to extract the main ideas, which could be useful for further developments in nuclear shape phase transitions and maybe in related areas of research as atomic or molecular physics.

## 2. DESCRIPTION OF THE MODEL

The low-lying quadrupole collective states of the even-even heavy nuclei can be described in the frame of the Bohr-Mottelson model [1,2] as β and γ vibrations coupled to some rotations of the nuclear shape in the ground state. In general, when both β and γ vibrations are considered (soft case), the system is described by the following Hamiltonian [1,2]:

$$H=-\frac{\hbar^2}{2B}\left[\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta}+\frac{1}{\beta^2\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma}-\frac{1}{\beta^2}\sum_{k=1}^{3}\frac{Q_k^2}{4\sin^2\left(\gamma-\frac{2\pi}{3}k\right)}\right]+V\left(\beta,\gamma\right). \tag{1}$$

In Eq. (1), $B$ is a mass parameter, $Q_k$ are angular momentum projections in the intrinsic reference frame, while $V\left(\beta,\gamma\right)$ is the energy potential. Particular Hamiltonians are obtained when some constraints are imposed on vibrations or on axial deformation. For example, when γ vibrations are “frozen” (γ-rigid) the Hamiltonian becomes [35]:

$$H=-\frac{\hbar^2}{2B}\left[\frac{1}{\beta^3}\frac{\partial}{\partial\beta}\beta^3\frac{\partial}{\partial\beta}-\frac{1}{\beta^2}\sum_{k=1}^{3}\frac{Q_k^2}{4\sin^2\left(\gamma_0-\frac{2\pi}{3}k\right)}\right]+V\left(\beta\right), \tag{2}$$

where $\gamma_0$ here is no more a variable, being seen as a parameter that specifies the non-axial deformation. If γ vibrations are frozen for an axial symmetric shape ($\gamma_0=0^0$), the corresponding Hamiltonian is [36]:

$$H=-\frac{\hbar^2}{2B}\left[\frac{1}{\beta^2}\frac{\partial}{\partial\beta}\beta^2\frac{\partial}{\partial\beta}-\frac{1}{3\beta^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\right]+V\left(\beta\right). \tag{3}$$

The separation of variables is easy to achieve for Hamiltonians given by Eqs. (2) and (3), while for Eq. (1), usually a separable form of the potential is adopted [15,16] as $V\left(\beta,\gamma\right)=V\left(\beta\right)+U\left(\gamma\right)/\beta^2$. In [29] it is shown that, after the separation of variables, a common Schrödinger equation in β variable can be written for the Hamiltonians (1–3):

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}\beta^2}+\frac{W}{\beta^2}+u(\beta)\right]\phi(\beta)=\varepsilon\phi(\beta),\ u(\beta)=\frac{2B}{\hbar^2}V(\beta),\ \varepsilon=\frac{2B}{\hbar^2}E, \tag{4}$$

where $W$ is the energy contribution coming from the γ variable and the Euler angles. The expression for $W$ is not unique [15, 16, 29] depending on the β and γ deformations, on the separation of variables if is exactly achieved or not, if the system is rigid regarding the β and γ vibrations, on the shape of the potential, etc. The expression for $W$ is a function of some quantum numbers such as seniority, total angular momentum, and projections of the total angular momentum.

## 3. ANHARMONIC OSCILLATOR POTENTIALS OF SITH ORDER

The most general expression for an anharmonic oscillator potential of sixth order, which will be discussed here is:

$$V(\beta)=A\beta^2+B\beta^4+C\beta^6, \tag{5}$$

with $A$, $B$, and $C$ real independent parameters. Depending on its parameters, the potential (5) can have a spherical minimum, a deformed one, simultaneously spherical and deformed minima or a flat shape. Therefore, due to its flexibility, the potential (5) is very appropriate to describe nuclear phase transitions from a spherical shape to a deformed one by crossing the critical point where the potential is flat or it has two minima, a spherical and a deformed one, separated by a maximum. Exact solutions for Eq. (4) with potential (5) are difficult to get when $W$ is not zero [21], but fortunately for some sets of parameters this kind of solutions are available, while when this is not possible some good approximations can be applied. The most simplest case, considered for Eq. (4) with potential (5), is when both $A$ and $B$ are equal to zero, while $C=1/2$ [30]. For this potential, one have, up to a scale factor, free parameter solutions, which are useful for numerical applications to experimental data because tables with energy spectra and electromagnetic $E2$ transitions, which do not depend of any parameter, are provided. This makes an easy task for any reader to take these data and to compare with other theoretical or experimental data. A careful analysis performed in Ref. [30] shows that with this potential ($A=B=0$, $C=1/2$), one can describe nuclei lying somewhere on the middle between a spherical shape and the critical point of the shape phase transition. Therefore, only a few nuclei will be available for this solution, which is not a surprise taking into account the drastic constraints imposed on the parameters describing the potential. Nevertheless, this solution is very useful as a testing tool when the constraints imposed above the parameters are relaxed.

