THE PUBLISHING HOUSE
OF THE ROMANIAN ACADEMY

*PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A,*
*Volume 19, Number 2/2018, pp. 337–344*

# ASYMPTOTIC QUANTUM CORRELATIONS IN OPEN QUANTUM SYSTEMS

Aurelian ISAR[1,2]

[1] National Institute for Physics and Nuclear Engineering, Department of Theoretical Physics,
P. O. Box MG-6, RO-077125, Bucharest-Magurele, Romania
[2] Academy of Romanian Scientists, 54 Splaiul Independentei, Bucharest 050094, Romania
E-mail: isar@theory.nipne.ro

**Abstract.** We describe the asymptotic behaviour of quantum correlations (quantum entanglement, quantum discord, and quantum steering) in a system composed of two coupled bosonic modes immersed in a thermal reservoir, in the framework of the theory of open systems. The time evolution of quantum correlations is described in terms of the covariance matrix for Gaussian initial states. We show that, depending on the values of the strength of interaction between the two bosonic modes, and independent of the initial state, in the limit of infinite time the system evolves asymptotically to an equilibrium state, which may be entangled or separable. We show also that for all non-zero values of the strength of interaction between the modes, Gaussian quantum discord tends asymptotically for large times to some definite non-zero value. The asymptotic values of entanglement and discord depend on the strength of interaction between the bosonic modes, temperature, and dissipation parameter. We calculate also the Gaussian quantum steering and find out that in the limit of large times it is zero for all values of the strength of interaction between the two modes.

***Key words***: quantum entanglement, quantum discord, quantum steering, open systems.

## 1. INTRODUCTION

The physical understanding of different kinds of quantum correlations, like nonlocality, steering, entanglement, and discord constantly advanced in the last decades [1]. The genuine properties of quantum states in comparison with classical ones consist in such quantum correlations [1–3], which have proven to be useful in quantum information processing. Consequently, characterizing and quantifying quantum correlations represent a key subject in the theory of quantum information [2]. In this respect, quantum entanglement is considered to be a strong physical resource for quantum information processing and communication tasks and protocols [4, 5]. However, not all non-classical properties of quantum correlations are described by entanglement. In this sense, Zurek [6, 7] introduced the quantum discord as a measure of quantum correlations that includes entanglement and which can also be present in separable states. Steering is also a type of quantum nonlocality first identified in the Einstein-Podolsky-Rosen paper [8], which is distinct from both nonseparability and Bell nonlocality, allowing for new practical applications such as one-sided device-independent quantum key distribution [9]. To infer the steerability between two parties is equivalent with verifying the shared entanglement distribution by an untrusted party, by performing local measurements and classical communications [10].

To implement quantum information tasks into the real quantum systems is a difficult procedure, since they are not isolated, but always interact with their own environment. Consequently, quantum coherence and quantum correlations are inevitably affected during the interaction of quantum systems with their external environment. Therefore, it is necessary to take decoherence and dissipation into consideration in order to obtain a realistic description of quantum processes. In the last years decoherence and dynamics of quantum correlations in open systems of continuous variables have been intensively studied [11–49].

Recently we studied, in the framework of the theory of open systems based on completely positive quantum dynamical semigroups, the dynamics of quantum correlations of two uncoupled bosonic modes embedded in a common thermal reservoir, for initial Gaussian states of the subsystem [50–55]. In Ref. [56]

we described the dynamics of the quantum entanglement of a subsystem composed of two coupled bosonic modes interacting with a common thermal reservoir and have shown that, for a separable initial squeezed thermal state, entanglement generation may take place, for definite values of the squeezing parameter, average photon numbers, temperature of the thermal bath, dissipation coefficient, and the strength of interaction between the two modes. In Ref. [57] we have shown that, for initial uni-modal squeezed states, the generation of Gaussian quantum discord takes place during the interaction with the thermal bath, for all nonzero values of the strength of interaction between the coupled bosonic modes. In Ref. [58] we considered the system of two uncoupled bosonic modes interacting with a common thermal reservoir and described the behaviour of the Gaussian quantum steering, when the initial state of the system is a squeezed thermal state. We have shown that the suppression of the Gaussian steering takes place in a finite time, for all temperatures of the thermal reservoir and all values of the squeezing parameter, this behaviour being similar to the well-known phenomenon of entanglement sudden death. This kind of evolution of Gaussian steering and entanglement is in contrast with the dynamics of quantum discord, which decreases to zero asymptotically in time for uncoupled modes. We studied also the dynamics of quantum correlations of two bosonic modes in the case when each mode is coupled to its own thermal reservoir, for Gaussian initial states [59, 60].

In the present work we describe, in the same framework of the theory of open systems, the asymptotic behaviour in the limit of large times of quantum correlations of a subsystem composed of two coupled bosonic modes interacting with a common thermal reservoir. The initial state of the subsystem is taken of Gaussian form, and the Gaussian form of the state is preserved during the evolution under the quantum dynamical semigroup.

The paper is organized as follows. In Sec. 2 we write the Markovian master equation for the density operator of the considered open system interacting with a general environment and solve the evolution equation for the covariance matrix of the state of the bimodal bosonic system. Then we describe in Sec. 3 the asymptotic behaviour of the logarithmic negativity, Gaussian quantum discord, and Gaussian quantum steering for the considered open system. We show that in the limit of infinite time the system evolves asymptotically to an equilibrium state, which may be entangled or separable. We also show that for all initial states, the Gaussian discord tends asymptotically for large times to a definite non-zero value, which depends on the parameters characterizing the thermal bath (temperature and dissipation coefficient), as well as on the strength of interaction between modes, which determines actually the preservation in time of the quantum discord. We calculate the asymptotic Gaussian quantum steering and find out that it is zero for all values of the strength of interaction between the modes in the limit of large times. A summary is given in Sec. 4.

## 2. MASTER EQUATION FOR BOSONIC MODES INTERACTING WITH THE ENVIRONMENT

In order to study the dynamics of the subsystem consisting of two coupled bosonic modes (harmonic oscillators) in weak interaction with a thermal reservoir, we use the axiomatic formalism based on completely positive quantum dynamical semigroups. In this framework, the Markovian irreversible time evolution of an open system is described by the following Kossakowski-Lindblad master equation for the density operator $\rho(t)$ [61–64]:

$$\frac{\mathrm{d}\rho(t)}{\mathrm{d}t}=-\frac{i}{\hbar}[H,\rho(t)]+\frac{1}{2\hbar}\sum_j(2B_j\rho(t)B_j^\dagger-\{\rho(t),B_j^\dagger B_j\}_+), \tag{1}$$

where $H$ denotes the Hamiltonian of the open system and the operators $B_j, B_j^\dagger$ are defined on the Hilbert space of $H$, and describe the interaction of the subsystem with a general environment.

The Hamiltonian of two coupled in coordinates non-resonant harmonic oscillators of identical mass $m$ and frequencies $\omega_1$ and $\omega_2$ is given by

$$H=\tfrac{1}{2m}(p_x^2+p_y^2)+\frac{m}{2}(\omega_1^2x^2+\omega_2^2y^2)+qxy, \tag{2}$$

where $x, y$ are the coordinates and $p_x, p_y$ are the momenta of the two quantum oscillators, and $q$ is the coupling parameter. We are interested to use the Gaussian states, therefore we introduce such quantum dynamical semigroups that preserve this set during the time evolution, by taking the operators $B_j$ as polynomials of the first degree in the canonical variables of coordinates and momenta.

The equations of motion for the quantum correlations of the canonical observables $x, y$ and $p_x, p_y$ are the following ($\mathrm{T}$ denotes the transposed matrix) [64]:

$$\frac{\mathrm{d}\sigma(t)}{\mathrm{d}t} = Z\sigma(t) + \sigma(t)Z^{\mathrm{T}} + 2D, \tag{3}$$

where

$$Z = \begin{pmatrix} -\lambda & 1/m & 0 & 0 \\ -m\omega_1^2 & -\lambda & -q & 0 \\ 0 & 0 & -\lambda & 1/m \\ -q & 0 & -m\omega_2^2 & -\lambda \end{pmatrix}, \tag{4}$$