A more general situation than that discussed in the previous paragraph, was proposed in [31], namely for $B=0$ and $A$, $C$ real and positive numbers. Equation (4) was approximately solved for this potential in [31] by using an JWKB quantization rule [37]. The energy spectra, up to a scale factor, depend in this case on a single free parameter of whose variation in a restricted finite interval can cover a narrow shape transition between a spherical shape and a critical point. Therefore, with this potential can be described more nuclei than in the previous case [30] and also more accurately, especially due to the free parameter. Unfortunately, due to the complexity of the JWKB method, the wave functions are difficult to obtain for this potential and therefore, some information, as density probability distribution or electromagnetic transitions, is not available in the frame of this model. For simplicity, in Ref. [31], γ-rigid prolate nuclei were chosen as applications, but the solution can be easily extended for γ-soft nuclei or other shapes, too. A great step forward was done in [22, 23], where all terms depending on β in Eq. (5) were retained but with some constraints for parameters. Actually, Eq. (4) with a potential (5) was reduced in [22] to the quasi-exactly solvable equation of the sextic oscillator potential with a centrifugal barrier [21]:

$$V(\beta)=\frac{4\left(s-\frac{1}{4}\right)\left(s-\frac{3}{4}\right)}{\beta^2}+\left[b^2-4a\left(s+\frac{1}{2}+M\right)\right]\beta^2+2ab\beta^4+a^2\beta^6,\ M\in N,\ a,b\in R,\ a>0. \quad (6)$$

By quasi-exactly solvable equation it is understood [21] that situation where the matrix of an Hermitian Hamiltonian can be written in a block structure, with one of the block being an *M+1* by *M+1* finite matrix, while the second block is an infinite dimensional matrix. The trick in that case is that the *M+1* by *M+1* finite matrix can be diagonalized without touching the infinite one. Thus, the quasi-exactly solvable equations can be seen as an intermediate class between the exactly solvable and non-exactly solvable ones. In Eq. (6), $s$ is related with $W$, which in turn depends on some quantum numbers, which implies that finally $s$ will depend on these quantum numbers when one will try to apply this potential to the β equation (4). If $s$ depends on some quantum numbers and the coefficient of $\beta^2$ depends on $s$, then it results that the potential will depend on some quantum numbers and therefore one will have a state-dependent potential. Because in the frame of the Bohr-Mottelson model, the potential is state independent, in [22] it was proposed a method to keep the coefficient of $\beta^2$ constant by using the natural number $M$. In [22, 23], the potential (6) was applied to describe the phase transition from a spherical vibrator to a γ-unstable rotor. Inspired by this study, the sextic potential (6) was further applied for phase transitions from an approximately spherical vibrator to a γ-soft triaxial rotor [24], from an approximately spherical vibrator to a γ-soft prolate rotor [26], from an approximately spherical vibrator to a γ-rigid triaxial rotor [27], and from an approximately spherical vibrator to a γ-rigid prolate rotor [28], respectively. Because all these solutions [22, 24, 26, 27, 28] have in common the using of the sextic potential [21], but differ between them by the application to other shape phase transitions, in Ref. [29] the authors have written a general equation for all of them:

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}y^2}+\frac{W}{y^2}+\left(\alpha^2-4c\right)y^2+2\alpha y^4+y^6\right]\eta(y)=\varepsilon_y\eta(y), \quad (7)$$

where the connection between Eq. (4) with potential (6) and Eq. (7) is given by these relations:

$$\alpha=\frac{b}{\sqrt{a}},\ \varepsilon_y=\frac{\varepsilon}{\sqrt{a}},\ y=\beta a^{\frac{1}{4}},\ c=s+\frac{1}{2}+M\ . \quad (8)$$