$D$ is the matrix of diffusion coefficients and $\lambda$ the dissipation coefficient. We introduced the $4\times 4$ bimodal covariance matrix *σ (t)*, the elements of which are defined as $\sigma_{ij} = \mathrm{Tr}[(A_iA_j + A_jA_i)\rho]/2, i, j = 1,..,4,$ with $A = \{x, p_x, y, p_y\},$ which fully characterize any Gaussian state of a bimodal system (up to local displacements). The time-dependent solution of Eq. (3) is given by [64]

$$\sigma(t) = S(t)[\sigma(0) - \sigma(\infty)]S^{\mathrm{T}}(t) + \sigma(\infty), \tag{5}$$

where the matrix $S(t) = \exp(Zt)$ has to fulfill the condition $\lim_{t\to\infty} S(t) = 0.$ The values at infinity are obtained from the equation $Z\sigma(\infty) + \sigma(\infty)Z^{\mathrm{T}} = -2D.$

## 3. DYNAMICS OF QUANTUM CORRELATIONS

### 3.1. Dynamics of quantum entanglement

For Gaussian states, whose statistical properties are fully characterized by second-order moments of quadrature operators, the positive partial transpose (PPT) criterion of separability is necessary and sufficient [65, 66]: a Gaussian state is separable if and only if the partial transpose of its density matrix is non-negative.

The covariance matrix, which is a real, symmetric and positive matrix entirely specifying a two-mode Gaussian state, has the following block structure:

$$\sigma(t) = \begin{pmatrix} A & C \\ C^{\mathrm{T}} & B \end{pmatrix}, \tag{6}$$

where $2\times 2$ Hermitian matrices $A$ and $B$ are the covariance matrices for the single modes, and $C$ contains the cross-correlations between the modes. The covariance matrix (6) (where all first moments can be set to zero by means of local unitary operations that do not affect the quantum entanglement and quantum discord) contains four local symplectic invariants in form of the determinants of the block matrices $A, B, C$, and covariance matrix $\sigma$.

We use the logarithmic negativity as a measure to quantify the degree of entanglement of the two-mode states. For a Gaussian state, the logarithmic negativity is completely determined by the symplectic spectrum of the partial transpose of the covariance matrix: $E = \max\{0, -\log_2 2\tilde{n}_-\}$, where $\tilde{n}_-$ is the smallest of the two symplectic eigenvalues of the partial transpose $\tilde{\sigma}$ of the two-mode covariance matrix $\sigma$ [18]: $2\tilde{n}_{\mp}^2 = \tilde{\Delta} \mp \sqrt{\tilde{\Delta}^2 - 4\det\sigma}$ and $\tilde{\Delta} = \det A + \det B - 2\det C.$ We obtain [67, 68]

$$E(t) = \max\{0, -\frac{1}{2}\log_2[4f(t)]\}, \tag{7}$$

where

$$f(t) = \frac{1}{2}(\det A + \det B) - \det C - \left(\left[\frac{1}{2}(\det A + \det B) - \det C\right]^2 - \det \sigma(t)\right)^{1/2}. \tag{8}$$

$E$ determines the strength of entanglement for $E(t) > 0,$ and if $E(t) = 0,$ then the state is separable. In Refs. [33, 50, 53, 54, 67–70] we described the time evolution of the logarithmic negativity $E(t)$ for two uncoupled bosonic modes interacting with a common environment.

Now we suppose that the only non-zero quantum diffusion coefficients have the following form (we put $\hbar = 1$) [63]:

$$m^2\omega_1^2 D_{xx} = D_{p_x p_x} = \frac{m\omega_1\lambda}{2}\coth\frac{\omega_1}{2kT}, \quad m^2\omega_2^2 D_{yy} = D_{p_y p_y} = \frac{m\omega_2\lambda}{2}\coth\frac{\omega_2}{2kT}, \tag{9}$$

where $k$ is Boltzmann constant and $T$ the temperature of the thermal reservoir. While in the case of independent bosonic modes, this form of the coefficients would determine an asymptotic product Gibbs state describing a thermal equilibrium of the two modes with the thermal bath at temperature $T,$ in the present model with coupled bosonic modes, the asymptotic state does not have anymore the form of a product state. Namely, all the elements of the asymptotic covariance matrix have non-zero values depending, besides the frequencies of the modes, on $T$, $\lambda$, and $q$. Independent of the initial state, in the limit of large times the system evolves asymptotically to an equilibrium state, which may be entangled or separable. The direct interaction between the two modes favors the generation or the preservation in time of the created entanglement, while the temperature of the thermal bath acts towards preventing the generation of entanglement, or suppressing it, once it was created. It is the competition between these two factors - mutual interaction between the two modes and interaction with the thermal bath - that determines the final state of being separable or entangled. Indeed, using the asymptotic covariance matrix, we obtain the following expression of the logarithmic negativity in the limit of large times, as a function of the strength of the interaction $q$, temperature of the thermal bath $T$, and dissipation coefficient $\lambda$:

$$E(\infty) = -\frac{1}{2}\log_2\{\frac{\coth^2\frac{1}{2kT}}{4}[4 + \frac{3(1+\lambda^2)q^2}{(1+\lambda^2)^2 - q^2} - \frac{q}{(1+\lambda^2)^2 - q^2}\sqrt{16(1+\lambda^2)^3 + 8(\lambda^4 - 1)q^2 + q^4}]\}, \tag{10}$$

where we have taken $\omega_1 = \omega_2 \equiv \omega = 1$. In Fig. 1 we represent the dependence of the asymptotic logarithmic negativity $E(\infty)$ (14) as a function of temperature $T$ and $q$. As expected, one can notice that the asymptotic entanglement is increasing with the strength of the interaction and is decreasing with increasing the temperature of the thermal bath.

### 3.2. Dynamics of Gaussian quantum discord

Quantum discord has been defined as the difference between two quantum analogues of classically equivalent expressions of the mutual information, which is a measure of total correlations in a quantum state. It was introduced [6, 7] as a measure of all quantum correlations in a bipartite state, including, but not restricted to entanglement. For pure entangled states quantum discord coincides with the entropy of entanglement. Quantum discord can have non-zero values also for some mixed separable states, and therefore the correlations in such separable states with non-zero discord can characterize the quantumness of these states. For bipartite continuous variable systems, closed formulas of the Gaussian quantum discord have been obtained for bipartite thermal squeezed states [71] and for all two-mode Gaussian states [72].

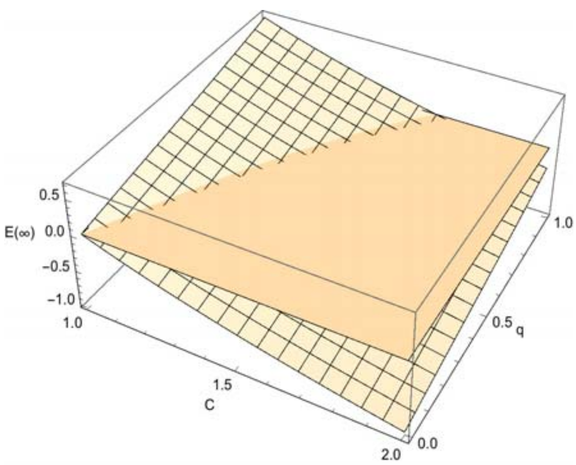


Fig. 1 – Asymptotic logarithmic negativity E(∞) *versus* temperature $T$ of the thermal bath (via C ≡ coth$\omega/2kT$) and interaction strength $q$, for a dissipation coefficient $\lambda = 0.1$ ( $\omega_1 = \omega_2 \equiv \omega = 1, m = \hbar = 1$ ).

The Gaussian quantum discord of a general two-mode Gaussian state $\rho_{12}$ can be defined as the quantum discord where the conditional entropy is restricted to generalized Gaussian positive operator valued measurements (POVM) on the mode 2. In terms of symplectic invariants it is given by (the symmetry between the two modes 1 and 2 is broken) [72] $D = f(\sqrt{\beta}) - f(\nu_-) - f(\nu_+) + f(\sqrt{\varepsilon})$, where

$$f(y) = \frac{y+1}{2}\log\frac{y+1}{2} - \frac{y-1}{2}\log\frac{y-1}{2}, \tag{11}$$

$$\varepsilon = \begin{cases} \dfrac{2\gamma^2 + (\beta-1)(\delta-\alpha) + 2|\gamma|\sqrt{\gamma^2 + (\beta-1)(\delta-\alpha)}}{(\beta-1)^2}, \\ \quad if \ \ (\delta-\alpha\beta)^2 \le (\beta+1)\gamma^2(\alpha+\delta) \\ \dfrac{\alpha\beta - \gamma^2 + \delta - \sqrt{\gamma^4 + (\delta-\alpha\beta)^2 - 2\gamma^2(\delta+\alpha\beta)}}{2\beta}, \\ \quad \text{otherwise} \end{cases} \tag{12}$$