By writing Eq. (7), in Ref. [29] the authors have compared these solutions and have underlined more clearly the similarities and differences between them, on one hand, but, on the other hand, they have also simplified the fitting procedure for experimental data by reducing the number of free parameters. Moreover, it was proved that in the critical point, free parameter solutions can be extracted [23, 27, 28, 29]. By using the sextic potential [21], in comparison with the potential used in [31], besides the possibility of getting also the wave functions, one can cover a full phase transition from a spherical shape to a deformed one by crossing also the critical point. Because of that, applications to the experimental data in this case are more interesting because new properties can be studied in detail. Nevertheless, due to the quasi-exactly solvability of the sextic potential [21], some properties of the nuclear shape phase transitions or other phenomena as shape coexistence still remain untouched. For example, the critical point of the phase transition between the spherical shape and prolate rotor needs a potential with two simultaneous minima, spherical and deformed, separated by a small maximum (barrier). Even if the sextic potential [21] is able to reproduce such a shape, the numerical results are far by the experimental data, and that is because of the conditions imposed on the parameters such that the quasi-exactly solvability be possible. Other problem in the case of the sextic potential [21, is that the spherical minimum is always above the deformed one, no matter how one chose its parameters. This is not a problem in general, but limits the possibility to get a better understanding of the shape coexistence.

The two drawbacks of the sextic potential [21] discussed at the end of the above paragraph have been removed in [33] by considering the general expression of the potential (5). In this situation, Eq. (4) with potential (5) is not quasi-exactly solvable; in [33] being solved through a numerical diagonalization in a basis of Bessel functions. Before this method to be proposed, an Algebraic Collective Model [38] was available to diagonalize matrices for Bohr Hamiltonian with potentials having a single minimum, flat or sharp, by using pseudo-harmonic oscillator functions. In [33], the diagonalization is made for a potential having simultaneously two minima, spherical and deformed, separated by a maximum (a barrier). The using of the Bessel functions as a basis, has been inspired by the fact that for E(5) [12] and X(5) [13], which describes critical points, Eq. (4) with an infinite square well potential is reduced to a Bessel equation. In [33], the study has been focused only on the situation when the two minima are degenerated, which is specifically for the critical point of the phase transition from a spherical shape to a prolate one [13], but also for shape coexistence. Moreover, degenerated minima request some constraints on the potential parameters, which of course, simplify the diagonalization procedure. Shortly, this particular situation is achieved as follows. After a scaling procedure of the energy [33], the potential (5) becomes:

$$V(\beta) = \beta^2 + \mu\beta^4 + \nu\beta^6 , \tag{9}$$

which has two minima if and only if $\mu < 0$ and $\nu > 0$:

$$\beta_{spheric} = 0, \quad \beta_{deformed} = \sqrt{\frac{\sqrt{\mu^2 - 3\nu} - \mu}{3\nu}} . \tag{10}$$

So far, these two minima (10) are not yet degenerated (are not positioned at the same energy level). The potential (9) has two degenerated minima (10) only if the following expression is adopted [33]:

$$V(\beta) = \beta^2 - 2q\beta^4 + q^2\beta^6 , \ \beta_{spheric} = 0, \ \beta_{deformed} = \frac{1}{\sqrt{q}} . \tag{11}$$

The main goal of the work [33] was to introduce the new method of diagonalization of the matrix associated with the Bohr-Mottelson Hamiltonian supplemented with a general potential (5), while as a test for the method was selected the critical point [13], by neglecting for the moment the γ band. Also, the possibility of describing shape coexistence properties by this method was suggested. Further developments and applications of this method will be discussed in the next section.

Another study, which used a potential in β of the form (5) was done in [32], where the wave functions have been expressed in terms of bi-confluent Heun functions, while for energy spectra have been used series forms of these functions. At the end, two free parameters are involved in fitting procedure. As applications the authors chose triaxial nuclei. The method can be easily adapted for other deformations.

## 4. DISCUSSIONS AND CONCLUSIONS

In the previous Sections, an introduction in the field of nuclear shape phase transitions has been made, followed by a discussion about the Bohr-Mottelson model, which plays an important role in this research area, while finally the attention was focused on a special class of Bohr-Mottelson Hamiltonian solutions, namely, for anharmonic oscillator potentials of sixth order. Due to the lake of space, these solutions have been only briefly introduced in Section 3, while here the goal is to see what are the present limits of these solutions and how these solutions can be improved and more important, how to extend their applications for experimental data and new physical phenomena.

The potentials proposed in [30, 31] for Eq. (4) are limited with respect to numerical applications for experimental data, the first one [30] due to its simplicity by considering only the term $\beta^6$, while the second one [31] because of the non-availability of the wave functions. Nevertheless, the potentials [30, 31] are particular cases for the more general ones considered in [22, 23, 24, 26–28, 32, 33], therefore, the first ones can be used as testing tools for the later ones when different applications for experimental data are done.

The most applications were done for the quasi-exactly solvable potential [21], in comparison with the other ones, and that because it was first time applied for Eq. (4) in 2004 [22], while the other ones have been quite recently proposed, but also because of the possibility of getting analytical solutions through the quasi-exactly solvability procedure. Thus, the sextic potential [21] was applied for the first time to the phase transition from the spherical vibrator to γ-unstable rotor [22]. This is one of the “lucky” situations for which an exactly separation of variables for Eq. (1) is possible [39, 40]. Nevertheless, the application of the sextic potential [21] for this case was not an easy task at all and that because of the *s* parameter that appears in Eq. (6) in both centrifugal term ($1/\beta^2$) and pure harmonic oscillator term ($\beta^2$). In order to apply the potential (6) for Eq. (4), in [22] *s* was expressed in terms of a quantum number $\tau$ [22], called seniority quantum number, which defines the eigenvalues of the Casimir operator of the SO(5) group. Therefore, the potential (6) depends on $\tau$ due to *s*, which gives a dependence of the potential on state being in conflict with the Bohr-Mottelson model, which requires a state independent potential. The great achievement of [22] was the solving of this problem by imposing the constraint that *c*, given by Eq. (8), remains constant. This constraint is satisfied if for every increase of $\tau$ by one unit, *M* is decreased by one unit also. Even so, finally two *c* constants result, one for $\tau$ odd and other for $\tau$ even, and therefore two potentials arise, which of course is better than a single potential for each state, but not enough. Thus, other two constants have been added to the potential (6), one for $\tau$ odd and other for $\tau$ even, which at the end are fixed such that the two potentials have the same minimum energy. These two additional constants, are not free parameters, depending on the initial parameters *a* and *b* that define the potential (6). In a next study [23], the same authors have shown how free parameter solutions can be obtained for the critical point of this shape phase transition and how appropriate is this solution to describe a phase transition from a spherical shape to a deformed one by doing several applications to different isotopic chains. On the other hand, the studies [22, 23] considered *M* not greater than one, while in [27] and more clearly in [29] it was proved that an increase of *M* shifts the location of the critical point and gives different signature values with respect to the case $M$=1. Also, in [29] a scaling procedure has been introduced for the energy, which leads to a decrease of the free parameters at least by one (7), which is very useful for numerical applications to experimental data. In conclusion, the studies started in [22, 23] can continue according to the indications given in [29], in order to see what are the new outcomes when the starting value of *M* is greater than one. For other research areas, where the problems are reduced to a radial Schrödinger equation in three dimensions as in (7), the methods introduced in [22, 23] to apply the sextic potential [21] could represent a starting point, while in the cases where some approximations are needed, the next discussions will be very useful.

Even if the first application of the sextic potential for the Bohr-Mottelson was done in 2004 [22], a next application for other deformation was done much later in 2011 for $\gamma$-soft triaxial nuclei [24] and this with some "sacrifices" because some approximations have been adopted for this realization. For example, the $\beta$ variable is not exactly separated by $\gamma$ and the three Euler angles in [24] as in [22], while also some approximations are introduced for the rotational term ($1/\beta^2$) such that a state independent potential to be obtained. Moreover, the potential for $\gamma$-soft triaxial nuclei depends also on the $\gamma$ variable, which is not the case for the $\gamma$-unstable nuclei. This leads to further complications in solving the equation for the $\gamma$ variable and to separate it by the other three Euler angles. A similar situation was found when the sextic potential was applied for $\gamma$-soft prolate nuclei [26]. These two first attempts [24, 26] have the merit of attacking of new deformations with the sextic potential [21] and indicating what are the problems and where improvements can be done. Despite of the approximations involved, the applications of [24, 26] to experimental data have shown an overall good agreement, but also that a better description can be done and more important that the fitting procedure is very difficult due to the many free parameters involved and due to the big size of the matrices. A great improvement of these solutions [24, 26], by removing some approximations and by reducing the number of free parameters, was offered in [29] opening the possibility for new applications for experimental data and moreover for new properties related with the nuclear shape phase transitions associated with $\gamma$-soft triaxial and prolate shapes. These examples [24, 26, 29] of applications of the sextic potential for problems where both radial and angular variables are involved in the description of the system could be very useful for similar problems from related research areas.