$$\alpha = 4\det A, \beta = 4\det B, \gamma = 4\det C, \delta = 16\det\sigma, \tag{13}$$

and $\nu_\mp$ are the symplectic eigenvalues of the state, given by $2\nu_\mp^2 = \Delta \mp \sqrt{\Delta^2 - 4\delta}$, where $\Delta = \alpha + \beta + 2\gamma$.

Recently [57] we have shown that Gaussian discord tends asymptotically for large times to some definite non-zero value. In the absence of coupling, the discord tends for large times to zero, corresponding to an asymptotic product state. In Fig. 2 it is represented the dependence of the asymptotic discord on the temperature and dissipation t and the strength of interaction between modes. It does not depend on the initial state, and the direct interaction between the two modes determines the generation of the quantum discord and its preservation at asymptotically large times. One can notice that the asymptotic discord is increasing with the strength of the interaction and decreases with increasing temperature and dissipation coefficient.

### 3.3. Dynamics of Gaussian quantum steering

It has been shown in Refs. [10, 73] that the steerability $A \to B$ is present if and only if the following relation does not hold:

$$\sigma + \frac{i}{2} 0_A \oplus \Omega_B \ge 0, \tag{14}$$

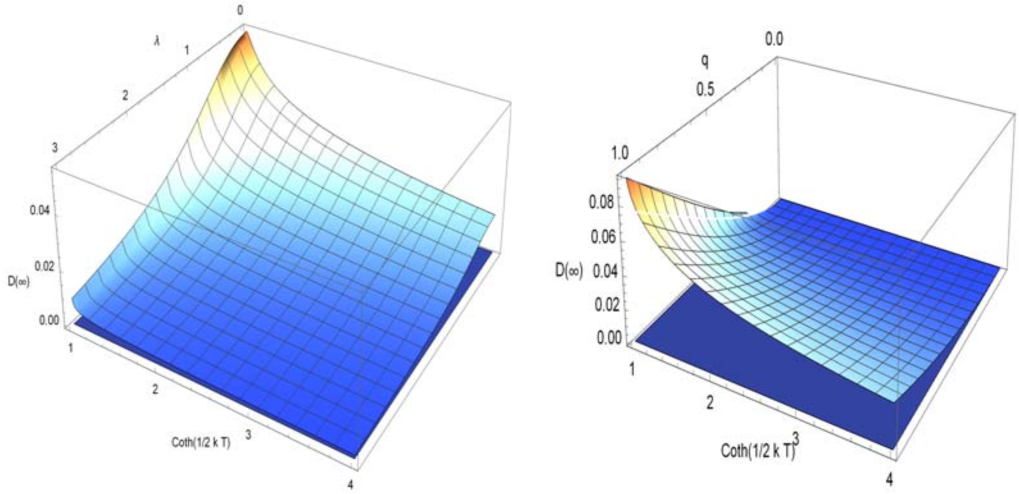


Fig. 2 – Asymptotic Gaussian quantum discord $D(\infty)$ *versus* temperature via $\coth(\omega/2kT)$ and a) dissipation coefficient $\lambda$, for strength of interaction between modes $q = 0.5$ (left) and b) strength of interaction between modes $q$ and dissipation coefficient $\lambda = 1$ (right) (we considered resonant modes with $\omega_1 = \omega_2 \equiv \omega = 1$).

or, equivalently

$$A > 0, \quad \text{and} \quad \Delta_\sigma^B + \frac{i}{2}\Omega_B \geq 0, \tag{15}$$

where $\Delta_\sigma^B = B - CA^{-1}C^{\mathrm{T}}$ is the Schur complement of $A$ in covariance matrix $\sigma$. A measure can be proposed of how much a state with covariance matrix $\sigma$ is $A \to B$ steerable with Gaussian measurements, by quantifying the amount by which the condition (14) is violated, as follows [74]:

$$G^{A\to B}(\sigma) = \max\{0, -\ln 2\nu^B\}, \tag{16}$$

where $\nu^B$ is the symplectic eigenvalue of $\Delta_\sigma^B$. The $B \to A$ steerability can be quantified in a similar manner by computing the symplectic eigenvalues of the Schur complement of $B$ in the covariance matrix $\sigma$. The quantity in Eq. (16) vanishes if and only if the state $\sigma$ is not steerable by Gaussian measurements and is invariant under local symplectic transformations. Note the similarity of this quantity with the negativity that measures the degree of entanglement of a state. In that situation the symplectic eigenvalues of the partially transposed covariance matrix $\sigma$ are relevant, which quantify the violation of the positive partial transpose (PPT) criterion. However, steering is fundamentally different from entanglement, being in general asymmetric with respect to the interchange between steerable parties.

The general quantity proposed in Refs. [74, 75], while easily computable for an arbitrary number of modes, has a particularly simple form when the steered party has one mode:

$$G^{A\to B}(\sigma) = \max\{0, \frac{1}{2}\ln\frac{\det A}{4\det\sigma}\} = \max\{0, S(A) - S(\sigma)\}, \tag{17}$$

where $S$ is the Renyi-2 entropy, which for Gaussian states reads $S = \frac{1}{2}\ln(16\det\sigma)$. For a symmetry between the two modes, both $A \to B$ and $B \to A$ steerability are equivalent. For large times we obtain the following asymptotic value of this expression ($C \equiv \coth\omega/2kT$):

$$G^{A\to B} = G^{B\to A} = \frac{1}{2}\ln\frac{16[(1+\lambda^2)^2 - q^2]}{C^2[16\lambda^4 + (q^2-4)^2 + 8\lambda^2(q^2+4)]}, \tag{18}$$

which has only negative values for non-zero temperatures and is 0 for $T = 0$. Therefore for large times the Gaussian quantum steering is completely missing, independent of the fact that the two bosonic modes are coupled or not.

## 4. SUMMARY

We investigated the Markovian dynamics of the quantum correlations for a system composed of two coupled bosonic modes, embedded in a common thermal bath, in the framework of the theory of open systems based on completely positive quantum dynamical semigroups.

The asymptotic behaviour of quantum correlations does not depend on the initial states of the system, and depends only on the parameters characterizing the thermal reservoir (temperature and dissipation coefficient) and the coupling between the modes. While in the case of uncoupled bosonic modes, for an initial entangled state we could notice only entanglement sudden death, and in the limit of large times the asymptotic state was always a separable state, in the present model we have a completely different scenario, determined by the appearance of the competition between the direct interaction between the two bosonic modes and the influence of the thermal environment. Independent of the initial state, in the limit of large times, the system evolves asymptotically to an equilibrium state, which is not necessarily anymore a separable state: depending on the values of the interaction strength of the two bosons, temperature, and dissipation coefficient, the asymptotic state may be separable or entangled. The asymptotic entanglement is increasing with the strength of the coupling and is decreasing with increasing the temperature of the bath.

Compared to the entanglement, Gaussian discord keeps for all times a non-zero value, including for asymptotic large times. Persistence of the non-zero discord for all times is the qualitatively novel feature of the model with two coupled bosonic modes embedded in a thermal environment, compared to the uncoupled bosonic modes model. The asymptotic discord is increasing with the strength of the coupling and decreases with increasing the temperature of the thermal bath and dissipation coefficient. In order to protect the quantum discord it is necessary to maintain the temperature of the environment and dissipation as low as possible. However, the permanent existence of the quantum discord during the whole time evolution of the system, and even more, its persistence for asymptotically large times, is itself a remarkable fact, compared to the finite-time existence of the quantum entanglement, which is much more sensitive to diffusion and dissipation effects of the environment. We investigated also the dynamics of the Gaussian quantum steering of two coupled bosonic modes immersed in a common thermal bath and have shown that in the limit of large times it is zero for all values of the temperature, dissipation, and strength of interaction between the modes.

## ACKNOWLEDGEMENTS

The author acknowledges the financial support received from the Romanian Ministry of Research and Innovation, through the Project PN 16 42 01 01/2016.