Special classes of quadrupole collective solutions are those involving some rigidities with respect to the $\gamma$ variable, as in Eqs. (2) and (3), because in these situations an exact separation of variables is possible. The potential depends only on the $\beta$ variable, while $\gamma$ plays a role of a parameter instead of a variable by specifying only the axial deformation. A drawback of these solutions is related with the description of the $\gamma$ band. For example, for $\gamma$-rigid prolate [36] only the ground and $\beta$ bands can be seen, while for $\gamma$-rigid triaxial nuclei [35, 41], a rotational band without a $\gamma$ vibration is associated with the $\gamma$ band with the help of the wobbling quantum number. Moreover, in this $\gamma$ band of the $\gamma$-rigid triaxial nuclei, the odd states up to $7^+$ are below their neighboring even states [27, 29, 41]. This reverse of the states in the $\gamma$ band was not yet observed experimentally, but for sure this behavior can be seen as a signature for these nuclei. Thus, by considering the γ rigidity, a great simplification is obtained for the separation of variables, but at the same time there appear limits for experimental applications. Nevertheless, these solutions are useful more from theoretical point of view and in cases where particular properties of the experimental data manifest such a behavior. Despite of the simplifications brought by the γ rigidity on the Bohr-Mottelson model, the application of the sextic potential [21] for these cases still remains difficult to achieve. For example, for γ-rigid triaxial nuclei, the rotational term depends on both total angular momentum ($L$) and its projection ($R$) on $x$-axes of the intrinsic reference frame. Therefore, in order that the sextic potential in β (6) remains state independent one have to keep $c$ constant by using $M$ to cancel the changes of the two quantum numbers $L$ and $R$ instead of one ($\tau$) as in [22]. This problem was solved in [27], but only for the ground and β bands of the γ rigid triaxial nuclei, while for the γ band a good approximation was involved. A similar success was achieved in [28], for γ-rigid prolate nuclei, where only the $0^+$ and $2^+$ of the ground and β bands are exactly described with the sextic potential [21]. Even if these solutions [27, 28] consider γ-rigidity when the sextic potential [21] is used, which limits their applications for experimental data of quadrupole collective states, their exact description of a part of the states represents a great achievement for the class of exactly solvable models in nuclear physics [42] and could serve as models of solving problems in other related research areas where more applications are possible. In [28] it was proved that these solutions are appropriate to evidence the symmetry properties in the critical point, while then by relaxing the γ rigidity to observe the structure evolution of these states. Also, in [27, 28] these solutions have been used to predict the energies for some states of uncertain parity or band, or some electromagnetic $E2$ transitions between some states, of course when the bands manifest a γ-rigid like structure.

The sextic potential [21] has been proved to be of great importance for the study of the nuclear shape phase transitions and their critical points, while by its many applications it resulted that one have to consider more general potentials in order to describe new properties. A step toward this direction was done recently in [33], by diagonalizing the matrix of the Bohr-Mottelson Hamiltonian (1) with a potential (5) using as a basis the Bessel functions. The applications from [33] are not done for a general situation when no constraints are imposed on the parameters, the goal being for the beginning to introduce only the method and to test it. Even so, degenerated spherical and deformed minima separated by a barrier (a local maximum) were considered for the critical point of the phase transition from spherical vibrator to γ-soft prolate rotor, while the possibility to describe shape coexistence properties was pointed out. This new method [33] opens a door for a considerable number of studies and applications. For example, the γ band, which was not considered in [33], will be included, while the constraints imposed on the parameters in [33] will be removed allowing to describe the nuclear shape phase transitions more accurately with respect to the experimental data. The method [33] opens also the possibility to solve the equation for the Bohr-Mottelson Hamiltonian amended with potentials that do not have a separable form $V(\beta,\gamma)=V(\beta)+U(\gamma)/\beta^2$ as it was used in most of the studies [15, 16], but a more complex one by allowing a mixture between terms, which depend on the β and γ variables. More important, the method [33] was proposed for quadrupole collective states, but it should be seen as a method of solving Schrödinger equation for such potential and in this way to extend its applications for other physical systems as atoms or molecules, membrane oscillations, etc.

As a final conclusion, applications of anharmonic oscillator potentials of sixth order to problems of nuclear structure [22–33,43–46] have demonstrated their usefulness in the cases where a physical system is described by a radial Schrödinger equation in three dimensions, different difficulties in applying these potentials have been identified and removed and more important new techniques of solving such problems have been proposed contributing in this way to the development of this research field and opening the door for new applications in nuclear structure and in related research areas.

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