## REFERENCES

1. G. ADESSO, T. R. BROMLEY, M. CIANCIARUSO, J. Phys. A: Math. Theor., **49**, 473001, 2016.
2. M. A. NIELSEN, I. L. CHUANG, *Quantum Computation and Quantum Information*, Cambridge Univ. Press, Cambridge, 2000.
3. C. WEEDBROOK *et al*., Rev. Mod. Phys., **84**, 621, 2012.
4. S. L. BRAUNSTEIN, P. VAN LOOCK, Rev. Mod. Phys., **77**, 513, 2005.
5. R. HORODECKI, P. HORODECKI, M. HORODECKI, K. HORODECKI, Rev. Mod. Phys., **81**, 865, 2009.
6. W. H. ZUREK, Annalen der Physik (Leipzig), **9**, 853, 2000.
7. H. OLLIVIER, W. H. ZUREK, Phys. Rev. Lett., **88**, 017901, 2001.
8. A. EINSTEIN, B. PODOLSKI, N. ROSEN, Phys. Rev. **47**, 777, 1935.
9. C. BRANCIARD *et al*., Phys. Rev. A, **85**, 010301, 2012.
10. H. M. WISEMAN, S. J. JONES, A. C. DOHERTY, Phys. Rev. Lett., **98**, 140402, 2007.
11. A. ISAR, Helv. Phys. Acta, **67**, 436, 1994.
12. A. ISAR, A. SANDULESCU, W. SCHEID, Int. J. Mod. Phys. B, **10**, 2767, 1996.
13. S. OLIVARES, M. G. A. PARIS, A. R. ROSSI, Phys. Lett. A, **319**, 32, 2003.
14. Y. ZHAO, G. H. CHEN, Physica A, **317**, 13, 2003.
15. A. SERAFINI, F. ILLUMINATI, M. G. A. PARIS, S. DE SIENA, Phys. Rev. A, **69**, 022318, 2004.
16. J. S. PRAUZNER-BECHCICKI, J. Phys. A: Math. Gen., **37**, L173, 2004.
17. P. J. DODD, J. J. HALLIWELL, Phys. Rev. A, **69**, 052105, 2004.
18. G. ADESSO, A. SERAFINI, F. ILLUMINATI, Phys. Rev. A, **70**, 022318, 2004.
19. A. ISAR, W. SCHEID, Physica A, **335**, 79, 2004.

20. A. V. DODONOV, V. V. DODONOV, S. S. MIZRAHI, J. Phys. A: Math. Gen., **38**, 683, 2005.
21. A. SERAFINI, M. G. A. PARIS, F. ILLUMINATI, S. DE SIENA, J. Opt. Soc. Am. B, **7**, R19, 2005.
22. G. ADESSO, A. SERAFINI, F. ILLUMINATI, Phys. Rev. A, **73**, 032345, 2006.
23. F. BENATTI, R. FLOREANINI, J. Phys. A: Math. Gen., **39**, 2689, 2006.
24. M. BAN, J. Phys. A: Math. Gen., **39**, 1927, 2006.
25. D. MCHUGH, M. ZIMAN, V. BUZEK, Phys. Rev. A, **74**, 042303, 2006.
26. S. MANISCALCO, S. OLIVARES, M. G. A. PARIS, Phys. Rev. A, **75**, 062119, 2007.
27. J. H. AN, W. M. ZHANG, Phys. Rev. A, **76**, 042127, 2007.
28. A. ISAR, W. SCHEID, Physica A, **373**, 298, 2007.
29. A. ISAR, Eur. J. Phys. Spec. Topics, **160**, 225, 2008.
30. A. ISAR, J. Russ. Laser Res., **30**, 458, 2009.
31. J. P. PAZ, A. J. RONCAGLIA, Phys. Rev. A, **79**, 032102, 2009.
32. A. ISAR, Phys. Scr., **82**, 038116, 2010.
33. A. ISAR, J. Russ. Laser Res., **31**, 182, 2010.
34. A. GHESQUIERE, T. DORLAS, Phys. Lett. A, **377**, 2831, 2013.
35. I. GHIU, P. MARIAN, T. A. MARIAN, Phys. Scr., T **153**, 014028, 2013.
36. P. MARIAN, I. GHIU, T. A. MARIAN, Phys. Rev. A, **88**, 012316, 2013.
37. I. GHIU, Phys. Scr., T **160**, 014013, 2014.
38. I. GHIU, P. MARIAN, T. A. MARIAN, Phys. Scr., T **160**, 014014, 2014.
39. Y. ZHAO, F. ZHENG, J. LIU, Y. YAO, Physica A, **423**, 80, 2015.
40. P. MARIAN, I. GHIU, T. A. MARIAN, Phys. Scr., **90**, 074041, 2015.
41. T. MIHAESCU, A. ISAR, Rom. J. Phys., **60**, 853, 2015.
42. S. SUCIU, A. ISAR, Rom. J. Phys., **60**, 859, 2015.
43. H. A. BOURA, A. ISAR, Rom. J. Phys., **60**, 1278, 2015.
44. P. MARIAN, T. A. MARIAN, J. Phys. A: Math. Theor., **48**, 115301, 2015.
45. P. MARIAN, T. A. MARIAN, Phys. Rev. A, **93**, 052330, 2015.
46. I. GHIU, C. GHIU, A. ISAR, Proc. Romanian Acad. A, **16**, 499, 2015.
47. I. GHIU, A. ISAR, Rom. J. Phys., **61**, 768, 2016.
48. H. A. BOURA, A. ISAR, Y. A. KOURBOLAGH, Rom. J. Phys., **61**, 1161, 2016.
49. H. A. BOURA, A. ISAR, Y. A. KOURBOLAGH, Rom. Rep. Phys., **68**, 19, 2016.
50. A. ISAR, Phys. Scr., T **140**, 014023, 2010.
51. A. ISAR, Open Sys. Inf. Dynamics, **18**, 175, 2011.
52. A. ISAR, Phys. Scr., T **143**, 014012, 2011.
53. A. ISAR, Phys. Scr., T **147**, 014015, 2012.
54. A. ISAR, Phys. Scr., T **153**, 014035, 2013.
55. S. SUCIU, A. ISAR, Rom. J. Phys., **61**, 1474, 2016.
56. A. ISAR, Open Sys. Inf. Dynamics, **23**, 1650007, 2016.
57. A. ISAR, T. MIHAESCU, Eur. Phys. J. D, **71**, 144, 2017.
58. T. MIHAESCU, A. ISAR, Rom. J. Phys. **62**, 107, 2017.
59. A. ISAR, Open Sys. Inf. Dynamics, **20**, 1340003, 2013.
60. A. ISAR, Phys. Scr., T **160**, 014019, 2014.
61. V. GORINI, A. KOSSAKOWSKI, E. C. G. SUDARSHAN, J. Math. Phys., **17**, 821, 1976.
62. G. LINDBLAD, Commun. Math. Phys., **48**, 119, 1976.
63. A. ISAR, A. SANDULESCU, H. SCUTARU, E. STEFANESCU, W. SCHEID, Int. J. Mod. Phys. E, **3**, 635, 1994.
64. A. SANDULESCU, H. SCUTARU, W. SCHEID, J. Phys. A: Math. Gen., **20**, 2121, 1987.
65. R. SIMON, Phys. Rev. Lett., **84**, 2726, 2000.
66. L. M. DUAN, G. GIEDKE, J. I. CIRAC, P. ZOLLER, Phys. Rev. Lett., **84**, 2722, 2000.
67. A. ISAR, Int. J. Quantum Inf., **6**, 689, 2008.
68. A. ISAR, Open Sys. Inf. Dynamics, **16**, 205, 2009.
69. A. ISAR, J. Russ. Laser Res., **28**, 439, 2007.
70. A. ISAR, Phys. Scr., T **135**, 014033, 2009.
71. P. GIORDA, M. G. A. PARIS, Phys. Rev. Lett., **105**, 020503, 2010.
72. G. ADESSO, A. DATTA, Phys. Rev. Lett., **105**, 030501, 2010.
73. S. J. JONES, H. M. WISEMAN, A. C. DOHERTY, Phys. Rev. A, **76**, 052116, 2007.
74. I. KOGIAS, A. R. LEE, S. RAGY, G. ADESSO, Phys. Rev. Lett., **114**, 060403, 2015.
75. I. KOGIAS, P. SKRZYPCZYK, D. CAVALCANTI, A. ACIN, G. ADESSO, Phys. Rev. Lett., **115**, 210401, 2015.

*Received August 2, 2017*